Bargaining Under Strategic Uncertainty:
The Role of Second-Order Optimism
(Online Appendix)

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Appendix F Comparative Statics

Fix two bargaining games $\mathcal{B}_*$ and $\mathcal{B}_{**}$. For each n, write $[\underline{x}_*^n, \overline{x}_*^n]$ for the interval associated with $\mathcal{B}_*$ and write $[\underline{x}_{**}^n, \overline{x}_{**}^n]$ for the interval associated with $\mathcal{B}_{**}$.

Discount Factor We now suppose $\mathcal{B}_*$ and $\mathcal{B}_{**}$ differ only in their discount factor: $\mathcal{B}_*$ is associated with δ_* and $\mathcal{B}_{**}$ is associated with δ_{**}. Remark D.1 implies the following:

Proposition F.1. *Let $\delta_{**} > \delta_*$.*

(i) If N is either infinite or odd, $\overline{x}_{**}^n > \overline{x}_*^n$.

(ii) If N is either infinite or even, $\underline{x}_{**}^n > \underline{x}_*^n$.

Finally, the DC must bind if δ is sufficiently high.

Remark F.1. Let $N < \infty$. There exists $\overline{\delta} \in (0, 1)$ so that, for all $\delta \in (\overline{\delta}, 1)$,

(i) $\delta^{N-n} > \frac{1-\delta}{\delta^{n-1}}$, if N is odd

(ii) $1 - \frac{\delta(1-\delta)}{\delta^{n-1}} > 1 - \delta^{N-n}$, if N is even.

Deadline We now suppose $\mathcal{B}_*$ and $\mathcal{B}_{**}$ differ only in their deadline: $\mathcal{B}_*$ is associated with N_* and $\mathcal{B}_{**}$ is associated with N_{**}.

Proposition F.2. *Fix $N_{**} \geq N_*$. Then, $[\underline{x}_*^n, \overline{x}_*^n] \subseteq [\underline{x}_{**}^n, \overline{x}_{**}^n]$ provided that either (i) N_* and N_{**} are both even; (ii) N_* and N_{**} are both odd; or (iii) $N_{**} = \infty$.*

Proof. Fix $N_{**} > N_*$. When $N_{**} = \infty$, the claim is immediate from the definitions. If N_{**} and N_* are even, then $\underline{x}_*^n = \underline{x}_{**}^n$ and

$$\overline{x}_*^n = \min\{1 - \frac{\delta(1-\delta)}{\delta^{n-1}}, 1 - \delta^{N_*-n}\} \leq \min\{1 - \frac{\delta(1-\delta)}{\delta^{n-1}}, 1 - \delta^{N_{**}-n}\} = \overline{x}_{**}^n.$$

If N_{**} and N_* are odd, then

$$\underline{x}_*^n = \max\{\frac{1-\delta}{\delta^{n-1}}, \delta^{N_*-n}\} \geq \max\{\frac{1-\delta}{\delta^{n-1}}, \delta^{N_{**}-n}\} = \underline{x}_{**}^n$$

and $\overline{x}_{**}^{(n,N_{**},\delta)} = \overline{x}_*^{(n,N_*,\delta)}$. ■

Appendix G Proofs for Appendix A

G.1 Proof of Proposition A.1

Proposition A.1 follows from the following:

Proposition G.1. *Fix an epistemic game $(\mathcal{B}, \mathcal{T})$ with a deadline N and suppose Bi proposes in the last period. Let (x_1^*, x_2^*, n) be the outcome induced by some (s_1^*, s_2^*) and suppose one of the following hold:*

(i) For some $k \geq 1$, $n = N - 2k + 1$ and $(s_i^*, t_i^*, s_{-i}^*, t_{-i}^*) \in R_i^2 \times R_{-i}^1$.

(ii) For some $k \geq 1$, $n = N - 2k$ and $(s_i^*, t_i^*, s_{-i}^*, t_{-i}^*) \in R_i^{2k+2} \times R_{-i}^{2k+1}$.

Then, $x_i^* \geq \delta^{N-n}$.

Lemma G.1. *Fix a game with a deadline $N < \infty$ and suppose Bi proposes in the last period. Let $\xi(\zeta(s_1^*, t_1^*, s_2^*, t_2^*)) = (x_1^*, x_2^*, N - 2k)$ for some $\frac{N-1}{2} \geq k \geq 1$. If $(s_i^*, t_i^*, s_{-i}^*, t_{-i}^*) \in R_i^{2k+2} \times R_{-i}^{2k+1}$, then $x_i^* \geq \delta^{N-(N-2k)}$.*

Proof. We will suppose the result is true for all j with $k > j \geq 1$ and will show that it is also true for k.[23] Throughout, we fix a state $(s_i^*, t_i^*, s_{-i}^*, t_{-i}^*) \in R_i^{2k+2} \times R_{-i}^{2k+1}$. We will show that $\xi(\zeta(s_1^*, t_1^*, s_2^*, t_2^*)) = (x_1^*, x_2^*, N - 2k)$ implies $x_i^* \geq \delta^{N-(N-2k)} = \delta^{2k}$. Note that, along the path induced by (s_1^*, s_2^*), there is a $(N - 2k)$-period history $h^* \in H_i^P$ with $s_i^*(h^*) = x_i^*$ and $s_{-i}^*(h^*, x_i^*) = \mathrm{a}$.

Case A: First, suppose $\beta_{i,h^*}(t_i^*)$ assigns probability one to

$$\mathsf{A}_{-i}[h^*, x_i^*] := \{r_{-i} \in S_{-i}(h^*) : r_{-i}(h^*, x_i^*) = \mathrm{a}\} \times T_{-i}.$$

Then, $\mathbb{E}\pi_i[s_i^*|t_i^*, h^*] = \delta^{N-2k-1} x_i^*$.

Note next that t_i^* strongly believes R_{-i}^1 and $R_{-i}^1 \cap [S_{-i}(h^*) \times T_{-i}] \neq \emptyset$ (in particular, $(s_{-i}^*, t_{-i}^*) \in R_{-i}^1 \cap [S_{-i}(h^*) \times T_{-i}]$). It follows that, for each $x \in [0, 1)$, t_i^* can secure $\delta^{N-1} x$ at h^*. (See Lemma C.5.) Since (s_i^*, t_i^*) is rational, it follows that, for each $x \in [0, 1)$, $\delta^{N-2k-1} x_i^* \geq \delta^{N-1} x$ or $x_i^* \geq \delta^{2k} x$. From this, $x_i^* \geq \delta^{2k}$.

Case B: Next, suppose $\beta_{i,h^*}(t_i^*)$ assigns strictly positive probability to

$$\mathsf{R}_{-i}[h^*, x_i^*] := \{r_{-i} \in S_{-i}(h^*) : r_{-i}(h^*, x_i^*) = \mathrm{r}\} \times T_{-i}.$$

Since t_i^* strongly believes R_{-i}^{2k+1} and $R_{-i}^{2k+1} \cap [S_{-i}(h^*) \times T_{-i}] \neq \emptyset$ (in particular, $(s_{-i}^*, t_{-i}^*) \in R_{-i}^{2k+1} \cap [S_{-i}(h^*) \times T_{-i}]$), it follows that $\beta_{i,h^*}(t_i^*)$ assigns strictly positive probability to $\mathsf{R}_{-i}[h^*, x_i^*] \cap R_{-i}^{2k+1}$.

[23]The base case of $k = 1$ follows the same proof with the following two amendments: In Case B.2, take $j = 0$, and Case B.3 does not obtain.

Fix some $(r_{-i}, u_{-i}) \in \mathsf{R}_{-i}[h^*, x_i^*] \cap R_{-i}^{2k+1}$. We will show that

$$\mathbb{E}\pi_{-i}[r_{-i}|u_{-i}^*, (h^*, x_i^*, \mathrm{r})] \le \delta^{N-2k}(1 - \delta^{2k-1}).$$

From this, the claim follows: Since (r_{-i}, u_{-i}) is rational,

$$\mathbb{E}\pi_{-i}[r_{-i}|u_{-i}, (h^*, x_i^*, \mathrm{r})] = \mathbb{E}\pi_{-i}[r_{-i}|u_{-i}, (h^*, x_i^*)] \ge \mathbb{E}\pi_{-i}[q_{-i}|u_{-i}, (h^*, x_i^*)]$$

for $q_{-i} \in S_{-i}(h^*, x_i^*)$ with $q_{-i}(h^*, x_i^*) = \mathrm{a}$. Thus,

$$\delta^{N-2k}(1 - \delta^{2k-1}) \ge \mathbb{E}\pi_{-i}[r_{-i}|u_{-i}, (h^*, x_i^*, \mathrm{r})] \ge \delta^{N-2k-1}(1 - x_i^*)$$

or $x_i^* \ge 1 - \delta(1 - \delta^{2k-1}) > \delta^{2k}$, as desired.

The remainder of the proof is devoted to showing $\mathbb{E}\pi_{-i}[r_{-i}|u_{-i}, (h^*, x_i^*, \mathrm{r})] \le \delta^{N-2k}(1 - \delta^{2k-1})$. For this, note that u_{-i} strongly believes R_i^{2k} and $R_i^{2k} \cap [S_i(h^*, x_i^*, \mathrm{r}) \times T_i] \ne \emptyset$. (Here, we used the fact that $(s_i^*, t_i^*) \in R_i^{2k} \cap [S_i(h^*, x_i^*, \mathrm{r}) \times T_i]$.) With this, $\beta_{-i,(h^*, x_i^*, \mathrm{r})}(u_{-i})$ assigns probability one to R_i^{2k}. As such, to show that $\mathbb{E}\pi_{-i}[r_{-i}|u_{-i}, (h^*, x_i^*, \mathrm{r})] \le \delta^{N-2k}(1 - \delta^{2k-1})$, it suffices to show the following:

> **Claim:** If $(r_i, u_i) \in R_i^{2k} \cap [S_i(h^*, x_i^*, \mathrm{r}) \times T_i]$ with $\xi(\zeta(r_i, r_{-i})) = (x_1, x_2, n)$, then $x_{-i} \le 1 - \delta^{2k-1}$ and $n \ge N - 2k + 1$.

We now turn to show this claim.

Fix $(r_i, u_i) \in R_i^{2k} \cap [S_i(h^*, x_i^*, \mathrm{r}) \times T_i]$ with $\xi(\zeta(r_i, r_{-i})) = (x_1, x_2, n)$. Certainly, $n \ge N - 2k + 1$. So, we focus on showing that $x_{-i} \le 1 - \delta^{2k-1}$. Write $h[n]$ for the n-period history in $H_1^P \cup H_2^P$ along the path induced by (r_1, r_2). There will be three subcases, based on whether $h[n] \in H_i^P$ or $h[n] \in H_{-i}^P$.

Subcase 1. Suppose $n = N$. Note that $(r_i, u_i) \in R_i^{2k} \subseteq R_i^2$ and $(r_{-i}, u_{-i}) \in R_{-i}^1 \cap (S_{-i}(h[n]) \times T_{-i})$. So, $\beta_{i,h[n]}(u_i)$ assigns probability one to

$$\{s_{-i} \in S_{-i}(h[n]) : s_{-i}(h[n], x) = \mathrm{a}, \text{ for all } x \in [0, 1)\} \times T_{-i}.$$

(Use Corollary B.2(i) to get that this set is Borel.) Since (r_i, u_i) is rational, $r_i(h[n]) = 1$ and so $x_{-i} = 0$.

Subcase 2. Suppose $n = N - 2j + 1$ for some j with $k \ge j \ge 1$, so that $h[n] \in H_{-i}^P$. Since $(r_i, u_i, r_{-i}, u_{-i}) \in R^{2k} \subseteq R^2$. It then follows from Lemma C.6(i) that $x_i \ge \delta^{N-(N-2j+1)} \ge \delta^{2k-1}$. Thus, $x_{-i} \le 1 - \delta^{2k-1}$, as desired.

Subcase 3. Suppose $n = N - 2j$ for some j with $k > j \ge 1$, so that $h[n] \in H_i^P$. Note, $(r_i, u_i, r_{-i}, u_{-i}) \in R^{2k} \subseteq R^{2(j+1)} \subseteq R_i^{2j+2} \times R_{-i}^{2j+1}$. It follows from the assumption that the claim holds for all $j < k$ that $x_i \ge \delta^{2j} \ge \delta^{2k-1}$. Thus, $x_{-i} \le 1 - \delta^{2k-1}$, as desired. ■

Proof of Proposition G.1. Immediate from Lemmata C.6(i) and G.1. ■

G.2 Proof of Proposition A.2

It will be useful to begin with a result that speaks to when the UC constraint must be satisfied.

Definition G.1. Call $Q_1 \times Q_2 \subseteq S_1 \times S_2$ a **constant set** if, for any $(s_1, s_2), (r_1, r_2) \in Q_1 \times Q_2$, $\pi_1(s_1, s_2) = \pi_1(r_1, r_2)$ and $\pi_2(s_1, s_2) = \pi_2(r_1, r_2)$.

Proposition G.2. *Fix an epistemic game $(\mathcal{B}, \mathcal{T})$ so that $\text{proj}_S R^\infty$ is a constant set. If (x_1^*, x_2^*, n) is an outcome induced by some $(s_1^*, s_2^*) \in \text{proj}_S R^\infty$, then*

(i) $x_1^* \geq \frac{1-\delta}{\delta^{n-1}}$, *and*

(ii) $x_2^* \geq \frac{\delta(1-\delta)}{\delta^{n-1}}$.

Remark G.1. If $\pi_1(s_1, s_2) = \pi_1(r_1, r_2)$ and $\pi_2(s_1, s_2) = \pi_2(r_1, r_2)$, then $\xi(\zeta(s_1, s_2)) = \xi(\zeta(r_1, r_2))$.

Proof of Proposition G.2. Throughout, fix some $(s_1^*, t_1^*, s_2^*, t_2^*) \in R^\infty$ with $\xi(\zeta(s_1^*, s_2^*)) = (x_1^*, x_2^*, n)$.

Since t_1^* strongly believes each R_2^m, $\beta_{1,\phi}(t_1^*)(R_2^m) = 1$ for each $m \geq 1$. From this, $\beta_{1,\phi}(t_1^*)(R_2^\infty) = 1$. Since $\text{proj}_S R^\infty$ is constant and $(s_1^*, s_2^*) \in \text{proj}_S R^\infty$, it follows from Remark G.1 that

$$R_2^\infty \subseteq \{r_2 : \xi(\zeta(s_1^*, r_2)) = \xi(\zeta(s_1^*, s_2^*))\} \times T_2.$$

Thus, by Lemma C.2, $x_1^* \geq \frac{1-\delta}{\delta^{n-1}}$.

If $n = 1$, then it follows from Lemma C.1(ii) that $x_2^* \geq \frac{\delta(1-\delta)}{\delta^{n-1}}$. So, we focus on the case of $n \geq 2$. Note that, along the path of play, there is a two-period history $h^* \in H_2^P$. Since t_2^* strongly believes each R_1^m and $(s_1^*, t_1^*) \in R_1^m \cap [S_1(h^*) \times T_1] \neq \emptyset$, $\beta_{2,h^*)}(t_2^*)(R_1^m) = 1$ for each $m \geq 1$. From this, $\beta_{2,h^*}(t_2^*)(R_1^\infty) = 1$. Since $\text{proj}_S R^\infty$ is constant and $(s_1^*, s_2^*) \in \text{proj}_S R^\infty$, it follows from Remark G.1 that

$$R_1^\infty \subseteq \{r_1 : \xi(\zeta(r_1, s_2^*)) = \xi(\zeta(s_1^*, s_2^*))\} \times T_1.$$

Thus, by Lemma C.3, $x_2^* \geq \frac{\delta(1-\delta)}{\delta^{n-1}}$. ■

Proof of Proposition A.2. Suppose the conclusion is false. Then, by Proposition G.2, $\text{proj}_{S_1} R_1^\infty \times \text{proj}_{S_2} R_2^\infty$ is not a constant set. Since the game has a deadline, this implies that we can find some B$(-i)$ and some history $h_{-i} \in H_{-i}$, so that the following holds:

(a) $\xi(\zeta(\text{proj}_S R^\infty(h_{-i})))$ contains at least two outcomes, but

(b) for any history $h' \in H$ that strictly follows h_{-i}, $\xi(\zeta(\text{proj}_S R^\infty(h')))$ contains, at most, one outcome.

Write $h_i \in H_i$ for the last history in H_i that precedes h_{-i}. So, if $h_{-i} \in H^R_{-i}$, then $h_{-i} = (h_i, x)$ for some $x \in [0,1]$. If $h_{-i} \in H^P_{-i}$, then $h_{-i} = (h_i, x, \text{r})$ for some $x \in [0,1]$.

We will show that, for any $(s_i, t_i) \in R^\infty_i(h_{-i})$, there is some (s_{-i}, t_{-i}) so that, at $(s_i, t_i, s_{-i}, t_{-i})$, B$i$ faces uncertainty about how B$(-i)$ breaks indifferences.

Step A: This step shows that any two outcomes in $\xi(\zeta(\text{proj}_S R^\infty(h)))$ are B$(-i)$ equivalent. Fix $(s_i, t_i, s_{-i}, t_{-i}), (r_i, u_i, r_{-i}, u_{-i}) \in R^\infty(h_{-i})$. Then, t_{-i} and u_{-i} strongly believe $R^1_i, R^2_i, \ldots$. It follows from the conjunction property of strong belief that t_{-i} and u_{-i} strongly believe R^∞_{-i}. Using the fact that (s_{-i}, t_{-i}) and (r_{-i}, u_{-i}) are rational, plus condition (b), it follows that

(i) $\mathbb{E}[s_{-i}|t_{-i}, h_{-i}] = \Pi_{-i}(\xi(\zeta(s_i, s_{-i}))) \geq \Pi_{-i}(\xi(\zeta(r_i, r_{-i}))) = \mathbb{E}[r_{-i}|t_{-i}, h_{-i}]$, and

(ii) $\mathbb{E}[r_{-i}|u_{-i}, h_{-i}] = \Pi_{-i}(\xi(\zeta(r_i, r_{-i}))) \geq \Pi_{-i}(\xi(\zeta(s_i, s_{-i}))) = \mathbb{E}[s_{-i}|u_{-i}, h_{-i}]$.

Thus, $\Pi_{-i}(\xi(\zeta(s_i, s_{-i}))) = \Pi_{-i}(\xi(\zeta(r_i, r_{-i})))$, as required.

Step B: First, suppose $h_{-i} = (h_i, x) \in H^R_{-i}$. Fix some $(s_i, t_i) \in R^\infty_i(h_{-i})$. Note that the sets

- $\mathsf{A}_{-i}[h_{-i}] := \{q_{-i} \in S_{-i}(h_{-i}) : q_{-i}(h_{-i}) = \text{a}\} \times T_{-i}$, and
- $\mathsf{R}_{-i}[h_{-i}] := \{q_{-i} \in S_{-i}(h_{-i}) : q_{-i}(h_{-i}) = \text{r}\} \times T_{-i}$

are both Borel. (Lemma B.1(ii).) Notice, by construction of h_{-i}, there exists $(s_{-i}, t_{-i}), (r_{-i}, u_{-i}) \in R^\infty_{-i}(h_{-i})$ with $s_{-i}(h_{-i}) = \text{a} \neq \text{r} = r_{-i}(h_{-i})$. States in $\{(s_i, t_i)\} \times \mathsf{A}_{-i}[h_{-i}]$ (resp. $\{(s_i, t_i)\} \times \mathsf{R}_{-i}[h_{-i}]$) necessarily induce distinct outcomes from $(s_i, t_i, r_{-i}, u_{-i})$ (resp. $(s_i, t_i, s_{-i}, t_{-i})$), since they result in the bargaining game concluding in different bargaining phases. It follows that, if $\beta_{i,h_i}(t_i)(\mathsf{A}_{-i}[h_{-i}]) > 0$ (resp. $\beta_{i,h_i}(t_i)(\mathsf{R}_{-i}[h_{-i}]) > 0$), then at $(s_i, t_i, r_{-i}, u_{-i})$ (resp. $(s_i, t_i, s_{-i}, t_{-i})$) B$i$ faces uncertainty about how B$(-i)$ breaks indifferences.

Step C: Finally, suppose $h_{-i} = (h_i, x, \text{r}) \in H^R_{-i}$. Fix some $(s_i, t_i) \in R^\infty_i(h_{-i})$. If $(s_1, \beta_{i,h}(t_i))$ does not have a distinguished outcome, then at any state in $\{(s_i, t_i)\} \times R^\infty_{-i}(h_{-i})$, B$i$ is uncertain about how B$(-i)$ breaks indifferences. So, we will suppose $(s_i, \beta_{i,h}(t_i))$ has a distinguished outcome—i.e., there exists some event $E_{-i} \subseteq S_{-i} \times T_{-i}$ with $\beta_{i,h}(\{s_i\} \times E_{-i}) > 0$ and $\xi(\zeta(\{s_i\} \times \text{proj}_{S_{-i}} E_{-i})) = \{(x^*_1, x^*_2, n)\}$. Note that, by (a), there exists some $(s_{-i}, t_{-i}) \in R^\infty_{-i}(h_{-i})$ with $\xi(\zeta(s_i, s_{-i})) \neq (x^*_1, x^*_2, n)$. Then, at $(s_i, t_i, s_{-i}, t_{-i})$, B$i$ faces uncertainty about how B$(-i)$ breaks indifferences. ■

Appendix H Model Extensions

H.1 Frequent Offers

Observe that we can write the gap function as $\text{gap}(n,\Delta) = 1 - e^{r(n-1)\Delta}(1-e^{-2r\Delta})$. Let $\overline{n} : (0,\infty) \to \mathbb{R}_+$ be defined by $\overline{n}(\Delta) = 1 - \frac{\ln(1-e^{-2r\Delta})}{r\Delta} > 1$.

Lemma H.1.

(i) $n \leq \overline{n}(\Delta)$ *if and only if* $\text{gap}(n,\Delta) \geq 0$.

(ii) $n \geq \overline{n}(\Delta)$ *if and only if* $\text{gap}(n,\Delta) \leq 0$

Proof. Observe that $\text{gap}(n,\Delta) \geq 0$ if and only if

$$\ln(1) \geq \ln(e^{r(n-1)\Delta}) + \ln(1-e^{-2r\Delta})$$

or if and only if $-\ln(1-e^{-2r\Delta}) \geq r(n-1)\Delta$. Thus, $n \leq \overline{n}(\Delta)$ if and only if $\text{gap}(n,\Delta) \geq 0$. Reversing the inequalities gives that $n \geq \overline{n}(\Delta)$ if and only if $\text{gap}(n,\Delta) \leq 0$. ■

Define $\text{del} : (0,\infty) \to \mathbb{R}_+$ so that $\text{del}(\Delta) = (\lfloor \overline{n}(\Delta) \rfloor - 1)\Delta$. Also define functions $\overline{\text{del}} : (0,\infty) \to \mathbb{R}_+$ and $\underline{\text{del}} : (0,\infty) \to \mathbb{R}_+$, so that $\overline{\text{del}}(\Delta) = (\overline{n}(\Delta) - 1)\Delta$ and $\underline{\text{del}}(\Delta) = (\overline{n}(\Delta) - 2)\Delta$. Observe that, for each Δ, $\overline{\text{del}}(\Delta) \geq \text{del}(\Delta) > \underline{\text{del}}(\Delta)$.

Lemma H.2. $\overline{\text{del}}(\Delta)$ *is strictly decreasing in* Δ *and convex.*

Proof. Notice that $\overline{\text{del}}(\Delta) = -\frac{1}{r}\ln(1-e^{-2r\Delta})$. So,

$$\frac{\partial \overline{\text{del}}}{\partial \Delta} = -\frac{2e^{-2r\Delta}}{(1-e^{-2r\Delta})} < 0$$

Since $e^{-2r\Delta} < 1$, $\partial \overline{\text{del}} / \partial \Delta < 0$. Moreover,

$$\frac{\partial^2 \overline{\text{del}}}{\partial \Delta^2} = 4re^{-2r\Delta}\frac{(1-e^{-2r\Delta}) + e^{-2r\Delta}}{(1-e^{-2r\Delta})^2}.$$

Again using the fact that $e^{-2r\Delta} \in (0,1)$, $\partial^2 \overline{\text{del}} / \partial \Delta^2 > 0$. ■

Lemma H.3. $\lim_{\Delta \to 0^+} \text{del} = \infty$.

Proof. Fix $\varepsilon > 0$. Since $\lim_{\Delta \to 0^+} \underline{\text{del}} = \infty$, there exists some $\rho > 0$ so that $\text{del}(\Delta) \geq \underline{\text{del}}(\Delta) > \varepsilon$ whenever $\Delta \in (0,\rho)$. ■

H.2 Outside Options

Fix some finite time period $n \geq 2$ and some $x^* \in [\underline{\underline{x}}^n, \overline{\overline{x}}^n]$. We want to show that there exists an epistemic game and a state thereof, viz. $(s_1^*, t_1^*, s_2^*, t_2^*)$, at which there is RCSBR, on-path strategic certainty and the outcome is $(x_1^*, x_2^*, n) = (x^*, 1 - x^*, n)$. The argument follows Appendix C.2. The key step is to redefine the strategy profile (s_1^*, s_2^*).

Recall from Appendix C.2 that $h^* = (1, \mathrm{r}, \ldots, 1, \mathrm{r})$; that is, it is a history in which there are $(n-1)$ offers of 1 followed by $(n-1)$ rejections. The strategy s_i^* satisfies the following properties: First, for any history $h \in H_i^P$, set (i) $s_i^*(h) = x_i^*$ if $h = h^*$, and (ii) $s_i^*(h) = 1$ if $h \neq h^*$. Second, for any history $h \in H_{-i}^P$, let $s_i^*(h, x) = \mathrm{a}$ if and only if either (i) $x \in [0, \min\{1 - w_i, 1 - \delta\})$, or (ii) $h = h^*$ and $x = x_{-i}^*$. Third, for any history $h \in H_{-i}^P$, let $s_i^*(h, x)$ take the outside option if and only if $x \in [1 - w_i, 1 - \delta)$. Fourth, for all other histories $h \in H_i^P$, let $s_i^*(h, x)$ reject the offer and continue negotiations if and only if $x \in [1 - \delta, 1]$.

The construction of the type structure is as in Appendix C.2. The arguments follow verbatim, with the exception of the following change to Lemma C.8:

Lemma H.4. *Fix $h \in H_i$. If $s_i^*, r_i \in S_i(h)$, then*

(i) $\mathbb{E}\pi_i[s_i^* | t_i^*, h] \geq \mathbb{E}\pi_i[r_i | t_i^*, h]$, *and*

(ii) $\mathbb{E}\pi_i[r_i | t_i^*, h] = \mathbb{E}\pi_i[s_i^* | t_i^*, h]$ *implies that either*

- $r_i(h) = s_i^*(h)$,
- $h = (\cdot, 1 - \delta) \in H_i^R$, $w_i \neq \delta$, *and* $r_i(h) = \mathrm{a}$,
- $h = (\cdot, 1 - w_i) \in H_i^R$, $w_i > \delta$, *and* $r_i(h) = \mathrm{a}$, *or*
- $h = (\cdot, 1 - \delta) \in H_i^R$, $w_i = \delta$, *and* $r_i(h)$ *is either* a *or exercise the outside option.*

The proof is analogous to the proof of Lemma C.8 and so omitted.

H.3 Discrete Grid of Feasible Allocations

Suppose the set of feasible allocations is constrained to lie in the discrete grid

$$\mathcal{A} = \left\{ (x_1, x_2) \text{ is an allocation and } x_1 \in \left\{ \frac{0}{K}, \frac{1}{K}, \ldots, \frac{K-1}{K}, \frac{K}{K} \right\} \right\}.$$

That is, Bi can offer an allocation (x_1, x_2) if and only if it lies in $\mathcal{A}$. Say that $\mathcal{A}$ has a grid of size K.

Recall that the UC is driven by the fact that, upfront, Bi reasons that B$(-i)$ will accept any offer (x_1, x_2) with $x_{-i} \in (\delta, 1]$. In the case where $\mathcal{A}$ is a continuum, rationality implies that Bi must offer an allocation with $x_{-i} = \delta$, expecting that offer to be accepted. Notice that this conclusion is driven by the fact that, in the continuum case, no $x_{-i} < \delta$ can maximize Bi's subjective expected utility. But, in the case where $\mathcal{A}$ is a discrete grid with some $\frac{j+1}{K} > \delta > \frac{j}{K}$, $\frac{j+1}{K}$ may well maximize Bi's subjective expected utility (if she expects B$(-i)$ to reject offers with $\delta > x_{-i}$). In light of this, write

$$\delta^+ = \min\left\{\frac{j}{K} : \frac{j}{K} \geq \delta\right\}.$$

Bi can only conclude that (x_1, x_2) will be accepted if $x_{-i} \geq \delta^+$. This relaxes Bi's UC: B1's UC is given by $\delta^{n-1}x_1^* \geq (1-\delta^+)$ and B2's UC is given by $\delta^{n-1}(1-x_1^*) \geq \delta(1-\delta^+)$.

Suppose Bi makes the proposal in the last period. When she accepts an earlier allocation, she reasons that, in the last period, B$(-i)$ would accept any (x_1, x_2) with $x_{-i} > 0$. In the case where $\mathcal{A}$ is a continuum, this would allow Bi to anticipate getting the full share of the pie, if the final period were reached. But, when $\mathcal{A}$ is a discrete grid, this only allows Bi to anticipate (for sure) getting $\frac{K-1}{K}$. (She may or may not assign probability 1 to getting the full pie.) Thus, Bi's deadline constraint is given by $\delta^{n-1}x_i^* \geq \delta^{N-1}\frac{K-1}{K}$.

Thus, the B1-B2 UCs and the DC are relaxed. If the grid is coarse, this can lead to new possibilities for delay. In particular, choose K so that $\delta \in (\frac{K-1}{K}, 1)$. (If δ is small, this may require choosing $K = 1$; however, if δ is large, this may involve choosing a somewhat finer grid.) In that case, $1 - \delta^+ = 0$, and so any allocation trivially satisfies the two UCs. Moreover, for any period $n \leq N$, there is an n-period outcome (x_1^*, x_2^*, n) that would satisfy the DC; simply take $x_i^* = \frac{K-1}{K}$ for the Bi with the deadline bargaining power.

However, when the grid is sufficiently fine—that is, when K is large—the limitations and possibilities for delay correspond to those in the case of the continuum. For example, take the case of a three-period deadline. When K is large, there cannot be delay until the last period. If there is delay until the penultimate period, then the agreed upon allocation (x_1^*, x_2^*) must satisfy $x_1^* \in [\max\{\delta\frac{K-1}{K}, \frac{(1-\delta^+)}{\delta}\}, \delta^+]$. This is a weaker requirement than the case of the continuum; but, as K gets large, it converges to the requirement that x_1^* is δ.

Appendix I Bargaining in Learnable Environments

Many negotiations take place in the context of long-run relationships.[24] Arguably, there is a subclass of these long-term relationships where the parties' preferences (over outcomes) are "learnable." The main text (page 3) argued that in these cases, negotiation failures are not easily explained by incomplete information. Here we elaborate.

Suppose the inefficiencies are a consequence of uncertainty about strategic posture—i.e., uncertainty about whether the other party is *capable* of accepting bad offers or making favorable offers. Over the course of a long-term relationship, the parties are likely to have

[24]Some examples: Labor unions (re)negotiate contracts with the same set of firms or government agencies. Divorce agreements involve marital partners that (at times) have a long history with one another. Nations have long-term negotiations, renegotiating issues of property, immigration, etc. Legislators will often negotiate policies amongst the same set of political actors. See the online appendix, for a discussion.

observed past concessions. If a party has ever observed a concession, the party would have to conclude that the other was, at the time, capable of making concessions. So if, in later negotiations, there is uncertainty about strategic posture, then it must be that the parties reason that capabilities change over time and, in particular, that they diminish over time.

Of course, "capability" may be a shorthand for the preferences or incentives of a particular negotiator. For instance, a union may give its leader incentives to take particular actions, and such incentives may well vary over time. But, at times, it is possible to obtain information about these incentives: Presumably, when the parties are involved in long-term relationships, they will make it their business to gather information about the incentives of key negotiators, etc. Analogously, at the start of the relationship, there may be quite a bit of uncertainty about the preferences of the parties. For instance, a firm may not understand how union members value wages versus other benefits. However, over the course of a long-term term relationship, the firm may come to understand union members' preferences over outcomes. (In the context of wars, Fearon (2004, page 290) and Powell (2006, page 172) make a similar argument.)

Bargaining inefficiencies in long-term "learnable" environments are arguably not well explained by incomplete information. At the same time, such learnable environments may lead the bargainers to have on-path strategic uncertainty and, even, common strong belief thereof. If so, RCSBR implies that delay must be an artifact of second-order optimism.

References

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