Bargaining Under Strategic Uncertainty:  
The Role of Second-Order Optimism

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Abstract

This paper shows that bargainers may reach delayed agreements even in environments where there is no uncertainty about payoffs or feasible actions. Under such conditions, delay may arise when bargainers face direct forms of strategic uncertainty—i.e., uncertainty about the opponent’s play. The paper restricts the nature of this uncertainty in two important ways. First, it assumes on-path strategic certainty: bargainers face uncertainty only after surprise moves. Second, it assumes Battigalli and Siniscalchi’s (2002) rationality and common strong belief of rationality (RCSBR)—a requirement that bargainers are “strategically sophisticated.” The main result characterizes the set of outcomes consistent with on-path strategic certainty and RCSBR. It shows that these assumptions allow for delayed agreement, despite the fact that the bargaining environment is one of complete information. The source of delay is second-order optimism: Bargainers do not put forward “good” offers early in the negotiation process because they fear that doing so will cause the other party to become more optimistic about her future prospects.

1 Introduction

Bargaining is an important feature of many economic and political relationships. Often, these interactions involve long periods of negotiation. This occurs despite the fact that

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delays in reaching agreements are typically costly for both parties. That is, there are often inefficient delays in reaching agreements. Examples include strikes, holdouts, legislative stalemates, renegotiations of debt contracts, wars, etc.

What is the source of such inefficient delays? The negotiations literature has long argued that they are a consequence of strategic uncertainty—i.e., uncertainty about how the others negotiate. (See, e.g., Walton and McKersie, 1965, pp. 37.) It is immediate that such uncertainty can lead to inefficient delays in reaching agreement: A bargainer may offer a split of the pie with the expectation that the offer will be accepted; this expectation may be incorrect and the offer may, in fact, be rejected.

This paper uses the tools of epistemic game theory to model strategic uncertainty. However, it restricts the nature of the uncertainty. Specifically, it studies strategic uncertainty that arises only after surprise moves in the negotiation process: At the start of the game, the bargainers have correct beliefs about how negotiations will unfold. They continue to hold those beliefs as long as there are no surprise moves. They face uncertainty only after surprise proposals or surprise rejections. This form of strategic uncertainty will be called on-path strategic certainty. Arguably, such strategic uncertainty is natural. In fact, the negotiations literature has recognized that it may arise (Stuart, 2004).

One might conjecture that this form of strategic uncertainty is inconsistent with inefficient delays in reaching agreement—at least if the bargainers are “strategically sophisticated” and there are no informational asymmetries: If the bargainers have correct ex ante beliefs about the outcome that will obtain, then they have the same expectation about the utility profile that will obtain. So, if delay were ex ante inefficient, strategically sophisticated bargainers would make mutually beneficial offers and avoid delay. However, this conjecture is incorrect. The goal of this paper shows that there may be inefficient delays in reaching agreement, even if the bargainers satisfy on-path strategic certainty, are strategically sophisticated, and face no uncertainty about the bargaining game.

To show this, we study the canonical alternating offers bargaining game (Ståhl, 1977; Rubinstein, 1982) and provide a model in which “strategically sophisticated” bargainers reason under strategic uncertainty. The bargainers are strategically sophisticated in the sense of satisfying Battigalli and Siniscalchi’s (2002) rationality and common strong belief of rationality (RCSBR). Theorems 6.1-6.2 characterize the set of outcomes consistent with on-path strategic certainty and RCSBR. The characterization shows the possibility of delay; if the discount factor is sufficiently high, there may be delays in reaching agreement.

The source of delay is second-order optimism: Bargainers do not put forward “good” offers early in the negotiation process because they fear that doing so will cause the other party to become more optimistic about her future prospects. To better understand the idea, suppose it is commonly understood that bargainers B1 and B2 will agree on an \( x : 1 - x \)
split of the pie in period \( n \geq 2 \). As delay is inefficient, they would prefer an \( x : 1 - x \) split earlier. But, bargainer B1 refrains from proposing \( x : 1 - x \) upfront—despite the fact that she (correctly) expects that B2 will also view the offer as beneficial. The key is that such a proposal (necessarily) causes B2 to become more optimistic relative to his initial expectations. The fact that there is delay in reaching agreement indicates that B1 fears B2’s optimistic-update will trigger further optimism on B2’s part—causing B2 to believe that he can do even better by rejecting the mutually beneficial offer. Proposition 7.1 establishes this fact. It shows that on-path strategic certainty, initial belief of on-path strategic certainty, and RCSBR, imply that delay must be an artifact of second-order optimism.

While Theorems 6.1-6.2 allow for delay, they also restrict the set of predicted outcomes. The restriction can be large when there is a significant gap between offers (i.e., a low discount factor) or when there is a tight deadline. For instance, with a deadline of three or four bargaining periods, there is no delay past period two. On the other hand, in an important limiting case, there is an ‘anything goes’ result. Specifically, when the time between offers is short and there is no deadline, essentially any outcome is consistent with RCSBR and on-path strategic certainty.\(^1\)

**Literature** The standard game-theoretic approach focuses on Nash equilibrium. In so doing, it implicitly assumes that bargainers have correct beliefs about future behavior—both on and off the path of play. Much like Nash equilibrium, the approach here assumes that there are correct beliefs along the path of play. However, unlike Nash equilibrium, it allows bargainers to have incorrect beliefs off the path of play. In this sense, it is closer in spirit to self-confirming equilibrium (Fudenberg and Levine, 1993).

Typically, the literature models strategic uncertainty implicitly: It assumes that strategic uncertainty is an artifact of incomplete information. Prominent examples are uncertainty about valuations (e.g., Admati and Perry, 1987; Sobel and Takahashi, 1983; Cramton, 1984; Fudenberg, Levine, and Tirole, 1985) or the cost of waiting (e.g., Rubinstein, 1985; Watson, 1998). We provide a rationale for inefficient agreements in a complete information environment.\(^2\) This is important, since certain negotiation failures are not well-explained by incomplete information—specifically, certain negotiations that take place in the context of long-run relationships. Fearon (2004, page 290) and Powell (2006, page 172) put forward this argument in the context of wars; Online Appendix D argues that their point applies to other long-term relationships.

At the surface, strategic uncertainty sounds similar to uncertainty about strategic pos-

\(^1\)This is in contrast to Nash equilibrium: That concept allows for incredible threats and, as a result, leads to an anything goes result in both situations.

\(^2\)The prevailing understanding of negotiation failures in complete information environments involves money burning in the course of negotiations (Avery and Zemsky, 1994; Haller and Holden, 1990).
ture, where there is uncertainty about whether a player is committed to implementing a particular strategy. (See, e.g., Chatterjee and Samuelson, 1987; Myerson, 1997; Abreu and Gul, 2000; Compte and Jehiel, 2002; Abreu and Pearce, 2007; Wolitzky, 2012; Abreu, Pearce, and Stacchetti, 2015a,b.) One might conjecture that modeling uncertainty about strategic posture (and focusing on the sequential equilibrium thereof) is sufficient to model any form of strategic uncertainty. However, the strategic incentives are quite different. When there is uncertainty about strategic posture, non-obstinate behavior reveals that a bargainer is strategic; in turn, this gives bargainers an incentive to signal that they they are obstinate or “irrational.” In contrast, here, bargainers fear that a “better-than-expected proposal” will signal irrationality and future conciliatory behavior; put differently, here, delay is an artifact of a bargainer’s attempt to not signal irrationality. Section 10.D shows that, in the presence of outside options, the two theories generate different predictions for delay.

2 Heuristic Exposition: Three-Period Deadline

This section provides a heuristic treatment for the case of a three-period deadline. It illustrates the model and provides intuition for the two main results.

Two bargainers negotiate on how to split a pie of size 1. Refer to Bargainer 1 (B1) as ‘she’ and Bargainer 2 (B2) as ‘he.’ The game is a three-period alternating offers bargaining problem: B1 begins the game by offering an allocation—i.e., a split of the pie $(x_1, x_2)$. If the offer is accepted, the game is over. If it is rejected, B2 offers an allocation $(y_1, y_2)$. And so on. If, in period $n \in \{1, 2, 3\}$, the bargainers agree to an allocation $(x_1, x_2)$, then their payoffs are given by $(\delta^{n-1} x_1, \delta^{n-1} x_2)$, where $\delta \in (0, 1)$ is a common discount factor. If there is perpetual disagreement, then their payoffs are $(0, 0)$.

**Strategic Uncertainty** The premise is that each bargainer faces strategic uncertainty—i.e., uncertainty about how the other plays the game. With this in mind, we will enrich the definition of the game to include the bargainers’ hierarchies of beliefs about the play of the game. In this expanded definition of a game, the bargainers are described by a state—i.e., a strategy-type pair $(s_1, t_1, s_2, t_2)$. The strategy $s_i$ describes $B_i$’s play and the type $t_i$ describes her hierarchies of conditional beliefs. Section 4 expands on this model.

We focus on the case in which strategic uncertainty arises only after surprise moves in the negotiation process. The idea is captured by a criterion that we call on-path strategic certainty: along the path of play, the bargainers have correct beliefs about the play of the game. This condition restricts the set of states that we focus on. To understand

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We model strategic uncertainty not just with first-order beliefs—i.e., beliefs about the play of the game—but hierarchies of beliefs. Doing so provides a language to capture two concepts of interest: “strategic sophistication” and second-order optimism.
how, consider two strategy profiles \((s_1, s_2)\) and \((s_1, r_2)\) that induce different paths of play. Suppose, at the start of the game, type \(t_1\) assigns probability 1 to the event that “B2 plays the strategy \(s_2\).” Then, at a state \((s_1, t_1, s_2, \cdot)\), type \(t_1\) has correct beliefs about the path of play; but, at a state \((s_1, t_1, r_2, \cdot)\), she has incorrect beliefs about the path of play. On-path strategic certainty rules out states of the latter form.

Note, if a state \((s_1, t_1, s_2, t_2)\) satisfies on-path strategic certainty, then the bargainers can have incorrect beliefs after a surprise move in the negotiation process. This is the particular form of strategic uncertainty that bargainers face.

**Strategically Sophisticated Bargainers** Strategic sophistication is captured by two criteria: *rationality* and *forward-induction reasoning*. \(B_i\) is rational if she chooses a strategy \(s_i\) that maximizes her subjective conditional expected utility; that is, \(s_i\) maximizes her expected utility at each information set, given her belief about how the other bargainer plays the game. Note, \(B_i\)’s conditional beliefs are specified by her type. Thus, the event that \(B_i\) is rational is a subset of the strategy-type pairs for \(B_i\).

Rationality requires that a bargainer maximizes given her beliefs. It does not restrict the beliefs themselves to be “strategically sophisticated.” Forward-induction reasoning asks that beliefs be “strategic sophisticated,” in the sense that they rationalize past behavior when possible. We follow Battigalli and Siniscalchi’s (2002) formalization: Type \(t_i\) rationalizes \(B_i\)’s behavior when possible, if \(t_i\) assigns probability 1 to the event “\(B_i\) is rational,” whenever she reaches a history consistent with that event. In that case, say that \(t_i\) strongly believes that “\(B_i\) is rational.” (See Definition 4.3.) Moreover, we require that \(t_i\) strongly believes that “\(B_i\) is rational and strongly believes that ‘\(B_i\) is rational.’” And so on. The full force of forward-induction reasoning is captured by *rationality and common strong belief of rationality* (RCSBR). Remark 5.1 further discusses the idea.

**Big Picture** Before turning to the details, we provide an overarching view of the results. We first characterize the outcomes consistent with RCSBR and on-path strategic certainty. The characterization will be captured by three constraints expressed *entirely* in terms of outcomes. Put differently, the constraints themselves will not make reference to strategies and types—the bread and butter of the epistemic analysis. Nonetheless, the constraints capture all the implications of RCSBR and on-path strategic certainty.

We use the constraints to analyze the implications for delayed agreement. They rule out delay until the final period, but allow for delay until the penultimate period. This raises the question: What is the source of delay? We show that, under an additional mild requirement, the mechanism for delay must be second-order optimism.
The Upfront and Deadline Constraints  Fix a state \((s^*_1, t^*_1, s^*_2, t^*_2)\) that results in delay—i.e., in an outcome \((x^*_1, x^*_2, n)\) with \(n \geq 2\). Suppose, at this state, there is rationality, strong belief of rationality, and on-path strategic certainty. We argue that these conditions alone imply that the outcome \((x^*_1, x^*_2, n)\) must satisfy three constraints: the Deadline Constraint and the B1-B2 Upfront Constraints.

We begin with the Deadline Constraint (DC). Consider the path of play induced by \((s^*_1, t^*_1, s^*_2, t^*_2)\). Along that path, there is an \(n\)-period history at which B1 moves. (If \(n = 2\), this is a history at which B1 accepts the proposal of \((x^*_1, x^*_2)\); if \(n = 3\), this is a history at which she proposes \((x^*_1, x^*_2)\).) At that history, \(t^*_1\) correctly expects the outcome to be \((x^*_1, x^*_2, n)\). (This follows from on-path strategic certainty.) Note, at that point, she continues to maintain the hypothesis that B2 is rational. Thus, she maintains a hypothesis that, if the final bargaining phase is reached, B\((-i)\) will accept any strictly positive share of the pie. The fact that she does not wait implies that she prefers the allocation of \((x^*_1, x^*_2)\) in period \(n\) versus waiting for essentially the full pie in the third period. Since the value of the former is \(\delta^{n-1}x^*_1\) and the value of the latter is \(\delta^2\), the DC requires:

\[
\text{DC. } B1 \text{ does not have an incentive to wait for the deadline: } \delta^{n-1}x^*_1 \geq \delta^2.
\]

Next, we turn to the B1-B2 Upfront Constraints (UCs). Consider the path of play induced by \((s^*_1, t^*_1, s^*_2, t^*_2)\). Since \(n \geq 2\), along that path, there is some history at which Bi makes a proposal. Consider the first such history—i.e., Bi’s upfront history. At that upfront history, \(t^*_i\) correctly anticipates that the outcome will be \((x^*_i, x^*_j, n)\). (This follows from on-path strategic certainty.) Upfront, \(t^*_i\) also believes that B\((-i)\) is rational. So, upfront, each \(t^*_i\) anticipates that B\((-i)\) would accept an offer that gives B\((-i)\) more than the discounted total pie—i.e., more than \(\delta\). Thus, each Bi must prefer getting the allocation of \((x^*_1, x^*_2)\) in period \(n\) to making an offer that, upfront, would give B\((-i)\) a \(\delta\) share of the pie.

Note, the idea of an “upfront offer” is implemented differently for the two bargainers. An upfront offer occurs in period 1 for B1 and in period 2 for B2. As such, B1’s constraint requires that receiving \(x^*_1\) in period \(n\) is preferred to \((1-\delta)\) in period 1, while B2’s constraint requires that receiving \(x^*_2\) in period \(n\) is preferred to \((1-\delta)\) in period 2. As such, the UCs require:

\[\text{B1’s UC. } B1 \text{ has no incentive to secure an acceptance upfront: } \delta^{n-1}x^*_1 \geq (1-\delta).\]

\[\text{B2’s UC. } B2 \text{ has no incentive to secure an acceptance upfront: } \delta^{n-1}x^*_2 \geq \delta(1-\delta).\]

Taken together, the constraints imply that there can be no delay until the final period. (When \(n = 3\), the DC conflicts with B2’s UC.) However, they allow for delay until period 4

\footnote{Note a subtlety: At the \(n\)-period history, Bi \textit{anticipates} that she would be able to get the full pie in the final bargaining phase. However, as we will soon see, \(n < 3\), so the final bargaining phase will never be met. Thus, if the final bargaining phase is met, Bi may very well rethink such an assessment.}
n = 2. In that case, the DC and B2’s UC imply that \((x^*_1, x^*_2, n) = (\delta, 1 - \delta, 2)\). Adding B1’s UC, we must additionally have \(\delta^2 + \delta \geq 1\). This is a requirement that the discount factor must be sufficiently large so that waiting is profitable.

**Characterization**  We argued that if an outcome is consistent with rationality, strong belief of rationality, and on-path strategic certainty, then it satisfies the DC and the B1-B2 UCs. So “minimal conditions” on strategic reasoning suffice to deliver these three constraints on outcomes. In turn, the constraints allow for delayed agreement.

One might conjecture that the delayed agreement is an artifact of limited strategic reasoning—i.e., we have only required that Bi strongly believes B(−i) is rational. A main result of the paper shows that a stronger assumption on strategic sophistication—specifically, RCSBR—does not refine the restrictions imposed by the DC and the B1-B2 UCs. That is, if an outcome satisfies the both the DC and the B1-B2 UCs, then it is consistent with RCSBR and on-path strategic certainty. In the context of the three-period deadline, this is shown as Example 6.1. However, the conclusion holds beyond the three-period deadline.

**Main Characterization (Theorems 6.1-6.2)**

(i) If an outcome is consistent with rationality, strong belief of rationality and on-path strategic certainty, then it satisfies the DC and the B1-B2 UCs.

(ii) If an outcome satisfies the DC and the B1-B2 UCs, then it is consistent with rationality, common strong belief of rationality, and common strong belief of on-path strategic certainty.

Taken together, the DC and the B1-B2 UCs characterize the outcomes consistent with RCSBR and on-path strategic certainty across all type structures. Thus, the analyst can use these constraints—expressed on outcomes alone—to study the predictions of RCSBR and on-path strategic certainty.

**Second-Order Optimism**  The mechanism for delay is second-order optimism. To see this, return to the example of a three-period deadline. Fix a state \((s^*_1, t^*_1, s^*_2, t^*_2)\) at which there is rationality, strong belief of rationality, on-path strategic certainty, and delay in reaching agreement. We have seen that the state induces the outcome \((x^*_1, x^*_2, n) = (\delta, 1 - \delta, 2)\). We now argue that—under an additional mild requirement—type \(t^*_1\) must exhibit second-order optimism. This requirement is that type \(t^*_1\) also initially believes that there is on-path strategic certainty. We now explain.

At the start of the game, B1’s expected payoffs are \(\delta^2\), and B2’s are \(\delta(1 - \delta)\). With the additional requirement on B1’s beliefs, \(t^*_1\) initially correctly anticipates both her and B2’s expected payoffs. Thus, in the first period, she anticipates that there is a mutually
beneficial offer to be made—i.e., some \((x_1, x_2) \gg (\delta^2, \delta(1 - \delta))\). Why is such an offer not made? We will argue that B1 anticipates that the very act of making a mutually beneficial offer makes B2 more optimistic about her future prospects.

Suppose B1 initially proposes an offer \((x_1, x_2)\). Recall, type \(t_1^*\) strongly believes that B2 is rational. As such, she must anticipate that, conditional upon receiving the offer, B2’s expected payoffs must be least \(x_2\): If \(t_1^*\) expects B2 to accept the offer, she anticipates that B2’s expected payoffs are exactly \(x_2\); if \(t_1^*\) expects B2 to reject the offer, she anticipates that B2 does so because B2’s expected payoffs, after receiving the offer, are greater than \(x_2\). When the offer is a mutually beneficial offer, \(x_2 > \delta(1 - \delta)\). Thus, \(t_1^*\) anticipates that, conditional upon receiving a mutually beneficial offer, B2’s expected payoffs are greater than \(\delta(1 - \delta)\). Put differently, she reasons that the very act of making the mutually beneficial offer makes B2 more optimistic about his future prospects relative to B2’s initial expectation of \(\delta(1 - \delta)\). We call this second-order optimism.

A subtle definitional point: If B1 were to make an alternate offer, then type \(t_2^*\) would actually be more optimistic about his future prospects, relative to his expectations at the start of the negotiation. However, at the state \((s_1^*, t_1^*, s_2^*, t_2^*)\), such an offer is not made, and so this form of optimism—a form of first-order optimism—never occurs.

**Overview** The paper proceeds as follows. Sections 3-4 introduce the bargaining game and the epistemic model of strategic uncertainty. Section 5 defines the key condition of strategic sophistication—namely, RCSBR. Section 6 characterizes the outcomes consistent with RCSBR and on-path strategic certainty. In so doing, it points to the possibility for delayed agreement. Section 7 shows that such delay must be an artifact of second-order optimism. Sections 8-9 revisit key ingredients of the analysis: Section 8 revisits the nature of the characterization result, highlighting that the result assumes that the analyst has no information about the bargainers’ type structure. Section 9 revisits the extent to which on-path strategic certainty is required for the analysis. Finally, Section 10 concludes by drawing (behavioral) comparisons to the literature.\(^5\)

### 3 The Bargaining Game

We study the canonical alternating offers bargaining model. Two bargainers, viz. B1 (‘she’) and B2 (‘he’), negotiate on how to split a pie of size 1. Write \(i\) for a particular bargainer and \(-i\) for the other bargainer. An **allocation** is a division of the pie—i.e., some \((x_1, x_2) \in [0, 1]^2\) with \(x_2 = 1 - x_1\).\(^6\)

\(^5\)Proofs for Sections 6-7 are in the Appendix; proofs for subsequent sections are in the Online Appendix.

\(^6\)We implicitly assume that all allocations are feasible; this is not crucial. See Online Appendix II.
In each bargaining phase, some $B_i$ assumes the role of the proposer, and the other $B(-i)$ assumes the role of the responder. $B_i$ proposes an allocation $(x_1, x_2)$. $B(-i)$ chooses to accept (a) or reject (r) the proposal. If $B(-i)$ accepts, the game is over and the resulting allocation is $(x_1, x_2)$. If $B(-i)$ rejects, then a new bargaining phase begins. In the new bargaining phase, $B(-i)$ is in the proposer’s role.

Period 1 begins with $B_1$ in the proposer role; if the game does not conclude, Period 2 continues with $B_2$ in the proposer role, etc. There are (at most) $N \in \mathbb{N}^+ \cup \{\infty\}$ periods with $N \geq 2$. If $N$ is finite, there is a deadline. If the bargainers agree to the allocation $(x_1, x_2)$ in period $n$, we say that $(x_1, x_2, n)$ is an n-period outcome. If there is perpetual disagreement, the disagreement outcome results; this is written as $(x_1, x_2, N) = (0, 0, N)$.

There is a common discount factor of $\delta \in (0, 1)$. Each $B_i$’s utility function is $\Pi_i(x_1, x_2, n) = \delta^{n-1}x_i$ if $n < \infty$ and $\Pi_i(x_1, x_2, n) = 0$ if $n = \infty$.

We can identify information sets with histories (i.e., sequences of moves). Say that $h$ is an n-period history if it occurs in an $n^{th}$-bargaining phase. The bargainer who moves at $h$ takes on either the proposer role (and chooses an allocation) or the responder role (and accepts or rejects an offer). Write $H^P_i$ for the set of histories at which $B_i$ takes on the proposer role and $H^R_i$ for the set of histories at which $B_i$ takes on the responder role. Then, $H_i = H^P_i \cup H^R_i$ is the set of histories at which $i$ moves. If $h = (\phi, x_1, r, x_2, \ldots, r) \in H^P_i$, we often write $(h, x'_i) \in H^R_i$ for a history of the form $(\phi, x_1, r, x_2, \ldots, r, x'_i) \in H^P_i$.

Write $s_i : H_i \rightarrow [0, 1] \cup \{a, r\}$ for a strategy of player $i$, where $s_i(h) \in \{a, r\}$ if $h \in H^R_i$ and $s_i(h) \in [0, 1]$ if $h \in H^P_i$. Note, carefully, we take the following convention: If $B_i$ takes on the role of the proposer at $h$, then $s_i(h)$ specifies $B_i$’s share of the allocation $x_i$. Write $S_i$ for the set of strategies of $B_i$ and endow $S_i$ with the product topology (Appendix A).

Each strategy profile induces a terminal history. Write $Z$ for the set of terminal histories, and write $\zeta : S_1 \times S_2 \rightarrow Z$ for the mapping from strategy profiles to terminal histories. In turn, each terminal history $z \in Z$ induces an outcome. Write $\xi$ for the mapping from terminal histories to outcomes. So, $(s_1, s_2)$ induces the outcome $(x_1, x_2, n)$ if $\xi(\zeta(s_1, s_2)) = (x_1, x_2, n)$. The strategic-form payoff function for $B_i$ is $\pi_i = \Pi_i \circ \xi \circ \zeta$.

4 Modeling Strategic Uncertainty

We use the language of conditional probability systems to model strategic uncertainty. Let us preview why this is the appropriate language: Because $B_i$ faces uncertainty about the strategy $B(-i)$ plays, she begins the game with a belief about $S_{-i}$. However, over the course of playing the game, $B_i$ may be forced to revise her beliefs. For instance, $B_2$ may begin the game assigning probability 1 to $B_1$ offering $(x_1, x_2) = (\frac{1}{4}, \frac{3}{4})$ upfront, only to instead receive $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$. At that point, $B_2$ will need to form a new assessment about $B_1$’s future
play. With this in mind, we view bargainers as having beliefs at each information set; the system of beliefs satisfies the rules of conditional probability, when possible.

**Conditional Probability Systems** We begin with abstract definitions and later apply the definitions to model strategic uncertainty. Fix a topological space \( \Omega \) and the Borel sigma-algebra thereof. Refer to Borel subsets of \( \Omega \) as events. Write \( \mathcal{P}(\Omega) \) for the set of Borel probability measures on \( \Omega \). A **conditional probability space** is some \((\Omega; \mathcal{E})\), where each element of \( \mathcal{E} \) is a Borel subset of \( \Omega \). The collection \( \mathcal{E} \) is a set of conditioning events.

Call \( \mu = (\mu(\cdot|E) : E \in \mathcal{E}) \) an array on \((\Omega; \mathcal{E})\) if, for each conditioning event \( E \in \mathcal{E} \), \( \mu(\cdot|E) \in \mathcal{P}(\Omega) \) with \( \mu(E|E) = 1 \). So, an array of conditional measures specifies a belief for each conditioning event. The array \( \mu \) is countable if, for each \( E \in \mathcal{E} \), there is a countable event \( F \subseteq \Omega \), with \( \mu(F|E) = 1 \). Throughout the main text, we restrict attention to countable arrays. Appendix A shows that the restriction is without loss of generality.

**Definition 4.1.** Call \( \mu = (\mu(\cdot|E) : E \in \mathcal{E}) \) a **conditional probability system** (CPS) (on \((\Omega; \mathcal{E})\)) if it is an array (on \((\Omega; \mathcal{E})\)) that satisfies the conditioning requirement: if \( E \subseteq F \subseteq G \), \( E \) is Borel, and \( F, G \in \mathcal{E} \), then \( \mu(E|G) = \mu(E|F)\mu(F|G) \).

Write \( \mathcal{C}(\Omega; \mathcal{E}) \) for the set of CPS’s on \((\Omega; \mathcal{E})\).

An array of conditional measures specifies a belief for each conditioning event. A CPS is an array that satisfies the rules of conditional probability when possible. To better understand the conditioning requirement, fix conditioning events \( F \) and \( G \) so that \( F \subseteq G \). Informally, this can be viewed as a situation where \( G \) precedes \( F \). The conditioning requirement says that, if \( \mu(F|G) > 0 \), then \( \mu(\cdot|F) \) must be derived from the ‘prior belief’ \( \mu(\cdot|G) \)—i.e., \( \mu(\cdot|F) = \mu(\cdot|G)/\mu(F|G) \). (See Remark A.1.)

**First-Order Conditional Beliefs** Bi’s first-order beliefs are conditional beliefs about the strategy that B(\( -i \)) chooses: Bi begins the game with an initial hypothesis about \( S_{-i} \). At each history at which Bi moves, she has a belief consistent with the history being reached.

Let \( S(h) \) be the set of strategy profiles \((s_1, s_2)\) so that the path induced by \((s_1, s_2)\) passes through \( h \). Then, \( S_{-i}(h) = \text{proj}_{S_{-i}} S(h) \) is the set of strategies \( s_{-i} \) that allow \( h \).\(^7\) Observe that \( S_{-i}(\phi) = S_{-i} \). Bi’s conditioning events correspond to members of

\[ \mathcal{S}_i = \{S_{-i}(h) : h \in \{\phi\} \cup H_i\}. \]

Each conditioning event \( S_{-i}(h) \) is closed in \( S_i \). (See Lemma A.1.)

Bi’s first-order beliefs are described by a CPS \( \mu_i = (\mu_i(\cdot|F_{-i}) : F_{-i} \in \mathcal{S}_i) \) on \((S_{-i}; \mathcal{S}_i)\). To understand how the conditioning requirement applies, consider histories \( h, h' \in H_i \) so that \( h \)

\(^7\)Here, and throughout the paper, we write \( \text{proj}_X Y \) for the projection of \( Y \) onto \( X \).
precedes \( h' \). In that case, \( S_{-i}(h') \subseteq S_{-i}(h) \); so, if \( \mu_i(\cdot|S_{-i}(h)) \) assigns positive probability to \( S_{-i}(h') \), then \( \mu_i(\cdot|S_{-i}(h)) \) is derived from \( \mu_i(\cdot|S_{-i}(h)) \) by the rules of conditional probability.

**Hierarchies of Conditional Beliefs** We will follow the literature and adapt Harsanyi’s (1967) type structure model to capture hierarchies of conditional beliefs about the play of the game. (See Ben-Porath, 1997 and Battigalli and Siniscalchi, 1999.)

**Definition 4.2.** A type structure \( T \) specifies three objects for each \( B_i \):

(i) a finite type set \( T_i \);

(ii) a set of conditioning events \( S_i \otimes T_{-i} \) and

(iii) a belief map \( \beta_i : T_i \rightarrow \mathcal{C}(S_{-i} \times T_{-i}; S_i \otimes T_{-i}) \), where each \( \beta_i(t_i) \) is a countable CPS.

Definition 4.2 restricts attention to type structures with finite type sets. (This is relaxed in Definition A.1 of Appendix A.) The belief map \( \beta_i \) associates each type, viz. \( t_i \), with a countable CPS on \( (S_{-i} \times T_{-i}; S_i \otimes T_{-i}) \), viz. \( \beta_i(t_i) \). For notational convenience, write \( \beta_{i,h}(t_i) \) for \( \beta_i(t_i)(\cdot|S_{-i}(h) \times T_{-i}) \).

A type structure induces hierarchies of conditional beliefs about the strategies played. To see this, observe that each type \( t_i \) has a system of conditional beliefs on the strategies and types of \( B(-i) \), viz. \( \beta_i(t_i) \). By marginalizing onto \( S_{-i} \), each type \( t_i \) then has a system of first-order conditional beliefs on the strategies of \( B(-i) \). Since each type \( t_{-i} \) is also associated with a system of first-order beliefs, \( \beta_i(t_i) \) induces a system of second-order beliefs—i.e., beliefs about the strategies and first-order beliefs of \( B(-i) \). And so on.

**Epistemic Game** An epistemic game is given by a pair \( (\mathcal{B}, \mathcal{T}) \), where \( \mathcal{B} \) is the bargaining game and \( \mathcal{T} \) is a type structure (associated with \( \mathcal{B} \)). It induces states: A state \( (s_1, t_1, s_2, t_2) \) describes each bargainer’s play (i.e., \( s_i \)) and beliefs (i.e., \( \beta_i(t_i) \)). An outcome is consistent with a set of states \( X \subseteq S_1 \times T_1 \times S_2 \times T_2 \) if there exists some \( (s_1, s_2) \in \text{proj}_{S_1 \times S_2} X \) that induces the outcome.

**On-Path Strategic Certainty** As discussed in Section 2, on-path strategic certainty captures the idea that strategic uncertainty arises only after surprise moves in the negotiation process. To formalize the idea, fix an epistemic game and a state thereof, viz. \( (s_1, t_1, s_2, t_2) \). The state induces a path of play to a particular terminal node \( \zeta(s_1, s_2) = z \).

We want to capture the idea that, at each history \( h \) along this path of play, \( B_i \) assigns probability 1 to the terminal node \( \zeta(s_1, s_2) = z \) being reached. Write

\[
\mathcal{Z}_{-i}(s_1, s_2) = \{ r_{-i} : \zeta(s_i, r_{-i}) = \zeta(s_1, s_{-i}) \} \times T_{-i}.
\]

\(^{8}\)In models of incomplete information, type structures are used to both induce payoff functions and model hierarchies of beliefs about the payoffs. Here, types do not affect payoffs.
This consists of the strategies of $B(-i)$ that will induce the same terminal node as $(s_i, s_{-i})$, viz. $\zeta(s_1, s_2) = z$, when $B_i$ plays $s_i$. If there is on-path strategic certainty at $(s_1, t_1, s_2, t_2)$, then, for each $i = 1, 2$ and each information set along the path of play to $\zeta(s_1, s_2) = z$, $t_i$ assigns probability 1 to $Z_{-i}[s_1, s_2]$.

**Definition 4.3** (Battigalli and Siniscalchi, 2002). A type $t_i$ **strongly believes** $E_{-i} \subseteq S_{-i} \times T_{-i}$ if $E_{-i}$ is an event satisfying the following requirement: For each $S_{-i}(h) \in S_i$ with $E_{-i} \cap [S_{-i}(h) \times T_{-i}] \neq \emptyset$, $\beta_{t_i}(t_i)(E_{-i}) = 1$.

If $t_i$ strongly believes an event $E_{-i}$, it assigns probability 1 to $E_{-i}$ at the start of the game (since $S_{-i}(\phi) = S_{-i}$) and at each history (of $B_i$) consistent with $E_{-i}$. The information sets of $B_i$ consistent with $Z_{-i}[s_1, s_2]$ are precisely the information sets along the path of play to $\zeta(s_1, s_2) = z$. Moreover, $Z_{-i}[s_1, s_2]$ is an event (i.e., Borel). (See Lemma A.1.) Thus:

**Definition 4.4.** Say that, at the state $(s_1, t_1, s_2, t_2)$, there is **on-path strategic certainty** if, for each $i$, $t_i$ strongly believes $Z_{-i}[s_1, s_2]$.

Write $C$ for the set of states at which there is on-path strategy certainty.

### 5 Strategic Sophistication: RCSBR

We next restrict the set of states by requiring that bargainers are “strategically sophisticated,” in the sense of satisfying rationality and common strong belief of rationality.

**Definition 5.1.** Call $(s_i, t_i)$ is **rational** if, for each $h \in H_i$ with $s_i \in S_i(h)$,

$$\sum_{s_{-i}}[\pi_i(s_i, s_{-i}) - \pi_i(r_i, s_{-i})] \beta_{t_i}(t_i)(\{s_{-i}\} \times T_{-i}) \geq 0,$$

for all $r_i \in S_i(h)$.

So, if $(s_i, t_i)$ is rational, then at each history it allows, $s_i$ maximizes $B_i$’s expected payoffs under the marginal (or first-order) beliefs of $t_i$. This maximization is done relative to all strategies that allow the history. Write $R_{i}^{1}$ for the set of rational strategy-type pairs for $i$ and $R_{i}^{1} = R_{i}^{1} \times R_{i}^{2}$ for the set of states at which each bargainer is rational.

Write

$$\text{SB}_i(E_{-i}) = S_i \times \{t_i \in T_i : t_i \text{ strongly believes } E_{-i}\}.$$

Inductively define sets $R_{i}^{m}$ so that $R_{i}^{m+1} = R_{i}^{m} \cap \text{SB}_i(R_{-i}^{m})$. Set $R_{i}^{\infty} = \bigcap_{m \in \mathbb{N}_{+}} R_{i}^{m}$. The set $R_{i}^{m} = R_{i}^{1} \times R_{i}^{2}$ is the set of states at which there is **rationality and (m – 1)th-order strong belief of rationality**. The set $R_{i}^{\infty} = R_{i}^{1} \times R_{i}^{\infty}$ is the set of states at which there is rationality and common strong belief of rationality (RCSBR).
Remark 5.1. In Section 2, we argued that common strong belief of rationality captures forward-induction reasoning. To better understand the connection, consider two situations. First, suppose type $t_2$ is consistent with RCSBR—i.e., $t_2 \in \text{proj}_{T_2} R_2^\infty$. Then type $t_2$ begins the game assigning probability 1 to $R_1^{\infty}$. Now suppose B1 makes an unexpected initial offer of $(x_1, x_2)$—i.e., an offer that $t_2$ initially assigned zero probability to. If that offer is consistent if RCSBR for B1—i.e., if there exists some $(s_1, t_1) \in R_1^{\infty}$ so that $s_1$ initially proposes $(x_1, x_2)$—then B2 must continue to assign probability 1 to $R_1^{\infty}$.

Second, suppose B1 makes an initial offer of $(x_1, x_2)$ with $x_2 > \delta$. In that case, B1 cannot both be rational and strongly believe that B2 is rational: If she strongly believes that B2 is rational, then she believes that B2 would accept both $(x_1, x_2)$ and an offer $(z_1, z_2)$ with $x_2 > z_2 > \delta$; after all, in both cases, B2 gets more than the discounted value of the pie. If B1 were also rational, she would prefer offering $(z_1, z_2)$ over $(x_2, x_2)$. So, conditional on such an offer, B2 cannot continue to believe that “B1 is rational and strongly believes rationality.” But, B2 may well be able to believe that “B1 is rational.” This is what common strong belief of rationality requires: Specifically, if $t_2 \in \text{proj}_{T_2} R_2^3$, then conditional on observing such an offer, she must assign probability 1 to $R_1^1$.

6 Characterization Theorem

This section takes the perspective of an analyst that observes the bargaining game, but not the bargainers’ type structure. Thus, it seeks to characterize the predictions of RCSBR and on-path strategic certainty across all associated type structures. Following Section 2, this will correspond to the outcomes that satisfy the Upfront and Deadline Constraints. We begin by reviewing the constraints. Then, we show that the set of outcomes consistent with RCSBR and on-path strategic certainty is the set of outcomes that satisfy these constraints.

The Constraints Recall, an outcome satisfies the Bi Upfront Constraint if Bi prefers the outcome to $(1 - \delta)$ upfront. When there is a deadline $N < \infty$, an outcome satisfies the Deadline Constraint if the proposer in period $N$ prefers the outcome to 1 in period $N$.

Definition 6.1. An outcome $(x_1^*, x_2^*, n)$ satisfies the Upfront Constraints (UCs) if $\delta^{n-1}x_1^* \geq 1 - \delta$ and $\delta^{n-1}x_2^* \geq \delta(1 - \delta)$.

Definition 6.2. An outcome $(x_1^*, x_2^*, n)$ satisfies the Deadline Constraint (DC) if (i) $N < \infty$ odd implies $x_1^* \geq \delta^{N-n}$ and (ii) $N < \infty$ even implies $x_2^* \geq \delta^{N-n}$.

Observe, if $N = \infty$, any outcome trivially satisfies the Deadline Constraint.

9For instance, she would be able to believe this if there were a type $t_1$ that initially believes “B2 rejects all offers, irrespective of the history, and B2 responds with $(y_1, y_2) = (1, 0)$ if and only if B1 offers $(x_1, x_2)$.”
Necessity of the Constraints  Any state at which there is RCSBR and on-path strategic certainty induces an outcome that satisfies the UCs and DC. In fact, these constraints must be satisfied under weaker requirements.

**Theorem 6.1.** Fix an epistemic game \((B, T)\) and a state \((s^*_1, t^*_1, s^*_2, t^*_2) \in R^2 \cap C\). The outcome induced by \((s^*_1, s^*_2)\), viz. \((x^*_1, x^*_2, n)\), satisfies the UCs and DC.

Theorem 6.1 says that if, at a state, there is rationality, strong belief of rationality, and on-path strategic certainty, then the associated outcome satisfies the UCs and DC. Thus, in any type structure, weak requirements on strategic reasoning lead to an outcome that satisfies the constraints.

Sufficiency of the Constraints  The constraints are sufficient for an outcome to be consistent with RCSBR and on-path strategic certainty. More specifically:

**Theorem 6.2.** Fix a bargaining game \(B\) and an outcome \((x^*_1, x^*_2, n)\) that satisfies the UCs and DC. There is an epistemic game \((B, T)\) and a state \((s^*_1, t^*_1, s^*_2, t^*_2)\) thereof, so that:

1. \((s^*_1, t^*_1, s^*_2, t^*_2) \in R^\infty \cap C\), and
2. the strategy profile \((s^*_1, s^*_2)\) induces the outcome \((x^*_1, x^*_2, n)\).

Theorem 6.2 says that if an outcome is consistent with the UCs and DC, the outcome is also consistent with RCSBR and on-path strategic certainty. So, outcomes consistent with these constraints are also consistent with the full force of strategic reasoning in some type structure. (See Section 8 for further discussion.) The example of a three-period deadline illustrates the proof of Theorem 6.2.

**Example 6.1.** Suppose \(N = 3\) and \(\delta^2 + \delta \geq 1\). Note, the outcome \((x^*_1, x^*_2, n) = (\delta, 1 - \delta, 2)\) satisfies the UCs and DC. We will construct a type structure and a state \((s^*_1, t^*_1, s^*_2, t^*_2)\) so that

1. \((s^*_1, t^*_1, s^*_2, t^*_2) \in R^\infty \cap C\), and
2. \((s^*_1, s^*_2)\) induces the outcome \((x^*_1, x^*_2, n) = (\delta, 1 - \delta, 2)\).

Toward that end, let \((s^*_1, s^*_2)\) have the following features:

- **B1’s Strategy, \(s^*_1\):** At the initial node, B1 offers \((x_1, x_2) = (1, 0)\). If this offer is rejected and, subsequently, B2 offers \((y_1, y_2)\), B1 accepts if and only if \(y_1 \geq \delta\). At each third-period history, B1 offers \((w_1, w_2) = (1, 0)\).

- **B2’s Strategy, \(s^*_2\):** B2 accepts an initial offer of \((x_1, x_2)\) if and only if \(x_2 > \delta\). If the initial offer is \((x_1, x_2) = (1, 0)\), B2 subsequently offers \((y_1, y_2) = (\delta, 1 - \delta)\). If the initial offer is \((x_1, x_2)\) with \(x_2 \in (0, \delta]\), B2 subsequently offers \((y_1, y_2) = (0, 1)\). At each third-period history, B2 accepts an offer of \((w_1, w_2)\) if and only if \(w_2 > 0\).
Set each $T_i = \{t^*_i\}$. The belief maps $\beta_i$ are described as follows: If $s^*_{-i} \in S_{-i}(h)$, $t^*_i$ assigns probability 1 to $(s^*_{-i}, t^*_{-i})$ at $h$. At any other history $h$, $t^*_i$ assigns probability 1 to a so-called “$h$-accommodating strategy” of $B(-i)$. This strategy allows $h$ and, subsequently, accommodates $Bi$ by offering to take zero share of the pie and accepting any offer.

It is immediate that, at $(s^*_1, t^*_1, s^*_2, t^*_2)$, there is on-path strategic certainty. In fact, there is common strong belief of on-path strategic certainty. (See Appendix B.) Moreover, there is also RCSBR. We now provide the intuition. (The formal argument is more subtle.)

First, $(s^*_1, t^*_1)$ is rational. At the start of the game, both $t^*_1$ and $t^*_2$ correctly anticipate the outcome $(x^*_1, x^*_2, 2)$ and believe that the other does. Moreover, there is an outcome $(x^*_1, x^*_2, 1)$ that both bargainers strictly prefer to $(x^*_1, x^*_2, 2)$. But, offering $(x^*_1, x^*_2)$ upfront would give $t^*_1$ a strictly lower expected payoff. Type $t^*_1$ expects that if she were to make such offer, then $B2$ would reject it and propose to take the full pie.

Second, $(s^*_2, t^*_2)$ is rational. This may seem peculiar: $B2$ rejects an initial offer of $(x^*_1, x^*_2)$ with $x^*_2 \in [0, \delta]$, even in the case where $x^*_2 > \delta x^*_1$—i.e., even if he strictly prefers $(x^*_1, x^*_2)$ to the outcome that $t^*_2$ expected to get at the start of negotiations. However, when such an offer is made, type $t^*_1$ no longer expects the outcome to be $(x^*_1, x^*_2, 2)$. Instead, $t^*_2$ expects the outcome to be $(y_1, y_2, 2) = (0, 1, 2)$, which he prefers to $(x^*_1, x^*_2, 1)$.

How, then, can $t^*_2$ strongly believe $R^1_1$? After all, conditional on $B1$ proposing a mutually beneficial offer $(x^*_1, x^*_2)$, $t^*_2$ believes that $B1$ will subsequently accept a zero share of the pie. The key is that, when $B1$ offers some such $(x^*_1, x^*_2)$ upfront, $B2$ must maintain a hypothesis that $B1$ is irrational: At the initial node, every type of $B1$ believes that $B2$ rejects such a mutually beneficial offer and responds by offering $(y_1, y_2) = (0, 1)$. Thus, conditional on $B1$ making such an offer, $B2$ must believe that $B1$ has not maximized her expected payoffs. At that point, he may reason—as $t^*_2$ does—that $B1$ will again fail to maximize her expected payoffs in the future. For this reason, $t^*_2$ strongly believes the event $R^1_1$. Iterating this argument, at the state $(s^*_1, t^*_1, s^*_2, t^*_2)$, there is RCSBR.

We point to two features of Example 6.1. First, the construction satisfies a no indifference property: Along the path of play, no bargainer is indifferent between any two actions. (See Remark B.4.) Second, under the construction, bargainers make tough offers and counteroffers prior to agreement. While it is important that the offers be ‘tough’ relative to the final allocation, there is no requirement on the trajectory of offers. To see this, take $N = \infty$ and $\delta = .8$. The outcome $(x^*_1, x^*_2, n) = (.5, .5, 4)$ satisfies the UCs and DC. The proof assumes that, along the path of play, each bargainer offers to take the full pie for herself. However, we can instead have a situation in which the bargainers’ offers are increasing over time. For instance, $B1$ may offer the allocation $(.8, 2)$; $B2$ may respond with $(.15, .85)$; and $B1$ may counteroffer with $(.9, 1)$. But, likewise, we can also have a situation in which the bargainers’ offers are decreasing over time. For instance, $B1$ may offer the allocation $(.8, 2)$; $B2$ may
respond with (.3,.7); and B1 may counteroffer with (.65,.35).

Characterization  Taken together, Theorems 6.1-6.2 characterize the set of outcomes consistent with RCSBR and on-path strategic certainty across all type structures.

Corollary 6.1. Fix a bargaining game $\mathcal{B}$. An outcome $(x_1^*, x_2^*, n)$ satisfies the UCs and DC if and only if there exists an epistemic game $(\mathcal{B}, T)$ so that $(x_1^*, x_2^*, n)$ is consistent with $R^\infty \cap C$.

This raises the question: To what extent do the UCs and DC allow for delayed agreement? For any given $(\delta, N)$, there exists some $\overline{\pi}(\delta, N) \geq 1$ so that $\overline{\pi}(\delta, N) \geq n$ if and only if there exists $(x_1^*, x_2^*, n)$ that satisfies the UCs and DC. Thus, $\overline{\pi}(\delta, N)$ is the maximum length of delay. In fact, $N > \overline{\pi}(\delta, N)$, and, when $\infty > N \geq 4$, $N - 2 \geq \overline{\pi}(\delta, N)$. So, at the point of agreement, the bargainers must each be able to make at least one future offer. This stands in contrast to papers with private information (Fuchs and Skrzypacz, 2013) or with overconfidence in the ability to make future offers (Yildiz, 2004; Simsek and Yildiz, 2014).\(^\text{10}\)

It may well be that $\overline{\pi}(\delta, N) = 1$, implying that there is no delay. (In particular, this will happen if $\delta$ is low.) However, for each $N \geq 3$, there exists a $\delta$ sufficiently large so that $\overline{\pi}(\delta, N) \geq 2$.\(^\text{11}\) (See Proposition B.1.) Sections 10 B-C further discuss behavior.

7 Source of Delay: Second-Order Optimism

This section shows that second-order optimism is a necessary requirement for delayed agreements. The key step is definitional: to define what is meant by an offer causing B2 to “become more optimistic” about his future prospects. This means that B2’s expected future payoffs increase after receiving the offer.

Given a history $h \in H_2$ and a strategy $s_2 \in S_2(h)$, write

$$\mathbb{E}\pi_2[s_2|t_2, h] = \sum_{s_1 \in S_1(h)} \pi_2(s_1, s_2) \beta_2, h(t_2)(\{s_1\} \times T_1)$$

for $t_2$’s expected payoffs at $h$, when B2 plays the strategy $s_2 \in S_2(h)$. Then, a strategy-type pair $(s_2, t_2)$ is more optimistic at $h'$ than at $h$ if the type $t_2$’s expected continuation payoff from playing $s_2$ is higher at $h'$ than at $h$:

Definition 7.1. Fix $h, h' \in H_2$. Say that $(s_2, t_2)$ is more optimistic at $h'$ than at $h$ if $s_2 \in S_2(h) \cap S_2(h')$ and $\mathbb{E}\pi_2[s_2|t_2, h'] > \mathbb{E}\pi_2[s_2|t_2, h]$.

Definition 7.2. Fix a nonempty $H'_2 \subseteq H_2$ and a $h \in H_2$. Say that $t_1$ reasons that B2 is more optimistic at $H'_2$ than at $h$ if there exists an event $E_2 \subseteq S_2 \times T_2$, so that

\(^{10}\)Such overconfident beliefs are often interpreted as “optimistic beliefs.” Note that this notion of optimism is quite different from second-order optimism.

\(^{11}\)A sufficiently large $\delta$ may be “far” from 1: If $\delta \geq \sqrt{2} - \frac{1}{2}$, then $\overline{\pi}(\delta, N) \geq 2$ irrespective of the deadline.
(i) $\beta_{1,\phi}(t_1)(E_2) = 1$; and

(ii) for each $h' \in H'_2$ and each $(s_2, t_2) \in E_2$, $(s_2, t_2)$ is more optimistic at $h'$ than at $h$.

Definition 7.2 captures second-order optimism. Loosely, type $t_1$ reasons that $B_2$ is more optimistic at $H'_2$ than at $h$ if, at the start of the game, she assigns probability 1 to strategy-type pairs $(s_2, t_2)$, which are more optimistic at each $h' \in H'_2$ than at $h$.

Fix a state $(s^*_1, t^*_1, s^*_2, t^*_2)$ and suppose the state induces the outcome $(x^*_1, x^*_2, n)$. Let $[(x^*_1, x^*_2, n)]$ be the subset of one-period histories $H'_2$ that follow $B_1$ making some offer that Pareto dominates $(x^*_1, x^*_2, n)$. Observe that $[(x^*_1, x^*_2, n)] \neq \emptyset$ if and only if $n \geq 2$. We will argue that, if $n \geq 2$, then $t^*_1$ reasons that $B_2$ would be more optimistic at any history in $[(x^*_1, x^*_2, n)]$ than he would be if $B_1$ made the initial offer associated with $s^*_1$—i.e., if $B_1$ initially proposes the allocation $(s^*_1(\phi), 1 - s^*_1(\phi))$.

**Proposition 7.1.** Fix a state $(s^*_1, t^*_1, s^*_2, t^*_2)$ at which:

(i) there is on-path strategic certainty,

(ii) $\beta_{1,\phi}(t_1^*)$ assigns probability 1 to $R^1_2 \cap \text{SB}_2(Z_1[s^*_1, s^*_2])$ and

(iii) $\xi(\zeta(s^*_1, s^*_2)) = (x^*_1, x^*_2, n)$ for some $n \geq 2$.

Then, $t^*_1$ reasons that $B_2$ is more optimistic at $[(x^*_1, x^*_2, n)]$ than at the history at which $B_1$ makes an initial offer of $(s^*_1(\phi), 1 - s^*_1(\phi))$.

Fix a state $(s^*_1, t^*_1, s^*_2, t^*_2)$ at which there is on-path strategic certainty. Suppose, further, that at the initial node, $B_1$ assigns probability 1 to “$B_2$ is rational and $B_2$ satisfies on-path strategic certainty.” Proposition 7.1 says that, if that state induces an inefficient outcome, then $B_1$ must reason that $B_2$ would be more optimistic after any mutually beneficial offer than he would be after the initial offer associated with $s^*_1$.

## 8 Revisiting Type Structures

We took the description of the strategic situation as an epistemic bargaining game $(B, T)$. The type structure $T$ represents the hierarchies of conditional beliefs that the bargainers consider possible. The bargainers then engage in forward-induction reasoning relative to those beliefs. Section 6 focused on the extreme case where the analyst does not have information about the bargainers’ type structure. As a consequence, Theorems 6.1-6.2 characterized the outcomes consistent with RCSBR and on-path strategic certainty across all type structures: Theorem 6.1 shows that, in any type structure, rationality, strong belief of rationality and on-path strategic certainty imply that the outcome must satisfy the UCs

\[ \beta_{1,\phi}(t_1)(E_2) = 1; \]

\[ \text{for each } h' \in H'_2 \text{ and each } (s_2, t_2) \in E_2, \text{ } (s_2, t_2) \text{ is more optimistic at } h' \text{ than at } h. \]
and DC. Theorem 6.2 shows that any outcome that satisfies these constraints is consistent with RCSBR and on-path strategic certainty in some type structure. The proof of Theorem 6.2 constructs a type structure that is incomplete—i.e., that does not induce all possible beliefs. This raises the question: Are incomplete type structures central to the consistency between RCSBR, on-path strategic certainty and delayed agreement? If so, part of the explanation for delay involves the bargainers having a limited set of beliefs. We show that, when there is a three-period deadline, such incompletenesses are central to the result. Then, we discuss difficulties in extending the result beyond three-periods. Section 10.E discusses the interpretation of Theorem 6.2 if incompletenesses are central for the result.

Call a CPS $\mu$ on $(\Omega, \mathcal{E})$ degenerate if, for each $E \in \mathcal{E}$, $\mu(\cdot | E)$ is a Dirac measure. Extend the definition of a type structure as in Definition A.1 of Appendix A. A type structure $T$ is degenerately complete if, for each $i = 1, 2$ and each degenerate CPS $\mu_{-i} \in \mathcal{C}(S_{-i} \times T_{-i}; S_i \otimes T_{-i})$, there is a type $t_i$ with $\beta_i(t_i) = \mu_{-i}$. So, a degenerately complete type structure has all point beliefs, but may also have non-point beliefs. A complete type structure (Brandenburger, 2003)—i.e., a type structure where $\beta_1$ and $\beta_2$ are onto—is a specific example of a degenerately complete type structure.

**Proposition 8.1.** Let $N = 3$ and $T$ be degenerately complete. If $(s_1, t_1, s_2, t_2) \in \bigcap_m R^m \cap C$, $(s_1, s_2)$ results in immediate agreement on the subgame perfect allocation.

The proof can be found in Online Appendix E. To understand the result, begin with an arbitrary finite game. Battigalli and Siniscalchi (2002) show that, if $T$ is complete, then RCSBR is behaviorally equivalent to the solution concept of extensive-form rationalizability (Pearce, 1984, EFR). Here we have an analogous relationship: When $T$ is degenerately complete, the behavioral implications of RCSBR corresponds to EFR. In turn, EFR implies that B1 initially proposes the subgame perfect allocation. On-path strategic certainty further implies that the offer is accepted immediately.

There is a question of whether Proposition 8.1 holds when $N > 3$. The answer is unknown because EFR behavior is not known beyond $N = 3$. At first, this may be surprising: It would appear that two known results in the literature would allow us to infer EFR behavior. As we now point out, this is not easily achieved.

**Remark 8.1.** In a finite game satisfying a no-ties condition, EFR is outcome equivalent to subgame perfection. (See Battigalli, 1997 and Chen and Micali, 2012.) So, one might be tempted to conclude that EFR behavior is known in discretized versions of the bargaining game. But, the bargaining game has ties; discretized versions will have ties when the grid is fine. Thus, one cannot easily apply these results. \[\square\]

Note, unlike Battigalli and Siniscalchi (2002), we do not know if there exists a degenerately complete $T$ in which $\bigcap_m R^m \neq \emptyset$. Online Appendix E explains why the message still holds if no such structure exists.
Remark 8.2. Fudenberg and Tirole (1991) provide a dominance procedure for extensive-form games and apply it to the bargaining game without a deadline. In conjunction with an equilibrium assumption, the dominance procedure leads to the subgame perfect outcome. The Fudenberg-Tirole procedure is defined on actions; as an output, it delivers a set of strategies that is a product of the action sets. EFR is defined on strategies and—at least on low rounds—is not a product of actions. This fact leads to computational difficulties absent from Fudenberg and Tirole’s (1991) analysis. Can the Fudenberg-Tirole procedure, nonetheless, be used to characterize EFR outcomes (even though, round-for-round, it does not characterize EFR strategies)? This is an open question.

9 Revisiting On-Path Strategic Certainty

On-path strategic certainty is, arguably, best understood from the analyst’s perspective: An outside observer can verify whether the bargainers have correct beliefs along the path of play. However, Bi cannot herself verify this fact, since she does not know the actual strategy employed by B(−i). Moreover, on the surface, on-path strategic certainty does not appear to be an implication of introspective strategic reasoning. This stands in contrast to strong belief of rationality, which is as an assumption about introspective strategic reasoning.

Can we replace the assumption of on-path strategic certainty with the assumption of RCSBR? The answer is yes, provided we rule out a degenerate form of strategic uncertainty. Return to the three-period deadline example. Suppose B1 initially proposes the SPE allocation \((x_1^*, x_2^*) = (1 - \delta(1 - \delta), \delta(1 - \delta))\), believing that her offer will be accepted. But she is incorrect. B2 rejects the offer and subsequently proposes \((y_1^*, y_2^*) = (\delta, 1 - \delta)\), believing that his offer will be accepted. He, too, is incorrect. B1 rejects the offer and proposes \((w_1^*, w_2^*) = (1, 0)\). Although this situation is consistent with RCSBR, it violates the constraints and results in perpetual disagreement. (See Section F.4.) But, arguably, it is a degenerate form of delay: When B2 offers \((y_1^*, y_2^*)\), he anticipates that any offer of \((y_1, y_2)\) with \(y_1 < \delta\) (resp. \(y_1 > \delta\)) will be rejected (resp. accepted). B2 offers \((y_1^*, y_2^*)\) only because he has incorrect beliefs about how B1 acts when indifferent (between accept/reject).13

This section shows that the message holds whenever the game has a deadline: Fix a state at which there is RCSBR, and no bargainer faces uncertainty about how the other breaks indifferences. Then, the UCs and DC must be satisfied. In fact, we will see that the converse also holds. To formalize the idea, we will need several definitions.

13Note, in the epistemic model, we can—and typically do—have strategic uncertainty even when agents face no uncertainty about how other agents act when indifferent. This stands in contrast to an equilibrium analysis of a complete information model, where strategic uncertainty must be an artifact of uncertainty about how others behave when indifferent. The assertion is that behavior that is an artifact of uncertainty about how others break indifferences is of lesser interest.
Definition 9.1. Call two outcomes, \((x_1^0, x_2^0, n^*)\) and \((x_1^{**}, x_2^{**}, n^{**})\), \textbf{Bi-equivalent} if
\[
\Pi_i(x_1^0, x_2^0, n^*) = \Pi_i(x_1^{**}, x_2^{**}, n^{**}).
\]

Note, two distinct outcomes that are Bi-equivalent will not be B\((-i)\)-equivalent. We are interested in whether there are distinct Bi-equivalent outcomes consistent with both RCSBR and a history \(h\).

Definition 9.2. Fix some \(s_i \in S_i\) and some \(\mu_i \in \mathcal{P}(S_{-i} \times T_{-i})\). Say that \((s_i, \mu_i)\) has a \textbf{distinguished outcome} if there exists some event \(E_{-i} \subseteq S_{-i} \times T_{-i}\) with \(\mu_i(E_{-i}) > 0\) so that \(\xi(\{s_i\} \times \text{proj}_{S_{-i}} E_{-i}))\) is a singleton.

Suppose \((s_i, \beta_i, h(t_i))\) has a distinguished outcome. Then, we can find an outcome \((x_1, x_2, n)\) and an event \(E_{-i}\) so that (i) \(\beta_i, h(t_i)(E_{-i}) > 0\), and (ii) if \((s_i, t_{-i}) \in E_{-i}\), then \((s_i, s_{-i})\) induces the outcome \((x_1, x_2, n)\). To better understand the definition, suppose that \((s_i, \beta_i, h(t_i))\) does not have a distinguished outcome. Then, at \(h, t_i\) faces an extreme form of uncertainty about the outcome that will obtain: For each \((x_1, x_2, n)\) consistent with \(h, \beta_i, h(t_i)\) assigns zero probability to the situation that “\((x_1, x_2, n)\) obtains when Bi plays \(s_i\)” (Formally, if \(E_{-i} \subseteq \{\{s_i\} \times S_{-i}(h)\} \cap (\xi \circ \zeta)^{-1}(\{(x_1, x_2, n)\})\) is Borel, \(\beta_i, h(t_i)(E_{-i} \times T_{-i}) = 0\).)

Definition 9.3. Fix an epistemic game \((B, T)\) and a state \((s_1, t_1, s_2, t_2)\) at which there is RCSBR. At \((s_1, t_1, s_2, t_2)\), \textbf{Bi is uncertain about how B\((-i)\) breaks indifferences} if there is a history \(h \in H_i\) with \((s_1, s_2) \in S_1(h) \times S_2(h)\), so that the following hold:

(i) There are distinct B\((-i)\)-equivalent outcomes in \(\xi(\{s_i\} \times \text{proj}_{S_{-i}} E_{-i}))\).

(ii) If \((s_i, \beta_i, h(t_i))\) has a distinguished outcome, then there exists some event \(E_{-i}\) with \(\beta_i, h(E_{-i}) > 0\) so that, for each \((r_{-i}, t_{-i}) \in E_{-i},\) the following is satisfied: (a) \(r_{-i} \in S_{-i}(h),\) (b) \(\xi(\{s_i, r_{-i}\}) \neq \xi(\{s_i, s_{-i}\})\), and (c) \(\pi_{-i}(s_i, r_{-i}) = \pi_{-i}(s_i, s_{-i}).\)

Suppose, at \((s_1, t_1, s_2, t_2)\), Bi is uncertain about how B2 breaks indifferences. Then, \((s_1, s_2)\) allows a history \(h\) at which \(t_1\) faces uncertainty about distinct outcomes that are payoff-equivalent for B2. The nature of this uncertainty can take one of two forms. At \(h, t_1\) may assign positive probability to an event in which “the wrong B2-equivalent outcome” obtains. Alternatively, at \(h, t_1\) may not have a distinguished outcome; in that case, she cannot assign probability 1 to the event that the correct outcome will obtain.

Proposition 9.1. Fix a game with a deadline and a state \((s_1^*, t_1^*, s_2^*, t_2^*)\) that induces the outcome \((x_1^*, x_2^*, n)\). If \((s_1^*, t_1^*, s_2^*, t_2^*) \in R^\infty\) and, at this state, no Bi is uncertain about how B\(-i)\ breaks indifferences, then \((x_1^*, x_2^*, n)\) satisfies the UCs and DC.

The converse is also true. Fix an outcome \((x_1^*, x_2^*, n)\) that satisfies the UCs and DC. We can construct a type structure and a state \((s_1^*, t_1^*, s_2^*, t_2^*)\) so that (i) \((s_1^*, t_1^*, s_2^*, t_2^*)\) induces the outcome \((x_1^*, x_2^*, n)\), (ii) \((s_1^*, t_1^*, s_2^*, t_2^*) \in R^\infty\) and (iii) at \((s_1^*, t_1^*, s_2^*, t_2^*)\), no Bi is uncertain about
how B(−i) breaks indifferences. (See Remark B.4, which shows a stronger no-indifference property.)

10 Discussion

10.A Nash Equilibrium Theorem 6.1 and the proof of Theorem 6.2 give the following: If \((s_1, t_1, s_2, t_1) \in R^\infty \cap C\), then there exists some Nash equilibrium in sequentially optimal strategies, viz. \((r_1, r_2)\), so that \((s_1, s_2)\) and \((r_1, r_2)\) are outcome-equivalent. Thus, RCSBR and on-path strategic certainty give Nash outcomes in sequentially optimal strategies.

It is useful to connect to the literature. Fix a finite game of perfect information. If the game satisfies a no-ties condition called transference of decision-maker indifference (Marx and Swinkels, 1997, TDI), then the RCSBR strategies are sequentially optimal and give a Nash outcome. (See Battigalli and Friedenberg, 2012a, Lemma 8. Note, this statement is weaker than the statement in the previous paragraph.) More specifically, in a game satisfying TDI, RCSBR alone implies a constant payoff condition; in turn, TDI plus that condition imply on-path strategic certainty. However, the bargaining game violates TDI, and this generates a more subtle relationship: While RCSBR alone does not imply the constant payoff condition, RCSBR plus no uncertainty about breaking indifferences (Definition 9.3) does. This delivers a Nash outcome, but does not imply on-path strategic certainty. Thus, there is a gap between the results in Sections 6 and 9.

10.B Subgame Perfect Equilibrium For any given \((\delta, N)\), the subgame perfect outcome \((x_1^{SPE}, x_2^{SPE}, 1)\) satisfies the UCs and DC. Thus, the subgame perfect no-delay outcome is consistent with \(R^\infty \cap C\). In the specific case of no deadline, \((x_1^{SPE}, x_2^{SPE}, n)\) satisfies the UCs and DC if and only if \(n \leq \pi(\delta, \infty)\). As such, for any given \(\delta\), the longest length of delay involves the parties agreeing either on the SPE allocation or a nearby allocation.\(^{14}\) When \(\delta\) is large, the delay is arbitrarily long, only to agree on a close to 50:50 split of the pie.

10.C No Deadline: Anything Goes Result Consider the case of no deadline. For any given \(\delta, \overline{\pi}(\delta, \infty) < \infty\). Thus, for a given discount factor, there is a bound on delay. However, for any given \(n\),

\[
\lim_{\delta \to 1} \{x_1 : (x_1, x_2, n) \text{ satisfies the UCs and DC}\} = [0, 1].
\]

This provides an anything goes result in an important limiting case: When the discount factor is close to 1, there can be arbitrarily long delays and the bargainers can agree on

\(^{14}\)If \(\ln(1-\delta^2)/\ln(\delta)\) is an integer, then they would agree on the SPC allocation. Otherwise, they would agree on a nearby allocation.
essentially any allocation. This constrasts with Myerson’s (1997) one-sided reputational bargaining model. (See Compte and Jehiel, 2002, Proposition 2.)

We can reinterpret this limiting case in terms of frequent offers: Consider a continuous time variant of the model, where the bargainers are restricted to make offers at intervals of length \( \Delta > 0 \). The original model can be embedded into this one, by taking \( \delta = e^{-r\Delta} \) for a common discount rate \( r \). Thus, a fixed discount factor can be viewed as the case in which the time between offers \( \Delta \) is significant. When the time between offers gets small, the length of delay gets large. See Online Appendix H.

10.D Extension: Outside Options In the context of the reputational bargaining, Compte and Jehiel (2002) show that outside options typically serve to cancel out the possibility of obstinacy in bargaining—the parties seeks to reveal themselves as ‘rational’ as soon as possible. (The term ‘typical’ reflects a minor condition on parameters. The result is true both for one- and two-sided uncertainty about strategic posture.) We now show that, under the analysis here, there may be long delays, even in the presence of outside options.

The model follows Compte and Jehiel (2002): There is no deadline. In any bargaining phase, the responder can either accept the proposal, reject the proposal and continue negotiations, or reject the proposal and exercise the outside option. In the last case, the bargaining terminates and the bargainers obtain an outside option \((w_1, w_2)\) with \(w_1 + w_2 \leq 1\).

At any state at which there is RCSBR, on-path strategic certainty, and delay, no \( B_i \) exercises her outside option: If this were not the case, \( B_2 \) would anticipate that his payoff is some \( \delta^{n-1} w_2 < w_2 \); he would instead exercise the outside option immediately. With this in mind, states at which there is RCSBR, on-path strategic certainty, and delay involve agreement on some allocation—i.e., an outcome \((x_1^*, x_2^*, n)\) with \( n \geq 2 \).

Outcomes consistent with RCSBR, on-path strategic certainty and delay can be characterized by Outside Option Constraints (OOCs) and Generalized Upfront Constraints (GUCs). (See Online Appendix H.) An outcome \((x_1^*, x_2^*, n)\) satisfies the OOCs if each \( B_i \) prefers the outcome to getting the outside option in the first feasible period (i.e., period 2 for \( B_1 \) and period 1 for \( B_2 \)). With this, the OOCs require \( \delta^{n-1} x_1^* \geq \delta w_1 \) and \( \delta^{n-1} x_2^* \geq w_2 \). An outcome \((x_1^*, x_2^*, n)\) satisfies the GUCs if each \( B_i \) prefers the outcome to what she can secure upfront. However, now, \( B_i \) need not think that \( B(\bar i) \) will accept offers in \((\delta, w_{-i})\); thus, \( B_i \) can secure \( 1 - \max\{\delta, w_{-i}\} \) upfront. With this, the GUCs require \( \delta^{n-1} x_1^* \geq 1 - \max\{\delta, w_2\} \) and \( \delta^{n-1} x_2^* \geq \delta(1 - \max\{\delta, w_1\}) \).

If a delayed outcome \((x_1^*, x_2^*, n)\) with \( n \geq 3 \) satisfies the OOCs and GUCs, then it satisfies the UCs. But the converse need not hold. Thus, the presence of outside options can limit
the scope of delay.\footnote{If \((x_1^*, x_2^*, 2)\) satisfies the OOCs and the GUCs, then either it satisfies the UCs or \((x_1^*, x_2^*) = (w_1, 1 - w_1)\).} Moreover, for each \(n \geq 2\),
\[
\lim_{\delta \to 1} \{x_1 : (x_1, x_2, n) \text{ satisfies the OOCs and GUCs}\} = [w_1, 1 - w_2] \neq \emptyset.
\]
In this limiting case, there can be arbitrarily long delays. But, the resulting set of allocations is bounded—substantially bounded if the outside options are close to efficient.

10.E Incomplete vs. Complete Type Structures To understand the relationship between Theorem 6.2 and Proposition 8.1, it will be useful to review the nature of forward-induction reasoning as implemented here.

Refer to Remark 5.1. RCSBR captures the idea that the bargainers adopt a hypothesis consistent with the highest form of strategic sophistication possible. In this sense, it formalizes forward-induction reasoning. But, importantly, the “highest form of strategic sophistication possible” depends on both the parameters of the bargaining game and the bargainers’ type structure. This implies that forward-induction reasoning can be both \textit{pure} and \textit{contextualized}. We now explain.

Fix a type structure. The structure can be associated with an event about the players’ beliefs: Each type in that structure \textit{fully believes} the event—that is, each type does not give up on the hypothesis that the event is true irrespective of the evidence.\footnote{Importantly, because the event is only about beliefs, that event will never be contradicted by evidence. So, players can hold onto the hypothesis irrespective of observed play. (If the event involved strategies, it may be contradicted by some observed play; if so, it is impossible to fully believe the event. For instance, in the bargaining game, it is impossible to fully believe the event \(R_{i,j}^1\) (i.e., irrespective of the type structure).} Moreover, the event is common full belief amongst the types in that structure. When RCSBR is applied to the type structure, players engage in forward-induction reasoning relative to the common full belief event. (See Appendix A in Battigalli and Friedenberg, 2012b for a finite game result.) So, in the case of a surprise move, a player never gives up on the hypothesis that the event is true, nor on the hypothesis that “the other player never gives up on the hypothesis that the event is true,” etc. This may well imply that, after a surprise move, the player may not be able to rationalize past behavior. That is, the common full belief event limits the players’ ability to rationalize past behavior. Incomplete type structures are typically associated with a \textit{non-trivial} common belief event—i.e., an event that is a strict subset of the set of all hierarchies of beliefs. By constrast, complete type structures are typically associated with a \textit{trivial} event—i.e., an event that corresponds to the set of all hierarchies of beliefs.\footnote{Results in Friedenberg (2010) and Friedenberg and Keisler (2011) explain why we say “typically.”} As such, in incomplete type structures, the common full belief event typically limits the players’ ability to rationalize past behavior. By contrast, in complete type structures, the ability to rationalize past behavior is typically not limited by the common full belief event.
The distinction between RCSBR applied to incomplete versus complete type structures is what Battigalli, Friedenberg, and Siniscalchi (2012) refer to as contextualized-forward-induction reasoning versus pure-forward-induction reasoning. (See, also, Brandenburger, Friedenberg, and Keisler, 2008, pg. 319.) The idea is that the social situation may exist in the context of history, cultural norms, etc.; that context can impact the players’ beliefs. An incomplete structure corresponds to a situation in which the context gives rise to a non-trivial common full belief event. In that case, RCSBR gives rise to a contextualized-forward-induction reasoning. By contrast, a complete structure corresponds to a context-free situation. In this sense, RCSBR gives rise to a more pure-forward-induction reasoning.

With this in mind, suppose that \( N = 3 \) and return to Proposition 8.1 and Theorem 6.2. Proposition 8.1 says that, if the bargainers’ type structure is complete, then RCSBR and on-path strategic certainty imply immediate agreement. By contrast, Theorem 6.2 gives that RCSBR and on-path strategic certainty are consistent with delayed agreement. Thus, when \( N = 3 \), part of the explanation for delay involves incomplete type structures: The bargainers can satisfy on-path strategic certainty, engage in contextualized-forward-induction reasoning, and exhibit delayed agreements. But, if they satisfy on-path strategic certainty and engage in pure-forward-induction reasoning, they cannot have delayed agreements.

This raises the question: Which common full belief events give rise to delayed agreements. A better understanding of these events is important for understanding the ingredients of delayed agreement. This is left as an open question.

References


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Appendix A  Preliminaries

This Appendix generalizes the framework in Sections 4-5. The purpose is to be sure that our analysis does not hinge on simplifying assumptions. Throughout, we identify a history with a sequence of nodes. For instance, the history that coincides with an initial offer of \((x_1, x_2)\) is written as \((\phi, x_1)\). As such, if \(h'\) precedes \(h\), then \(h'\) is an initial segment of \(h\).

**Topology**  Endow \([0,1]\) with the Euclidean metric and \(\{a,r\}\) with the discrete metric. For each \(h \in H_i\), there is a choice set \(C_h\), where \(C_h = [0,1]\) if \(h \in H_i^P\) and \(C_h = \{a,r\}\) if \(h \in H_i^R\). Then, \(S_i = \prod_{h \in H_i} C_h\). Endow \(S_i\) with the product topology—i.e, the weak topology generated by the projection maps \(\text{proj}_h : S_i \to C_h : h \in H_i\). Then, \(S_i\) is a compact Hausdorff space.

Let \(Z_D\) be the set of disagreement terminal histories. Note, any history in \((H \cup Z)\setminus Z_D\) is a finite history and any history in \(Z_D\) is a countable history.

**Lemma A.1.**

(i) For each \(h \in (H \cup Z)\setminus Z_D\), \(S_i(h)\) is closed.

(ii) For each \(h \in Z_D\), \(S_i(h)\) is Borel.
(iii) For each $z \in Z$ and each $s_i \in S_i$, the set $\{s_{-i} : \zeta(s_i, s_{-i}) = z\}$ is Borel.

**Proof.** Part (i) (resp. (ii)) is immediate from the fact that a finite (resp. countable) intersection of sets $(\text{proj}_h)^{-1}(\{c_h\})$ is closed (resp. Borel). For part (iii), fix $z \in Z$ and $s_i \in S_i$. If $s_i \notin S_i(z)$, then $\{s_{-i} : \zeta(s_i, s_{-i}) = z\} = \emptyset$ is Borel. So suppose $s_i \in S_i(z)$. By parts (i)-(ii), it suffices to show $\{s_{-i} : \zeta(s_i, s_{-i}) = z\} \subseteq S_{-i}(z)$. Given $s_{-i} \in S_{-i}(z)$, there exists $r_i \in S_i$ with $\zeta(r_i, s_{-i}) = z$. It follows that, along the path from the root to the terminal node $z$, $r_i$ and $s_i$ must specify the same choices. So $\zeta(s_i, s_{-i}) = \zeta(r_i, s_{-i})$. □

**Type Structures** We adapt the approach in Ben-Porath (1997) and Battigalli and Siniscalchi (1999) to provide a general definition of type structures for infinite games. It coincides with the literature on a variant of the game where the set of feasible allocations is discrete.

Given a compact Hausdorff space $\Omega$, endow $\mathcal{P}(\Omega)$ with the weak* topology. Endow the set of arrays, $\prod_{E \in E}\{\mu \in \mathcal{P}(\Omega) : \mu(E) = 1\}$, with the product Borel sigma algebra and $\mathcal{C}(\Omega; \mathcal{E})$ with the relative sigma-algebra.

**Definition A.1.** A type structure $T$ specifies three objects for each $B_i$:

(i) $T_i$ is a compact Hausdorff type set;
(ii) $\mathcal{S}_i \otimes \mathcal{T}_{-i} = \{S_{-i} \times T_{-i}\} \cup \{S_{-i}(h) \times T_{-i} : h \in H_i\}$ is the set of conditioning events;
(iii) $\beta_i : T_i \to \mathcal{C}(S_{-i} \times T_{-i}; \mathcal{S}_i \otimes \mathcal{T}_{-i})$ is a measurable belief map.

We will abuse notation and write $\beta_i(h, t_i)$ for $\beta_i(t_i)(|S_{-i}(h) \times T_{-i}$) (resp. $\beta_i, \mathcal{E}(t_i)$ for $\beta_i(t_i)(|S_{-i} \times T_{-i}$)). We will say that the type structure has countable beliefs if each $\beta_i(t_i)$ is a countable CPS. Definition 4.2 is an example of a type structure with countable beliefs.

**Remark A.1.** A conceptual issue underlies the definitions: If a countable array satisfies the conditioning requirement, then it satisfies the rules of conditional probability when possible. The same is true when the conditioning events are countable—i.e., even if the array is not countable. For more general arrays or conditioning events, the conditioning requirement should be viewed as a minimal criteria to capture the desired idea.

We don’t know if, in this more general case, Definition 4.1 is overly permissive. That said, this potential additional generality is immaterial. Ultimately, our behavioral result will be a Sandwich Theorem: Theorem 6.1 involves a permissive definition of beliefs, and Theorem 6.2 involves countable beliefs. We conjecture that such a Sandwich approach may be a useful technique in other epistemic analyses of infinite games. □

Many definitions in the main text apply to Definition A.1. But, two amendments are needed.
Notational Change} Fix some \( h \in H_i \cup \{\phi\} \). For each \( t_i \), write \( \text{marg}_{S_{-i}} \beta_{i,h}(t_i) \) for the marginal distribution of \( \beta_{i,h}(t_i) \) on \( S_{-i} \). Let \( s_i \in S_i(h) \) so that \( \pi_i(s_i, \cdot) \) is \( \text{marg}_{S_{-i}} \beta_{i,h}(t_i) \) integrable. Then, write

\[
\mathbb{E}\pi_i[s_i|t_i,h] = \int_{S_{-i}} \pi_i(s_i, \cdot) \text{d}\text{marg}_{S_{-i}} \beta_{i,h}(t_i).
\]

(Note, \( \mathbb{E}\pi_i[s_i|t_i,h] \) is defined only if \( \pi_i(s_i, \cdot) \) is \( \text{marg}_{S_{-i}} \beta_{i,h}(t_i) \) integrable.)

**Definition of Rationality** Begin with a definition.

**Definition A.2.** Fix a strategy \( s_i \) and an array \( \mu_i \) on \((S_{-i}; S_i)\). Say \( s_i \) is **sequentially optimal under** \( \mu_i \) if, for each information set \( h \in H_i \) with \( s_i \in S_i(h) \), the following hold:

(i) \( \pi_i(s_i, \cdot) : S_{-i} \to \mathbb{R} \) is \( \mu_i(\cdot|S_{-i}(h)) \)-integrable, and

(ii) if \( \pi_i(r_i, \cdot) : S_{-i} \to \mathbb{R} \) is \( \mu_i(\cdot|S_{-i}(h)) \)-integrable for \( r_i \in S_i(h) \), then

\[
\int_{S_{-i}} \pi_i(s_i, \cdot) \text{d}\mu_i(\cdot|S_{-i}(h)) \geq \int_{S_{-i}} \pi_i(r_i, \cdot) \text{d}\mu_i(\cdot|S_{-i}(h)).
\]

Fix a strategy \( s_i \) and a history \( h \) allowed by \( s_i \). Condition (i) says that, at \( h \), \( Bi \) must be able to evaluate her expected payoffs from \( s_i \) under the array \( \mu_i \). Condition (ii) says that, at \( h \), \( s_i \) must maximize \( Bi \)'s expected payoffs under the array \( \mu_i \). The maximization is relative to all strategies that allow \( h \), provided \( Bi \) can evaluate her expected payoffs from \( r_i \) under \( \mu_i \). When the array is countable, \( Bi \) can evaluate her expected payoffs from every strategy.

Write \( \text{marg}_{S_{-i}} \beta_i(t_i) \) for \( t_i \)'s **marginal array** on \( S_{-i} \)—i.e., an array \( \mu_i \) on \((S_{-i}; S_i)\) with \( \mu_i(\cdot|S_{-i}(h)) = \text{marg}_{S_{-i}} \beta_{i,h}(t_i)(\cdot) \) for each \( S_{-i}(h) \in S_i \). Then, \( \text{marg}_{S_{-i}} \beta_i(t_i) \) specifies \( t_i \)'s first-order belief. A strategy-type pair \((s_i, t_i)\) is **rational** if \( s_i \) is sequentially optimal under the marginal array \( \text{marg}_{S_{-i}} \beta_i(t_i) \). This corresponds to Definition 5.1 when the type structure is finite and the beliefs are countable.

**Appendix B**  **Proofs of Theorems 6.1-6.2**

Observe that \((x_1^n, x_2^n, n)\) satisfies the UCs and DC if and only if \( n < \infty \) and \( x_1^n \in [\mathcal{Z}^n; \mathcal{P}^n] \), where

\[
\mathcal{Z}^n = \begin{cases} 
\max\left\{ \frac{1-\delta}{\delta^n-t}, \delta^{N-n} \right\} & \text{if } N < \infty \text{ is odd} \\
\frac{1-\delta}{\delta^n-t} & \text{otherwise}
\end{cases}
\]

and

\[
\mathcal{P}^n = \begin{cases} 
\min\left\{ 1 - \delta(1-\delta), 1 - \delta^{N-n} \right\} & \text{if } N < \infty \text{ is even}, \\
1 - \frac{\delta(1-\delta)}{\delta^n-t} & \text{otherwise}.
\end{cases}
\]
In the proof of Theorem 6.1, we take a general definition of a type structure (Definition A.1). We show that, if a state is in $R^2 \cap C$, then it induces an outcome $(x_1^*, x_2^*, n)$, where $x_1^* \in [\mathbb{Z}_n, \mathbb{X}_n]$. In the proof of Theorem 6.2, we fix some outcome $(x_1^*, x_2^*, n)$, where $x_1^* \in [\mathbb{Z}_n, \mathbb{X}_n]$. We then show that we can construct a finite type structure with countable beliefs (Definition 4.2) and a state that induces the outcome $(x_1^*, x_2^*, n)$, so that the state is in $R^\infty \cap C$. Taken together, the two results provide a “sandwich” on behavior.

### B.1 Proof of Theorem 6.1

Here, we take a general definition of a type structure (Definition A.1).

**Definition B.1.** Say that type $t_i \in T_i$ can secure payoffs of $q$ at history $h \in H_i \cup \{\phi\}$ if there exists some strategy $s_i \in S_i(h)$ so that $\mathbb{E}\pi_i[s_i|t_i, h] \geq q$.

The first remark is immediate.

**Remark B.1.** Fix a state $(s_i, t_i) \in R_i^1$ and some $(h, x_{-i}) \in H^R_i$ allowed by $s_i$.

(i) If $x_{-i} \in [0, 1 - \delta)$, then $s_i(h, x_{-i}) = a$.

(ii) If $(h, x_{-i})$ is an $N$-period history and $x_{-i} \in [0, 1)$, then $s_i(h, x_{-i}) = a$.

An immediate implication is:

**Remark B.2.** Fix $t_i$ that strongly believes $R_{-i}^1$. Let $h \in H^P_i$ be such that $R_{-i}^1 \cap [S_{-i}(h) \times T_{-i}] \neq \emptyset$. Then, $\beta_{i,h}(t_i)(R_{-i}^1) = 1$. Moreover,

(i) $R_i^1 \subseteq \{ r_{-i} \in S_{-i}(h) : r_{-i}(h, x_i) = a, \text{ for all } x_i \in [0, 1 - \delta) \} \times T_{-i}$, and

(ii) if $h' \in H^P_i$ is an $N$-period that (weakly) follows $h$, $R_i^1 \subseteq \{ r_{-i} \in S_{-i}(h') : \text{ if } r_{-i} \in S_{-i}(h') \text{ then, } r_{-i}(h', x_i) = a \text{ for all } x_i \in [0, 1) \} \times T_{-i}$.

**Lemma B.1.** Fix some $(s_1^*, t_1^*, s_2^*, t_2^*) \in R^2$. Suppose $\xi(\zeta(s_1^*, s_2^*)) = (x_1^*, x_2^*, n)$, where $B_i$ is the proposer in period $n$.

(i) If $n \leq N - 1$, then $x_1^* \geq 1 - \delta$ and $x_2^* \geq \delta(1 - \delta)$.

(ii) If $n = N$, then there exists an $N$-period history $h^* \in H^P_i$ with $(s_1^*, s_2^*) \in S(h^*)$, $s_i^*(h^*) = 1$, and $x_{-i}^* = 0$.

**Proof.** First, let $n \leq N - 1$. By assumption, there is an $n$-period history $h^* \in H^P_i$ with $(s_1^*, s_2^*) \in S(h^*)$, $s_i^*(h^*) = x_i^*$, and $s_{-i}^*(h^*, x_i^*) = a$. So, $\mathbb{E}\pi_{-i}[s_{-i}^*|t_{-i}^*, (h^*, x_i^*)] = \delta^{n-1}x_{-i}^*$.

Note $t_i^*$ strongly believes $R_{-i}^1$ and, by assumption, $R_{-i}^1 \cap [S_{-i}(h^*) \times T_{-i}] \neq \emptyset$. Thus, applying Remark B.2, for each $x_i \in [0, 1 - \delta)$, $t_i^*$ can secure $\delta^{n-1}x_i$ at $h^*$. Since $(s_i^*, t_i^*)$ is
rational, it follows that \( x_i^* \geq 1 - \delta \). It suffices to show that, for any \( x_{-i} \in [0, \bar{\delta} - 1) \), \( t_{-i}^* \) can secure \( \delta^n x_{-i} \) at \((h^*, x_i^*)\). If so, since \((s_{-i}^*, t_{-i}^*)\) is rational, it follows that \( \delta^{n-1} x_{-i}^* \geq \delta^n (1 - \delta) \) or \( x_{-i}^* \geq \delta(1 - \delta) \).

Fix some \( x_{-i} \in [0, \bar{\delta} - 1) \). Since \( n \leq N - 1 \), we can construct a strategy \( r_{-i} \) so that 
\[
\begin{align*}
r_{-i}(h^*, x_i^*) &= r, \\
r_{-i}(h^*, x_i^*, r) &= x_{-i}, \text{ and } \\
r_{-i}(h) &= s_{-i}^*(h) \text{ for all } h \in H_{-i} \setminus \{(h^*, x_i^*), (h^*, x_i^*, r)\}.
\end{align*}
\]

Note \( t_{-i}^* \) strongly believes \( R_{1}^1 \) and, by assumption, \( R_{1}^1 \cap (S_{1}(h^*, x_i^*) \times T) \neq \emptyset \). So, by Remark B.2, \( \mathbb{E}_{\pi_{-i}}[r_{-i}(t_{-i}^*) (h^*, x_i^*)] = \delta^n x_{-i} \), as desired.

Next, suppose that \( n = N \). Note, \( t_{i}^* \) strongly believes \( R_{1}^1 \) and \( R_{1}^1 \cap (S_{1}(h^*) \times T_{-i}) \neq \emptyset \). So, using Remark B.2, \( t_{i}^* \) can secure \( \delta^{n-1} x_i \) at \( h^* \) for each \( x_i \in [0, 1) \). Since \((s_{i}^*, t_{i}^*)\) is rational, \( s_{i}^*(h^*) = 1 \) and so \( x_{i}^* = 0 \). ■

The next step is to establish the UCs. In fact, we will establish this under a somewhat weaker assumption. (We will later make use of this stronger result.)

**Lemma B.2.** Fix some \((s_{1}^*, t_{1}^*, s_{2}^*, t_{2}^*) \in R^2 \) with \( \xi(\zeta(s_{1}^*, s_{2}^*)) = (x_{1}^*, x_{2}^*, n) \). If \( \beta_{1, \phi}(t_{1}^*) \) assigns probability 1 to an event \( E_{2} \subseteq S_{2} \times T_{2} \) with

\[
E_{2} = \{r_{2} : \xi(\zeta(s_{1}^*, r_{2})) = (x_{1}^*, x_{2}^*, n)\} \times T_{2},
\]

then \( \delta^{n-1} x_{1}^* \geq 1 - \delta \) and \( n < \infty \).

**Proof.** Since \((s_{1}^*, t_{1}^*) \in R_{1}^1 \), it follows that \( \mathbb{E}_{\pi_{1}}[s_{1}^*|t_{1}^*, \phi] \geq \mathbb{E}_{\pi_{1}}[r_{1}|t_{1}^*, \phi] \) for all \( r_{1} \). Since \((s_{1}^*, t_{1}^*) \in SB_{1}(R_{2}^1) \), Remark B.2 gives that, for each \( r_{1} \) with \( r_{1}(\phi) \in [0, \bar{\delta} - 1) \), \( \mathbb{E}_{\pi_{1}}[r_{1}|t_{1}^*, \phi] = r_{1}(\phi) \). Thus, \( \mathbb{E}_{\pi_{1}}[s_{1}^*|t_{1}^*, \phi] \geq 1 - \delta \). Now, by assumption, \( \beta_{1, \phi}(t_{1}^*)(E_{2}) = 1 \) for some \( E_{2} \subseteq \{r_{2} : \xi(\zeta(s_{1}^*, r_{2})) = (x_{1}^*, x_{2}^*, n)\} \times T_{2} \). As such, (i) \( \mathbb{E}_{\pi_{1}}[s_{1}^*|r_{1}^*, \phi] = \delta^{n-1} x_{1}^* \) if \( n < \infty \), and (ii) \( \mathbb{E}_{\pi_{1}}[s_{1}^*|t_{1}^*, \phi] = 0 \) if \( n = \infty \). From this \( n < \infty \) and \( \mathbb{E}_{\pi_{1}}[s_{1}^*|t_{1}^*, \phi] = \delta^{n-1} x_{1}^* \geq 1 - \delta \). ■

**Lemma B.3.** Fix some \((s_{1}^*, t_{1}^*, s_{2}^*, t_{2}^*) \in R^2 \) with \( \xi(\zeta(s_{1}^*, s_{2}^*)) = (x_{1}^*, x_{2}^*, n) \). If \( n \geq 2 \) and \( \beta_{2, \phi}(s_{1}^*, \phi|x)(t_{2}^*) \) assigns probability 1 to an event \( E_{1} \subseteq S_{1} \times T_{1} \) with

\[
E_{1} = \{r_{1} : \xi(\zeta(r_{1}, s_{2}^*)) = (x_{1}^*, x_{2}^*, n)\} \times T_{2},
\]

then \( \delta^{n-1} x_{2}^* \geq \delta(1 - \delta) \) and \( n < \infty \).

**Proof.** By assumption, there is a two-period history \( h^* \in H_{2}^P \) with (i) \( (s_{1}^*, s_{2}^*) \in S(h^*) \), and (ii) \( \beta_{2, \phi}(h^*) \) assigns probability 1 to some event \( E_{1} \) with \( E_{1} = \{r_{1} : \xi(\zeta(r_{1}, s_{2}^*)) = (x_{1}^*, x_{2}^*, n)\} \times T_{2} \). Thus, (i) \( \mathbb{E}_{\pi_{2}}[s_{2}^*|t_{2}^*, h^*] = \delta^{n-1} x_{2}^* \) if \( n < \infty \), and (ii) \( \mathbb{E}_{\pi_{2}}[s_{2}^*|t_{2}^*, h^*] = 0 \) if \( n = \infty \).

Observe that \((s_{2}^*, t_{2}^*) \in SB_{2}(R_{1}^1) \) and \((s_{1}^*, t_{1}^*) \in R_{1}^1 \cap S_{1}(h^*) \). So, by Remark B.2, for any \( r_{2} \in S_{2}(h^*) \) with \( r_{2}(h^*) \in [0, 1 - \delta) \), \( \mathbb{E}_{\pi_{2}}[r_{2}|t_{2}^*, h^*] = \delta r_{2}(h^*) \). Since \((s_{2}^*, t_{2}^*) \in R_{1}^1 \), it follows that \( \mathbb{E}_{\pi_{2}}[s_{2}^*|t_{2}^*, h^*] \geq \delta x_{2} \) for all \( x_{2} \in [0, 1 - \delta) \). Thus, \( n < \infty \) and \( \delta^{n-1} x_{2}^* \geq \delta(1 - \delta) \). ■

The UCs are now (almost) a corollary of the previous results.
Lemma B.4. Fix \((s_1, t_1, s_2, t_2) \in R^2 \cap C\) with \(\xi(\zeta(s_1, s_2)) = (x_1, x_2, n)\). Then, \(n < \infty\). Moreover, (i) \(\delta^{n-1} x_1 \geq 1 - \delta\), and (ii) \(\delta^{n-1} x_2 \geq \delta(1 - \delta)\).

Proof. Since \((s_1, t_1) \in SB(1(Z_2[s_1, s_2]))\), it follows that \(\beta_{1, \phi}(t_1)\) assigns probability 1 to
\[
\{r_2 : \zeta(s_1, r_2) = \zeta(s_1, s_2)\} \times T_2 \subseteq \{r_2 : \xi(\zeta(s_1, r_2)) = (x_1, x_2, n^*)\} \times T_2.
\]
Thus, by Lemma B.2, \(n < \infty\) and part (i) holds.

If \(n = 1\), part (ii) follows from Lemma B.1(i). So, take \(n \geq 2\). Along the path of play induced by \((s_1, s_2)\), there is some two-period history \(h^* \in H^P_2\). Since \((s_2, t_2) \in SB(1(Z_1[s_1, s_2]))\), \(\beta_{2,h^*}(t_2)\) assigns probability 1 to
\[
\{r_1 : \zeta(r_1, s_2) = \zeta(s_1, s_2)\} \times T_1 \subseteq \{r_1 : \xi(\zeta(r_1, s_2)) = (x_1, 1 - x_1, n)\} \times T_1.
\]
Thus, by Lemma B.3, part (ii) holds.

We now turn to the DC. Begin with a preliminary result.

Lemma B.5. Let \(N < \infty\) and suppose \(Bi\) proposes in the last period. Let \(h^* \in H_1\) with \(R^i_{-i} \cap (S_{-i}(h^*) \times T_{-i}) \neq \emptyset\). If \(t_i\) strongly believes \(R^i_{-i}\) then, for each \(x_i \in [0, 1]\), \(t_i\) can secure \(\delta^{N-1} x_i\) at \(h^*\).

Proof. Fix some \(x_i \in [0, 1]\) and some \(h^* \in H_1\) with \(R^i_{-i} \cap (S_{-i}(h^*) \times T_{-i}) \neq \emptyset\). Construct \(s_i \in S_i(h^*)\) that satisfies the following properties: Set \(s_i(h) = x_i\) for each \(N\)-period history \(h \in H^P_i\) that (weakly) follows \(h^*\). Set \(s_i(h) = 1\) for all other histories \(h \in H^P_i\) that (weakly) follow \(h^*\). Set \(s_i(h, x_{-i}) = a\) if and only if \(x_{-i} \in [0, 1 - \delta]\), for each \((h, x_{-i}) \in H^R_i\) that (weakly) follows \(h^*\). Using Remark B.2, the fact that \(t_i\) strongly believes \(R^i_{-i}\), and the fact that \(R^i_{-i} \cap (S_{-i}(h^*) \times T_{-i}) \neq \emptyset\), \(E_{pi}[s_i|t_i, h^*] \geq \delta^{N-1} x_i\).

Lemma B.6. Let \(N < \infty\) and suppose \(Bi\) proposes in the last period. Fix some \((s_1^*, t_1^*, s_2^*, t_2^*) \in R^2\) and write \(\xi(\zeta((s_1^*, s_2^*))) = (x_1^*, x_2^*, n)\).

(i) If \(Bi\) is the responder in the \(n^{th}\) period, then \(x_i^* \geq \delta^{N-n}\).

(ii) If \(Bi\) is the proposer in the \(n^{th}\) period and \((s_1^*, t_1^*, s_2^*, t_2^*) \in C\), then \(x_i^* \geq \delta^{N-n}\).

Proof. If \(n = N\), the result follows from Lemma B.1. Fix \(n < N - 1\) and observe that there is some \(n\)-period history \(h^* \in H^P_i \cup H^P_{-i}\) with \((s_1^*, s_2^*) \in S(h^*)\). Write \(h_i = h^*\) if \(h^* \in H^P_i\) and \(h_i = (h^*, x_{-i}^*)\) if \(h^* \in H^R_{-i}\). In either case, \(E_{pi}[s_i^*|t_i^*, h_i] = \delta^{n-1} x_i^*\). This is immediate if \(h_i = (h^*, x_{-i}^*) \in H^R_{-i}\), as then \(s_i^*(h^*, x_{-i}^*) = a\). If \(h_i = h^* \in H^P_i\), then \(s_i^*(h^*) = x_i^*\) and \(s_{-i}(h^*, x_{-i}^*) = a\). As \((s_1^*, t_1^*, s_2^*, t_2^*) \in C\), it follows that \(\beta_{i,h_i}(t_i^*)\) assigns probability 1 to
\[
Z_{-i}[s_1, s_2^*] = \{r_{-i} \in S_{-i}(h^*) : r_{-i}(h^*, x_i^*) = a\} \times T_{-i}.
\]
Thus, \(E_{pi}[s_i^*|t_i^*, h_i] = \delta^{n-1} x_i^*\).
Now, use the fact that \( t_i^* \) strongly believes \( R_{-i}^1 \) and \( R_{-i}^1 \cap (S_{-i}(h_i) \times T_{-i}) \neq \emptyset \) (since \( (s^*_{-i}, t^*_{-i}) \in R_{-i}^1 \cap (S_{-i}(h_i) \times T_{-i}) \)). It follows from Lemma B.5 that, for each \( x_i \in [0, 1) \), \( t_i^* \) can secure \( \delta^{n-1} x_i \) at \( h_i \). Since \( (s^*_i, t^*_i) \in R_{i}^1 \), \( \mathbb{E}_{\pi}(s^*_i|t^*_i, h_i) = \delta^{n-1} x^*_i \geq \delta^{n-1} x_i \) for each \( x_i \in [0, 1) \). This implies that \( \delta^{n-1} x^*_i \geq \delta^{n-1} \) or \( x^*_i \geq \delta^{n-n} \).

**Proof of Theorem 6.1.** Immediate from Lemmata B.4-B.6. ■

### B.2 Proof of Theorem 6.2

Set \( C^1 = C \) and inductively define \( C^{m+1} = C^m \cap (SB_1(\text{proj}_{S_2 \times T_2} C^m) \times SB_2(\text{proj}_{S_1 \times T_1} C^m)) \).

The set of states at which there is on-path strategic certainty and common strong belief of on-path strategic certainty is \( C^\infty = \bigcap_m C^m \).

Throughout the exposition, we fix some finite time period \( n \) with \( n \leq N \) and some \( x^* \in [x^n, x^n] \). We begin by constructing particular strategies \( s^*_1 \) and \( s^*_2 \), so that \( (s^*_1, s^*_2) \) induces the outcome \( (x^*_1, x^*_2, n) = (x^*, 1-x^*, n) \). To do so, it will be convenient to fix a particular history \( h^* \in H^P \cup H^R \). If \( n = 1 \), \( h^* = \phi \). If \( n \geq 2 \), \( h^* = (1, r, \ldots, 1, r) \)—i.e., there are \( (n-1) \) offers of 1 followed by \( (n-1) \) rejections.

The strategy \( s^*_i \) satisfies the following properties. For any history \( h \in H^P_i \), set

\[
s^*_i(h) = \begin{cases} 
  x^*_i & \text{if } h = h^* \\
  1 & \text{if } h \neq h^*
\end{cases}
\]

For each \( h \in H^R_i \), set \( s^*_i(h, x_{-i}) = a \) if and only if one of the following holds: (i) \( x_{-i} \in [0, 1-\delta) \), (ii) \( h = h^* \) and \( x_{-i} = x^*_{-i} \), or (iii) \( h \) is an \( N \)-period history and \( x_{-i} \in [0, 1) \).

Note that the strategy profile \( (s^*_1, s^*_2) \) induces each bargainer to propose 1, reject, propose 1, reject, etc., up until the \( n^{th} \)-bargaining phase. In the \( n^{th} \)-bargaining phase, the proposer makes an offer that is accepted. This offer is \( x^*_1 = x^* \) if \( B_1 \) is the proposer and \( x^*_2 = 1-x^* \) if \( B_2 \) is the proposer. As such, in the \( n^{th} \)-bargaining phase, the bargainers agree on \((x^*, 1-x^*)\).

The construction of the type structure will be analogous to Example 6.1. \( B_i \) begins the game with a hypothesis that \( B(-i) \) plays the strategy \( s^*_{-i} \). If \( B_i \) observes a deviation from this behavior, she updates her belief and subsequently expects that \( B(-i) \) will act in an accommodating manner. Thus, it will be useful to have the concept of the accommodating strategy. The **accommodating strategy** for bargainer \( i \), written \( \alpha^i_t \), is a strategy such that \( \alpha^i_t(h) = 0 \) for all \( h \in H_i^P \) and \( \alpha^i_t(h, x_{-i}) = a \) for all \( (h, x_{-i}) \in H_i^R \). The **h-accommodating strategy** for bargainer \( i \), written \( \alpha^{i,h}_t \) is a strategy that allows \( h \) but otherwise agrees with the accommodating strategy.

Now, construct \( T = (B; T_1, T_2; S_1, S_2; \beta_1, \beta_2) \) so that, for each \( B_i, T_i = \{t_i^*\} \) and

- \( \beta^i_{t, h}(s^*_{-i}, t^*_{-i}) = 1 \), if \( s^*_{-i} \in S_{-i}(h) \), and

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• \( \beta_i(t_i^*)(\alpha^h_{-i}, t^*_{-i}) = 1 \), if \( s^*_{-i} \notin S_{-i}(h) \).

Notice that \( \beta_i(t_i^*) \) is a conditional probability system. This is immediate from the following fact: If \( S_{-i}(h') \times T_{-i} \subseteq S_{-i}(h) \times T_{-i} \) and \( \alpha^h_{-i} \in S_{-i}(h') \), then the \( h' \)-accommodating strategy \( \alpha^h_{-i} \) is the \( h \)-accommodating strategy \( \alpha^h_{-i} \). With this, \( T \) satisfies Definition 4.2.

Theorem 6.2 follows immediately from the following Lemma:

**Lemma B.7.** \( (s^*_1, t^*_1, s^*_2, t^*_2) \in R^\infty \subseteq C^\infty \).

The key to showing Lemma B.7 is establishing the following lemmata:

**Lemma B.8.** If \( (r, t^*_i) \in R^1_i \), then \( \zeta(r_i, s^*_{-i}) = \zeta(s^*_i, s^*_{-i}) \).

**Lemma B.9.** Each \( R^1_i \) is closed.

**Proof of Lemma B.7.** By Lemma B.8, for each \( i = 1, 2 \)

\[
\{(s^*_i, t^*_i)\} \subseteq R^1_i \subseteq Z_i[s^*_1, s^*_2].
\]  

We will show that, for each \( m \geq 2 \),

\[
\{(s^*_i, t^*_i)\} \subseteq R^m_i = R^{m-1}_i \subseteq Z_i[s^*_1, s^*_2] = \text{proj}_{S_{i} \times T_{i}} C^{m-1}.
\]

Observe that both \( R^1_{-i} \) and \( Z_{-i}[s^*_1, s^*_2] \) are Borel (Lemmata B.9 and A.1-(iii)). Moreover, by construction, \( t^*_i \) strongly believes \( \{(s^*_{-i}, t^*_{-i})\} \). So, by Equation (1), for each \( h \) with \( s^*_{-i} \in S_{-i}(h), \beta_i(h(t^*_i)(R^1_{-i}) = \beta_i(h(t^*_i)(Z_{-i}[s^*_1, s^*_2]) = 1 \). Observe that, if \( Z_{-i}[s^*_1, s^*_2] \cap (S_{-i}(h) \times T_{-i}) \neq \emptyset \), then \( s^*_{-i} \in S_{-i}(h) \) and, so, \( R^1_{-i} \cap (S_{-i}(h) \times T_{-i}) \neq \emptyset \). From this, \( t^*_i \) strongly believes both \( R^1_{-i} \) and \( Z_i[s^*_1, s^*_2] \). As such,

\[
\{(s^*_i, t^*_i)\} \subseteq R^2_i = R^1_i \subseteq Z_i[s^*_1, s^*_2] = \text{proj}_{S_{i} \times T_{i}} C^1.
\]

Repeating the argument for each \( m \) establishes the result inductively. \( \blacksquare \)

**Remark B.3.** In light of Lemma B.7, we could alternatively show that \( (s^*_1, t^*_1, s^*_2, t^*_2) \) satisfies the equilibrium dominance criterion in Battigalli and Siniscalchi (2002, Section 6.1).

**Lemma B.10.** For each \( r_i \in S_i \),

(i) \( \pi_i(s^*_i, s^*_{-i}) \geq \pi_i(r_i, s^*_{-i}) \), and

(ii) \( \pi_i(s^*_i, s^*_{-i}) = \pi_i(r_i, s^*_{-i}) \) if and only if \( \zeta(r_i, s^*_{-i}) = \zeta(s^*_i, s^*_{-i}) \).

**Proof.** Observe that \( \pi_i(s^*_i, s^*_{-i}) = \delta^{n-1} x^*_i \). Using the fact that \( x^*_i \in [x^n, \bar{x}^n] \),

\[
\pi_i(s^*_i, s^*_{-i}) = \begin{cases} 1 - \delta & \text{if } Bi=B1 \\ \delta(1 - \delta) & \text{if } Bi=B2 \end{cases}
\]
and, if $B_i$ is the proposer in period $N$, $\pi_i(s_i^*, s_{-i}^*) \geq \delta^{N-1}$. We use these facts below.

Suppose $\zeta(r_i, s_{-i}^*) \neq \zeta(s_i^*, s_{-i}^*)$. Then, there is some $k$-period history $h \in H_k$ so that $s_i^*, r_i \in S_i(h), s_{-i}^* \in S_{-i}(h)$, but $s_i^*(h) \neq r_i(h)$. (Note, $k \leq n$.) We will show that $\pi_i(s_i^*, s_{-i}^*) > \pi_i(r_i, s_{-i}^*), from which the result follows.

First, let $h \in H_i^P$ with $r_i(h) \in [0, 1 - \delta)$. Then, $s_{-i}^*(h, r_i(h)) = a$ and, so, $\pi_i(r_i, s_{-i}^*) = \delta^{k-1} r_i(h) < \delta^{k-1}(1 - \delta)$. Note, $\pi_i(s_i^*, s_{-i}^*) \geq \delta^{k-1}(1 - \delta)$, since $k \geq 2$ when $B_i = B2$. Thus, $\pi_i(s_i^*, s_{-i}^*) > \pi_i(r_i, s_{-i}^*).

Second, fix $h$ so that either (a) $h \in H_i^P$ with $r_i(h) \in [1 - \delta, 1]$ or (b) $h = (h^*, x_{-i}^*) \in H_i^R$. In (a) $s_{-i}^*(h, r_i(h)) = r$ and in (b) $r_i(h) = r$. So, in either case,

$$\pi_i(r_i, s_{-i}^*) < \begin{cases} \max\{\delta^{N-1}, \delta^{k+1}(1 - \delta)\} & \text{if } B_i \text{ is the proposer in } N \\ \delta^{k+1}(1 - \delta) & \text{if } B_i \text{ is the responder in } N. \end{cases}$$

(One possibility is that $(r_i, s_{-i}^*)$ results in agreement on some offer made by $B_i$ in period $\ell < N$; this can happen only if $x_i \in [0, 1 - \delta)$. A second possibility is that $(r_i, s_{-i}^*)$ results in agreement on some offer made by $B_i$ in period $N$; this can happen only if $x_i \in [0, 1]$. The final possibility is that $(r_i, s_{-i}^*)$ results in $x_i = 0$.) Thus, $\pi_i(s_i^*, s_{-i}^*) > \pi_i(r_i, s_{-i}^*)$.

Finally, let $h = (\cdot, x) \in H_i^R$ with $h \neq (h^*, x_{-i}^*)$. Then, $h = (\cdot, 1), s_i^*(h) = r \neq a = r_i(h)$. It follows that $\pi_i(s_i^*, s_{-i}^*) > 0 = \pi_i(r_i, s_{-i}^*)$. ■

**Lemma B.11.** Fix an $n$-period history $h \in H_i$ with $s_i^* \in S_i(h)$ but $s_{-i}^* \notin S_{-i}(h)$. For each $r_i \in S_i(h),\n
(i) \pi_i(s_i^*, \alpha_{-i}^h) \geq \pi_i(r_i, \alpha_{-i}^h), \text{ and} \\
(ii) \pi_i(s_i^*, \alpha_{-i}^h) = \pi_i(r_i, \alpha_{-i}^h) \text{ if and only if either} \\
\bullet \zeta(r_i, \alpha_{-i}^h) = \zeta(s_i^*, \alpha_{-i}^h), \\
\bullet n < N, h = (\cdot, 1 - \delta) \in H_i^R, \text{ and } r_i(h) = a, \text{ or} \\
\bullet n = N, h = (\cdot, 1) \in H_i^R, \text{ and } r_i(h) = a.$

**Proof.** Fix an $n$-period history $h \in H_i$ with $s_i^*, r_i \in S_i(h)$ but $s_{-i}^* \notin S_{-i}(h)$. Certainly, if $\zeta(r_i, \alpha_{-i}^h) = \zeta(s_i^*, \alpha_{-i}^h)$, then $\pi_i(s_i^*, \alpha_{-i}^h) = \pi_i(r_i, \alpha_{-i}^h)$. So, suppose that $\zeta(r_i, \alpha_{-i}^h) \neq \zeta(s_i^*, \alpha_{-i}^h)$. Then, there exists $h' \in H_i$ that (weakly) follows $h$, so that $(s_i^*, \alpha_{-i}^h), (r_i, \alpha_{-i}^h) \in S(h')$, but $s_i^*(h') \neq r_i(h')$. There are four cases.

First, suppose that $h \in H_i^P$. Then, for each $x_i, \alpha_{-i}^h(h, x_i) = a$. As such, $h' = h$ and $r_i(h) < 1$. So, $\pi_i(s_i^*, \alpha_{-i}^h) = \delta^{n-1} > \delta^{n-1} r_i(h) = \pi_i(r_i, \alpha_{-i}^h).$

Second, suppose that $n < N, h = (\cdot, x_{-i}) \in H_i^R$ and $x_{-i} \in [0, 1 - \delta)$. Then, $s_i^*(h) = a$ and, so, $h' = h$. Thus, $r_i(h) = r$ and, so, $\pi_i(s_i^*, \alpha_{-i}^h) = \delta^{n-1}(1 - x_{-i}) > \delta^n = \pi_i(r_i, \alpha_{-i}^h).

Third, suppose that $n < N, h = (\cdot, x_{-i}) \in H_i^R$ and $x_{-i} \in [1 - \delta, 1]$. Then, $s_i^*(h) = r$. If $h' \neq h$, then $h' = (h, r) \in H_i^P$. (This follows from the fact that $\alpha_{-i}^h(h, r, \cdot) = a.$) Thus, the
analysis in the first case gives that \(\pi_i(s_i^*, \alpha_i^h) > \pi_i(r_i, \alpha_i^h)\). So, suppose that \(h' = h\). Then, \(r_i(h) = a\) and, so, \(\pi_i(s_i^*, \alpha_i^h) = \delta^0a\) and \(\pi_i(r_i, \alpha_i^h) = \delta^{n-1}(1 - x_{-i})\). If \(x_{-i} \in (1 - \delta, 1]\), then \(\pi_i(s_i^*, \alpha_i^h) > \pi_i(r_i, \alpha_i^h)\). If \(x_{-i} = 1 - \delta\), then \(\pi_i(s_i^*, \alpha_i^h) = \pi_i(r_i, \alpha_i^h)\). If \(x_{-i} = 1\), \(\pi_i(s_i^*, \alpha_i^h) = 0 = \pi_i(r_i, \alpha_i^h)\). ■

Lemma B.12. \(R^1_i\) is the set of \((r_i, t_i^*) \in S_i \times T_i\) that satisfy the following criterion: If there exists some \(h \in H_i\) with \(s_i^*, r_i \in S_i(h)\) and \(r_i(h) \neq s_i^*(h)\), then \(s_i^* \not\in S_{-i}(h)\) and either

\[(i) n < N, h = (\cdot, 1 - \delta) \in H^R_i, \text{ and } r_i(h) = a, \text{ or}
(ii) n = N, h = (\cdot, 1) \in H^R_i, \text{ and } r_i(h) = a.
\]

Write \([s_i]\) for the set of strategies \(r_i\) that induce the same plan of action as \(s_i\). Note \(r_i \in [s_i^*]\) if and only if \(r_i(h) = s_i^*(h)\), for each \(h \in H_i\) with \(s_i^*, r_i \in S_i(h)\).

Proof of Lemma B.12. By Lemmata B.10-B.11, \([s_i^*] \times \{t_i^*\} \subseteq R^1_i\). Suppose that \((r_i, t_i^*) \in R^1_i\), but \(r_i \not\in [s_i^*]\). Then, there exists some \(h \in H_i\) so that \(s_i^*, r_i \in S_i(h)\) and \(r_i(h) \neq s_i^*(h)\). Since \((s_i^*, t_i^*), (r_i, t_i^*) \in R^1_i\), \(\pi_i[s_i^*|t_i^*, h] = \pi_i[r_i|t_i^*, h]\). So, Applying Lemmata B.10-B.11, \(s_i^* \not\in S_{-i}(h)\) and either (i) or (ii) holds. The converse follows from Lemmata B.10-B.11. ■


Proof of Lemma B.9. It suffices to show the following: If \((\tilde{s}_i, t_i^*) \not\in R^1_i\), then \((\tilde{s}_i, t_i^*)\) is not a limit point of \(R^1_i\)—i.e., there exists \(V_i \subseteq S_i \times T_i\) open so that \((\tilde{s}_i, t_i^*) \not\in V_i\), but \((V_i \setminus \{(\tilde{s}_i, t_i^*)\}) \cap R^1_i = \emptyset\). (See Aliprantis and Border, 2007, Lemma 2.8.) Let \(H_i\) be the set of all \(h \in H_i\) so that (i) \(h\) is not an \(n\)-period history with \(h = (\cdot, 1 - \delta)\) and \(n \leq N - 1\), and (ii) \(h\) is not an \(N\)-period history with \(h = (\cdot, 1)\). Since \((\tilde{s}_i, t_i^*) \not\in R^1_i\), we can find some \(\tilde{h} \in H_i\) with \(s_i, t_i^* \in S_i(\tilde{h})\) but \(s_i(\tilde{h}) \neq s_i^*(\tilde{h})\). (See Lemma B.12.) Let \(U_{\tilde{h}}\) be open in \(C_{\tilde{h}}\) with \(\tilde{s}_i(\tilde{h}) \in U_{\tilde{h}}\) but \(s_i^*(\tilde{h}) \not\in U_{\tilde{h}}\). Moreover, let \(\hat{H}_i\) be the set of histories \(h\) on the path from the root to \(\tilde{h}\) (not including \(\tilde{h}\)) so that \(C_h = \{a, r\}\). For each \(h \in \hat{H}_i\), set \(U_h = \{s_i^*(h)\} = \{\hat{s}_i(h)\}\) and define

\[V_i = \left( \bigcap_{h \in \hat{H}_i \cup \{\tilde{h}\}} (\text{proj}_h)^{-1}(U_h) \right) \times \{t_i^*\}.\]

Note, \(V_i\) is a finite intersection of open sets and, so, open. By construction, \((\hat{s}_i, t_i^*) \in V_i\). Fix \((s_i, t_i) \in V_i\) and we will argue that \((s_i, t_i) \not\in R^1_i\): If \(s_i \in S_i(\tilde{h})\), then Lemma B.12 gives that \((s_i, t_i) \not\in R^1_i\). If \(s_i \not\in S_i(\tilde{h})\), then there exists some \(h\) on the path from the root to \(\tilde{h}\) so that \(s_i \not\in S_i(h)\) but \(s_i^*(h) \neq s_i(h)\). By construction, \(h \not\in \hat{H}_i\) or, equivalently, \(h \in H^P_i\). Thus, Lemma B.12 gives that \((s_i, t_i) \not\in R^1_i\). ■
Remark B.4. The construction displays the following no-indifference property: Fix a history \( h \in H_i \) so that \( R_0^\infty \neq \emptyset \). If \( r_i \in S_i(h) \) and \( r_i \) is a best response at \( h \) under \( \beta_i, h(t_i^h) \), then \( r_i(h) = s_i^*(h) \). So, along the path of play, no bargainer is indifferent between any two actions. Lemmata B.10-B.11 point to a stronger no-indifference property: If \( t_i^* \) does not have a uniquely optimal action at some \( h \in H_i \) allowed by \( s_i^* \), then \( R_0^\infty \cap S_i(h) = \emptyset \). Bi is the receiver, and either (a) Bi received an offer of \( 1 - \delta \) in a non-deadline period or (b) Bi received an offer of 1 in a deadline period. Thus, at \( (s_i^*, t_1^*, s_2^*, t_2^*) \), no Bi is uncertain about how B(\(-i\)) breaks indifferences.

B.3 Implications for Behavior

For any given \((\delta, N)\), \([x^1, \bar{x}^1] \neq \emptyset \) and \([x^{n+1}, \bar{x}^{n+1}] \subseteq [x^n, \bar{x}^n] \). So, there is some \( \pi(\delta, N) \) with \([x^n, \bar{x}^n] \neq \emptyset \) if and only if \( n \leq \pi(\delta, N) \). Observe \( N > \pi(\delta, N) \geq 1 \). (Note, \([x^1, \bar{x}^1] \neq \emptyset \) and, when \( N < \infty \), \([x^N, \bar{x}^N] = \emptyset \).) In fact, when \( \infty > N \geq 4 \), \( \pi(\delta, N) \leq N - 2 \). (Note, when \( \infty > N \geq 4 \), \([x^{N-1}, \bar{x}^{N-1}] = \emptyset \).) Finally, Online Appendix G shows:

Proposition B.1. Fix \( n \) with \( N - 2 \geq n \geq 2 \). There exists \( \delta[N, n] \in (\frac{1}{2}, 1) \) so that \([x^n, \bar{x}^n] \neq \emptyset \) if and only if \( \delta \geq \delta[N, n] \).

Appendix C Proofs for Section 7

The key behind Proposition 7.1 is the following observation:

Observation C.1. Fix a strategy profile \((s_1^*, s_2^*)\) that induces the outcome \((x_1^*, x_2^*, n)\). Suppose, further, that \((s_2^*, t_2^*)\) is rational and strongly believes \( Z_1[s_1^*, s_2^*] \). If \((y_1, y_2, 1)\) Pareto dominates \((x_1^*, x_2^*, n)\), then \((s_2^*, t_2^*)\) is more optimistic at \((\phi, y_1)\) than at \((\phi, s_1^*(\phi))\). □

Fix a state \((s_1^*, t_1^*, s_2^*, t_2^*)\) that induces the outcome \((x_1^*, x_2^*, n)\), and suppose \((y_1, y_2, 1)\) Pareto dominates \((x_1^*, x_2^*, 2)\). Consider the history \( h \) at which B1 makes an initial offer of \((y_1, y_2)\). (This history is, of course, precluded by the state.) If \((s_2^*, t_2^*)\) is rational, type \( t_2^* \) must expect to get expected payoffs of at least \( y_2 \) by playing according to \( s_2^* \) (at \((\phi, y_1)\)). On the other hand, if \( t_2^* \) satisfies on-path strategic certainty, then, when B1 offers the initial allocation of \((s_1^*(\phi), 1 - s_1^*(\phi))\) (as she does in the given state), type \( t_2^* \) must get expected payoffs of \( \delta^{n-1}x_2^* \) by playing according to \( s_2^* \). Thus, type \( t_2^* \)'s expected payoffs at \((\phi, y_1)\) are strictly higher than \( t_2^* \)'s expected payoffs at \((\phi, s_1^*(\phi))\).

Proof of Proposition 7.1. Take \( E_2 = R_2^1 \cap \text{SB}_2(Z_1[s_1^*, s_2^*]) \). By assumption, \( \beta_{1, \phi}(t_1^*)|E_2) = 1 \). Suppose \([x_1^*, x_2^*, n] \neq \emptyset \). Fix \((r_2, u_2) \in E_2 \). Since \((r_2, u_2) \in Z_2[s_1^*, s_2^*] \), \( \zeta(s_1^*, r_2) = \zeta(s_1^*, s_2^*) \). Since \((r_2, u_2) \in R_2^1 \cap \text{SB}_2(Z_1[s_1^*, s_2^*]) \), Observation C.1 gives: For each \((y_1, y_2, 1)\)
that Pareto dominates $\xi(\zeta(s_1^*, r_2)) = \xi(\zeta(s_1^*, s_2^*))$, $(r_2, u_2)$ is more optimistic at $(\phi, y_1)$ than at $(\phi, s_1^*(\phi))$. ■