Online Appendix: The Context of the Game

Amanda Friedenberg* Martin Meier†

May 19, 2015

1 Online Appendix

This section illustrates how the results and arguments of Hellman (2014) apply to the setting of our main paper. There are two issues. First, our notion of a type structure is different from that in Hellman (2014). Second, we modify Hellman's (2014) example, so that it satisfies the injectivity condition. Notation is as in the main text, unless otherwise stated.

Background Write $X = \{-1, 1\}^{\mathbb{N}_0}$ for the Cantor space and $x = (x_0, x_1, \ldots)$ for an element of X. Endow X with the Borel sigma-algebra generated by the set of finite cylinders, written $\mathcal{B}(X)$. Take $\nu \in \Delta(X)$ to be the measure obtained taking the product measure based on a $\frac{1}{2} : \frac{1}{2}$ measure on $-1 : 1$. We can extend ν to $\nu^* : \mathcal{B}(X; \nu) \to [0, 1]$ so that $\nu^*(E) = \nu(E)$ for all Borel $E \subseteq X$.

Take $I = \{1, 2\}$ and, for each $i \in I$, let $\Omega_i = \{i\} \times X$. Set $\Omega = \Omega_1 \cup \Omega_2 = I \times X$. Take $\mu \in \Delta(\Omega)$ with $\mu = \upsilon \times \nu$ and $\upsilon \in \Delta(I)$ and $\upsilon(1) = \upsilon(2) = \frac{1}{2}$. Again, we can extend μ to $\mu^* : \mathcal{B}(\Omega; \mu) \to [0, 1]$ so that $\mu^*(E) = \mu(E)$ for all Borel E on Ω.

Define two operators: $\iota : \Omega \to \Omega$ and $S : \Omega \to \Omega$. For each $\omega = (i, x_0, x_1, \ldots)$, $\iota(\omega) = (i, -1x_0, x_1, \ldots)$ and $S(\omega) = (-i, x_1, x_2, \ldots)$. These are Borel measurable μ^*-preserving transformations.

Game of Incomplete Information The player set will consist of I. Each player $i \in I$ will have an action set $\{L_i, M_i\}$. Take $\Theta = \{-1, 1\}$.[1] Player i's payoff function π_i is described by Figure 1.1: It is assumed that $\kappa_i^1 > \kappa_i^3$, $\kappa_i^4 > \kappa_i^2$,$\lambda_i^3 > \lambda_i^1$, and $\lambda_i^2 > \lambda_i^4$, i.e., that player i prefers to coordinate with the other player if $\theta = 1$ and to miscoordinate with the other if $\theta = -1$. Moreover, it is further assumed that

$$\frac{\kappa_i^4 - \kappa_i^2}{(\kappa_i^1 - \kappa_i^3) + (\kappa_i^4 - \kappa_i^2)} \neq \frac{\lambda_i^4 - \lambda_i^2}{(\lambda_i^1 - \lambda_i^3) + (\lambda_i^4 - \lambda_i^2)}. \tag{1}$$

*Department of Economics, W.P. Carey School of Business, Arizona State University, amanda.friedenberg@asu.edu, http://www.public.asu.edu/ afrieden.

†Instituto de Análisis Económico - CSIC, Barcelona, and Institut für Höhere Studien, Vienna, martin.meier@iae.csic.es

[1] This corresponds to $\{\underline{\theta}, \overline{\theta}\}$ in the main text.

That is, the 'cut-point,' i.e., the probability that she assigns to L_{-i} that makes player i indifferent between her actions, is different in $\theta = 1$ vs. $\theta = -1$.

1	L_{-i}	M_{-i}
L_i	κ_i^1	κ_i^2
M_i	κ_i^3	κ_i^4

-1	L_{-i}	M_{-i}
L_i	λ_i^1	λ_i^2
M_i	λ_i^3	λ_i^4

Figure 1.1

Write $\mathbb{E}_p[\pi_i(\theta, c_i)] = p\pi_i(\theta, c_i, L_{-i}) + (1-p)\pi_i(\theta, c_i, M_{-i})$ for i's expected payoff when the parameter is θ, she chooses c_i, and assigns probability p to L_{-i}. Taken together, the assumptions on preferences are equivalent to the following: There exists $p[1], p[-1] \in (0,1)$ with $p[1] \neq p[-1]$ and:

- If $p > p[1]$, $\mathbb{E}_p[\pi_i(1, L_i)] > \mathbb{E}_p[\pi_i(1, M_i)]$.
- If $p < p[1]$, $\mathbb{E}_p[\pi_i(1, L_i)] < \mathbb{E}_p[\pi_i(1, M_i)]$.
- If $p > p[-1]$, $\mathbb{E}_p[\pi_i(-1, L_i)] < \mathbb{E}_p[\pi_i(-1, M_i)]$.
- If $p < p[-1]$, $\mathbb{E}_p[\pi_i(-1, L_i)] > \mathbb{E}_p[\pi_i(-1, M_i)]$.

Note, the payoffs in Hellman (2014) satisfy the above properties—these are the only properties his arguments make use of. The payoffs in the main text also satisfy the above conditions. In addition, the payoffs in the text satisfy the no ties condition required of simple games (which the above does not explicitly impose). More specifically,

It will be convenient to introduce a mapping $W : \Omega \to \Theta$ defined as the projection of the second coordinate of Ω onto Θ. That is, if $\omega = (i, x_0, x_1, \ldots)$, then $W(\omega) = x_0 \in \Theta$.

Bayesian Game The set of types of player i is $T_i = \{\{\omega\} \cup S^{-1}(\{\omega\}) : \omega \in \Omega_i\}$. Note, for each $t_i \in T_i$, there is a unique $\omega = (i, x_0, x_1, \ldots) \in \Omega$ so that

$$t_i = \{(i, x_0, x_1, \ldots), (-i, 1, x_0, x_1, \ldots), (-i, -1, x_0, x_1, \ldots)\}.$$

Thus, there is a bijective map $\tau_i : T_i \to \Omega_i$ defined so that $\tau_i(t_i) = (i, x_0, x_1, \ldots)$ if and only if $t_i = \{(i, x_0, x_1, \ldots), (-i, 1, x_0, x_1, \ldots), (-i, -1, x_0, x_1, \ldots)\}$.

We endow T_i with the smallest topology so that τ_i is continuous. So, for each open set $V \subseteq T_i$, there exists some open set $U \subseteq \Omega_i$ so that $(\tau_i)^{-1}(U) = V$. Using the fact that τ_i is bijective, for each open set $U \subseteq \Omega_i$, $\tau_i((\tau_i)^{-1}(U)) = U$. It follows that T_i is a compact metric space and τ_i is a closed mapping.

Define a belief map $\beta_i : T_i \to \Delta(\Theta \times T_{-i})$ as follows: Set $\beta_i(t_i)(\theta, t_{-i}) = 1$ if $W(\tau_i(t_i)) = \theta$ and $\tau_i(t_i) \in t_{-i}$. So, if $\tau_i(t_i) = (i, x_0, x_1, x_2, \ldots)$, then $\beta_i(t_i)$ assigns probability one to (x_0, t_{-i}^*) where $t_{-i}^* = \{(-i, x_1, x_2, \ldots), (i, x_0, x_1, x_2, \ldots), (i, -1x_0, x_1, x_2, \ldots)\}$.

Bayesian Equilibrium Throughout, we will fix a strategy profile (s_1, s_2) that is a Bayesian equilibrium. Note, $s_i : T_i \to \Delta(\{L_i, M_i\})$. We can derive a $\lambda_i : T_i \to [0,1]$ so that $\lambda_i(t_i) = s_i(t_i)(L_i)$. We will refer to this as the derived strategy of player i. We will then derive mappings $l_i : \Omega_i \to [0,1]$ so that $l_i(\omega) = \lambda_i((\tau_i)^{-1}(\omega))$. We call l_i the Ω_i-derived strategy of player i.

Theorem 1.1 (Hellman's Main Theorem)**.** *If (s_1, s_2) is a Bayesian Equilibrium, then, for some player $i \in I$, the Ω_i-derived strategy l_i is not μ-measurable.*

Note, μ induces measures on T_1 and T_2 via τ_1 and τ_2. Write $\hat{\mu}_i$ for the image measure of μ under τ_i. In the main text, we make use of a corollary of the Hellman result.

Corollary 1.1. *If (s_1, s_2) is a Bayesian Equilibrium, then, for some player $i \in I$, s_i is not $\hat{\mu}_i$-measurable.*

Proof. Note, there is a a measurable mapping $f : \Delta(\{L_i, M_i\}) \to [0,1]$ so that $s_i(t_i) = \sigma_i \in \Delta(\{L_i, M_i\})$ if and only if $f(\sigma_i)(L_i) = \lambda_i(t_i)$. So, if s_i is $\hat{\mu}_i$-measurable, then so is λ_i.

Thus, it suffices to show: If λ_i is $\hat{\mu}_i$-measurable, then l_i is μ-measurable. Fix some $E \subseteq \mathbb{R}$ Borel. Since λ_i is $\hat{\mu}_i$-measurable, there exists Borel sets $F, G \subseteq T_i$ with $F \subseteq (\lambda_i)^{-1}(E) \subseteq G$ with $\hat{\mu}_i(F) = \hat{\mu}_i(G)$. Note, τ_i maps measurable sets to measurable sets. So, $\tau_i(F) \subseteq \Omega_i$ and $\tau_i(G) \subseteq \Omega_i$ are Borel in Ω_i and so Borel in Ω. Thus, we have

$$\hat{\mu}_i(F) = \hat{\mu}_i((\tau_i)^{-1}(\tau_i(F))) = \mu(\tau_i(F))$$

and

$$\hat{\mu}_i(G) = \hat{\mu}_i((\tau_i)^{-1}(\tau_i(G))) = \mu(\tau_i(G)),$$

where the first equality uses injectivity of τ_i. So, notice, we have Borel sets $\tau_i(F), \tau_i(G) \subseteq \Omega$ with $\tau_i(F) \subseteq \tau_i((\lambda_i)^{-1}(E)) \subseteq \tau_i(G)$ and $\mu(\tau_i(F)) = \mu(\tau_i(G))$. Since $\tau_i((\lambda_i)^{-1}(E)) = (l_i)^{-1}(E)$, it follows that $(l_i)^{-1}(E) \in \mathcal{B}(\Omega; \mu)$, as required. ∎

Thus, we return to show the Hellman result. To do so, it will be convenient to introduce an Ω-derived strategy $l : \Omega \to [0,1]$ so that $l(\omega) = l_i(\omega)$ if $\omega \in \Omega_i$. It suffices to show that, if (s_1, s_2) is a Bayesian Equilibrium, then l is not μ-measurable.

Properties of a Bayesian Equilibrium Throughout this section, we fix a Bayesian equilibrium (s_1, s_2). Now we will derive two properties that (s_1, s_2) must satisfy. The properties will be stated in terms of the Ω-derived mapping l. In the next section, we apply a result due to Levy (2012), which shows that, if a mapping satisfies those properties, then it must not be measurable.

To get at the properties, note that, for each t_i there is a unique point (θ^*, t^*_{-i}) is the support of $\beta_i(t_i)$. A feature of the payoffs is:

- If $\theta^* = 1$ and $\lambda_{-i}(t^*_{-i}) = 1$ (resp. $\lambda_{-i}(t^*_{-i}) = 0$), then $\lambda_i(t_i) = 1$ (resp. $\lambda_i(t_i) = 0$).
- If $\theta^* = -1$ and $\lambda_{-i}(t^*_{-i}) = 1$ (resp. $\lambda_{-i}(t^*_{-i}) = 0$), then $\lambda_i(t_i) = 0$ (resp. $\lambda_i(t_i) = 1$).

This comes about from the following fact: If the parameter is $\theta^* = 1$, then type t_i has an incentive to match t^*_{-i} and, if the parameter is $\theta^* = -1$, then type t_i has an incentive to mis-match t^*_{-i}.

Let us phrase these facts in terms of the Ω-derived strategy l. Fix ω and suppose $\omega \in \Omega_i$, i.e., there is some t_i with $\tau_i(t_i) = \omega$. Then, $W(\omega) := \theta^*$ and $\tau_i(t_i) \in t^*_{-i}$ if and only if $\tau_{-i}(t^*_{-i}) = S(\omega)$. Thus, we can rephrase the facts as the following property of l:

Lemma 1.1.

- *If $\omega = (i, 1, \cdot) \in \Omega_i$ with $l(S(\omega)) = 1$ (resp. $l(S(\omega)) = 0$), then $l(\omega) = 1$ (resp. $l(\omega) = 0$).*
- *If $\omega = (i, -1, \cdot) \in \Omega_i$ with $l(S(\omega)) = 1$ (resp. $l(S(\omega)) = 0$), then $l(\omega) = 0$ (resp. $l(\omega) = 1$).*

We will need one more fact: Before stating the fact in terms of l, let's first state it in terms of (s_1, s_2). It will be convenient to write $B_i : T_i \to T_{-i}$ for the mapping that takes t_i and maps it to the t^*_{-i} that determines t_i's best response, i.e., $B_i(t_i) = t^*_{-i}$ where $\tau_i(t_i) \in t^*_{-i}$. Note, B_i is not injective. In particular, for each t^*_{-i}, there are two distinct types t_i and u_i with $B_i(t_i) = B_i(u_i) = t^*_{-i}$. For these t_i, u_i, $\tau_i(t_i) = (i, x_0, x_1, \ldots)$ if and only if $\tau_i(u_i) = (i, -1x_0, x_1, \ldots)$. Thus, while t_i, u_i assigns probability one to the same type t^*_{-i}, they assign probability one to distinct parameters θ, since $W(\tau_i(t_i)) \neq W(\tau_i(u_i))$.)

Note any equilibrium must satisfy:

If $B_i(t_i) = B_i(u_i)$ for $t_i \neq u_i$, then either $\lambda_i(t_i) \in \{0, 1\}$ or $\lambda_i(u_i) \in \{0, 1\}$.

Without loss of generality, take $W(\tau_i(t_i)) = 1$ and $W(\tau_i(u_i)) = -1$. Write $B_i(t_i) = B_i(u_i) = t^*_{-i}$ and consider the probability with which type t^*_{-i} plays L, viz. $\lambda_i(t^*_{-i})$.

- L_i (resp. M_i) is a unique best response for t_i if $\lambda_i(t^*_{-i}) > p[1]$ (resp. $\lambda_i(t^*_{-i}) < p[1]$).
- L_i (resp. M_i) is a unique best response for u_i if $\lambda_i(t^*_{-i}) < p[-1]$ (resp. $\lambda_i(t^*_{-i}) > p[-1]$).

(Recall the definition of $p[1]$ and $p[-1]$ from the definition of the payoff functions.) Since the payoff functions were chosen so that $p[1] \neq p[-1]$, it follows that, for any value of $\lambda_i(t^*_{-i})$, either t_i or u_i has a unique best response.

Now, let's convert the statement to a property of l. Fix some $\omega \in \Omega_i$ and write $t_i = (\tau_i)^{-1}(\omega)$. Then, $B_i(t_i)$ is a type t^*_{-i} with $\tau_{-i}(B_i(t_i)) = S(\omega)$. Thus, if $B_i(u_i) = B_i(t_i)$, then $\tau_i(u_i) = \omega'$ with $S(\omega') = S(\omega)$. In this case, either $\omega' = \omega$ or $\omega' = \iota(\omega)$.

Lemma 1.2. *Fix some $\omega \in \Omega_i$. Either $l(\omega) \in \{0, 1\}$ or $l(\iota(\omega)) \in \{0, 1\}$.*

Reduction to Levy's (2012) Result Note the output of Lemmas 1.1-1.2: We get a function $l : \Omega \to [0, 1]$ satisfying the following three properties:

P1 If $\omega = (i, 1, \cdot) \in \Omega_i$ with $l(S(\omega)) = 1$ (resp. $l(S(\omega)) = 0$), then $l(\omega) = 1$ (resp. $l(\omega) = 0$).

P2 If $\omega = (i, -1, \cdot) \in \Omega_i$ with $l(S(\omega)) = 1$ (resp. $l(S(\omega)) = 0$), then $l(\omega) = 0$ (resp. $l(\omega) = 1$).

P3 If $\omega \in \Omega_i$, either $l(\omega) \in \{0, 1\}$ or $l(\iota(\omega)) \in \{0, 1\}$.

The main result is Levy (2012) states:

Theorem 1.2 (Levy, 2012). *If $l : \Omega \to [0, 1]$ satisfies P1-P2-P3, then l is not μ-measurable.*

Proof of Theorem 1.1. Immediate from Lemmas 1.1-1.2 and Theorem 1.2. ∎

Overview of Levy, 2012 This section review's Levy's (2012) result: Theorem 1.2 is shown by introducing a function $F : \Omega \to \{-1, 1\}$ so that $F(\omega) = 1$ if and only if $l(\omega) = 1$. Also, define a counting function $N : X \times X \to 2^{\mathbb{N}_0}$ so that $N(x, x') = \{n \in \mathbb{N}_0 : x_n \neq x'_n\}$. The theorem follows from the following two lemmas.

Lemma 1.3. *Suppose $l : \Omega \to [0, 1]$ is a μ-measurable functions satisfying P1-P2-P3. For each $i \in I$, there exists some $Y_i \in \mathcal{B}(X; \nu)$ with $\nu^*(Y_i) = 1$ so that the following holds: For each $x \in Y_i$ and each $x' \in X$ with $|N(x, x')|$ finite,*

$$F(i, x) = (-1)^{|N(x,x')|} F(i, x').$$

Lemma 1.4. *Fix some $f : X \to \{-1, 1\}$ so that, there exists $Y \in \mathcal{B}(X; \nu)$ with $\nu^*(Y) = 1$ satisfying the following criteria: For each $x \in X$ and each $x' \in X$ with $|N(x, x')|$ finite,*

$$f(x) = (-1)^{|N(x,x')|} f(x').$$

Then f is not ν-measurable.

Proof of Theorem 1.2. Suppose l is μ-measurable. Then, by Lemma 1.3, for any given i, there is some ν-measurable $Y_i \subseteq X$ with $\nu^*(Y_i) = 1$ so that, for all $x \in Y_i$ and all $x' \in X$ with $|N(x, x')|$ finite, $F(i, x) = (-1)^{|N(x,x')|} F(i, x')$. By Lemma 1.4, $F(i, \cdot) : X \to \{-1, 1\}$ is not ν-measurable, i.e., there exists some $E_i \subseteq \{-1, 1\}$ so that $(F(i, \cdot))^{-1}(E_i) \subseteq X$ is not in $\mathcal{B}(X; \nu)$.

Now, define a mapping $g : [0, 1] \to \{-1, 1\}$ so that $g(k) = -1$ if and only if $k \neq 1$. This is a Borel measurable function and so $g^{-1}(E_i)$ is Borel measurable. Now, note that

$$l^{-1}(g^{-1}(E_i)) = [\{i\} \times (F(i, \cdot))^{-1}(g^{-1}(E_i))] \cup [\{-i\} \times (F(-i, \cdot))^{-1}(g^{-1}(E_i))].$$

Since l is μ-measurable, $l^{-1}(g^{-1}(E_i)) \in \mathcal{B}(\Omega; \mu)$. Using this plus the fact that $\Omega_{-i} \in \mathcal{B}(\Omega) \subseteq \mathcal{B}(\Omega; \mu)$, $l^{-1}(g^{-1}(E_i)) \backslash \Omega_{-i} \in \mathcal{B}(\Omega; \mu)$. But, by the display, $l^{-1}(g^{-1}(E_i)) \backslash \Omega_{-i} = \{i\} \times (F(i, \cdot))^{-1}(g^{-1}(E_i))]$. So, the fact $l^{-1}(g^{-1}(E_i)) \backslash \Omega_{-i} \in \mathcal{B}(\Omega; \mu)$ implies that $(F(i, \cdot))^{-1}(E_i) \in \mathcal{B}(X; \nu)$, contradicting the earlier lemma. ∎

A Useful Lemma Before coming to the proof of Lemmas 1.3-1.4, let us state a useful auxiliary Lemma.

Lemma 1.5. *Fix a ν-measurable set $E \subseteq X$ with $\nu^*(E) = 0$. Let*

$$F = \{x \in X : \text{ there exists } x' \in E \text{ with } N(x', x) < \infty\}.$$

Then F is ν-measurable with $\nu^*(F) = 0$.

To show this lemma, it will be useful to have the following: For each $n \in \mathbb{N}$, write $Z_n = \{-1, 1\}$. Given some $N \subseteq \mathbb{N}$, take $X_{-N} = \prod_{n \notin N} Z_n$. Given some $E \subseteq X$, write

$$E_n = \{z \in Z_n : \text{ there exists } x \in E \text{ with } x_n = z\}$$

and

$$E[N] = \prod_{n \in N} E_n \times X_{-N}.$$

Lemma 1.6. *Fix a ν-measurable set $E \subseteq X$. If $N \subseteq \mathbb{N}$ is finite, then $E[N]$ is ν-measurable.*

Proof. Since E is ν-measurable, there exists F, G Borel with $F \subseteq E \subseteq G$ and $\nu(F) = \nu(G)$. The sets $F[N]$ and $G[N]$ are also Borel and $F[N] \subseteq E[N] \subseteq G[N]$. It remains to show that $\nu(F[N]) = \nu(G[N])$.

Suppose $\nu(F[N]) \neq \nu(G[N])$, i.e., $\nu(F[N]) < \nu(G[N])$. Then $\nu^{|N|}(\prod_{n \in N} F_n) < \nu^{|N|}(\prod_{n \in N} G_n)$, where we write ν^n for the $|N|$-fold product of the measure that assigns $\frac{1}{2} : \frac{1}{2}$ to $-1 : 1$. Since each $x \in F$ is contained in G, it then follows that $\nu(F) < \nu(G)$ a contradiction. ∎

Proof of Lemma 1.5. Fix some $E \subseteq X$ with $\nu^*(E) = 0$. Consider the set $E[N]$. By Lemma 1.6, $E[N]$ is ν-measurable. Moreover, by construction of the measure ν, $\nu^*(E[N]) = 0$ if $\nu^*(E) = 0$. Thus,

$$0 = \sum_{\substack{N \subseteq \mathbb{N}: \\ |N| < \infty}} \nu^*(E[N]) \geq \nu^*(\bigcup_{\substack{N \subseteq \mathbb{N}: \\ |N| < \infty}} E[N]) = \nu^*(F),$$

as required. ∎

The next lemma follows from the earlier one.

Lemma 1.7. *Fix a μ-measurable set $E \subseteq \Omega$ with $\mu^*(E) = 0$. Let*

$$F = \{(i, x) \in \Omega : \text{ there exists } (i, x') \in E \text{ with } N(x', x) < \infty\} \subseteq E.$$

Then F is μ-measurable with $\mu^*(F) = 0$.

Proof of Lemma 1.3 To prove Lemma 1.3, we will need a number of auxiliary results. Begin by writing $M = \{\omega \in \Omega : l(\omega) \in (0, 1)\}$.

Lemma 1.8.

(i) $S(M) \subseteq M$.

(ii) If M is μ-measurable, $S(M)$ is μ-measurable.

Proof. Part (i) follows from Properties P1-P2: Suppose, $S(\omega) \notin M$. Then, by P1-P2, $\omega \notin M$.

For Part (ii): Recall that $S : \Omega \to \Omega$ is a measurable function from a Polish space to a Polish space, where each $S^{-1}(\{\omega\})$ is finite. So, by Purves's (1966) Theorem, S maps Borel sets to Borel sets. We will argue that, in fact, S maps $E \in \mathcal{B}(\Omega; \mu)$ to members of $\mathcal{B}(\Omega; \mu)$: If $E \in \mathcal{B}(\Omega; \mu)$ then there exists Borel F, G with $F \subseteq E \subseteq G$ and $\mu(G \backslash F) = 0$. Note, $S(F), S(G)$ are Borel with $S(F) \subseteq S(E) \subseteq S(G)$. Note, since S is a μ-preserving transformation, $\mu(S(G)\backslash S(F)) = \mu(S^{-1}(S(G)\backslash S(F)))$. By construction of ν and the definition of S, $\mu(S^{-1}(S(G)\backslash S(F))) \leq 2\mu(G\backslash F) = 0$ and so $\mu(S(G)\backslash S(F)) = 0$, as required. ■

Lemma 1.9.

(i) $M \cap \iota(M) = \emptyset$.

(ii) For each $E \subseteq \Omega$, $S^{-1}(S(E)) = E \cup \iota(E)$.

Proof. Part (i) follows from Property P3: Fix $\omega \in M$ and note that some $\omega' \in S^{-1}(\{\omega\})$ must be contained in $\Omega\backslash M$. So, if $\omega \in M$, $\iota(\omega) \in \Omega\backslash M$.

For Part (ii), fix $\omega \in S^{-1}(S(E))$, i.e., there exists $\omega' \in S(E)$ so that $S(\omega) = \omega'$. Thus, write $\omega = (i, x_0, x_1, \ldots)$ and then $\omega' = (-i, x_1, x_2, \ldots)$. Since $\omega' \in S(E)$, there exists $\omega'' \in E$ with $S(\omega'') = \omega'$. Thus, either $\omega'' = \omega$ or $\omega'' = \iota(\omega)$. This establishes $S^{-1}(S(E)) \subseteq E \cup \iota(E)$.

For the converse, fix $\omega \in E \cup \iota(E)$ with $\omega = (i, x_0, x_1, \ldots)$. Then, $S(\omega) = (-i, x_1, \ldots)$. So, certainly $\omega \in S^{-1}(\{S(\omega)\})$, as required. ■

Lemma 1.10. *If M is μ-measurable, then $\mu^*(M) = 0$*

Proof. Note:

$$\mu^*(M) \geq \mu^*(S(M)) = \mu^*(S^{-1}(S(M))) = \mu^*(M) + \mu^*(\iota(M)) = 2\mu^*(M),$$

where the inequality is by Lemma 1.8(i)-(ii), the first equality is by the fact that S is a μ^*-preserving transformation, the second inequality is by Lemma 1.9(i)-(ii), and the last equality is by the fact that ι is a μ^*-preserving transformation. This establishes that $\mu^*(M) = 0$. ■

Define the set $\overline{M}$ so that

$$\overline{M} = M \cup \{(i, x') \in \Omega : \text{ there exists } (i, x) \in M \text{ with } |N(x, x')| < \infty\}.$$

Note, if $(i, x) \in \overline{M}$ with $|N(x, x')| < \infty$, then $(i, x') \in \overline{M}$ as well. The following is an immediate consequence of Lemma 1.10 and Lemma 1.7:

Corollary 1.2. *If M is μ-measurable, then $\mu^*(\overline{M}) = 0$*

Lemma 1.11. *Suppose M is μ-measurable.*

(i) $S^{-1}(\Omega\backslash\overline{M}) \in \mathcal{B}(\Omega;\mu)$ *with* $\mu^*(S^{-1}(\Omega\backslash\overline{M})) = 1$.

(ii) *For each* $(i,x) \in S^{-1}(\Omega\backslash\overline{M})$, $F(i,x) = x_0F(S(i,x))$.

Proof. Begin with Part (i): If M is μ-measurable, then, by Corollary 1.2, $\overline{M}$ is μ-measurable. Thus, $\Omega\backslash\overline{M} \in \mathcal{B}(\Omega;\mu)$ and, by Lemma 1.10, $\Omega\backslash\overline{M}$ is a set of μ^*-measure 1. Since S is a measure preserving transformation, $S^{-1}(\Omega\backslash\overline{M})$ is also a set of μ^*-measure 1.

Turn to Part (ii): Fix $(i,x) \in S^{-1}(\Omega\backslash\overline{M}) \subseteq S^{-1}(\Omega\backslash M)$. Then $S(i,x) \in \Omega\backslash\overline{M} \subseteq \Omega\backslash M$. So, either $l(S(i,x)) = 1$ or $l(S(i,x)) = 0$. We have:

- $F(S(i,x)) = 1$ if and only if $l(S(i,x)) = 1$,
- $F(S(i,x)) = -1$ if and only if $l(S(i,x)) = 0$,

First suppose that $F(S(i,x)) = 1$. If $x_0 = 1$, then, by P1, $F(i,x) = 1$ and, of course, in this case, $x_0F(S(i,x)) = 1$. If $x_0 = -1$, then, by P2, $F(i,x) = -1$ and, of course, in this case, $x_0F(S(i,x)) = -1$. Next suppose that $F(S(i,x)) = -1$. This implies that $l(S(i,x)) = 0$. Thus, if $x_0 = 1$, then, by P1, $F(i,x) = -1$ and, of course, in this case, $x_0F(S(i,x)) = -1$. Likewise, if $x_0 = -1$, then, by P2, $F(i,x) = 1$ and, of course, in this case, $x_0F(S(i,x)) = 1$. ■

Inductively define sets $Y^k \subseteq \Omega$: Set $Y^0 = S^{-1}(\Omega\backslash\overline{M})$ and set $Y^{k+1} = S^{-1}(Y^k)$.

Lemma 1.12. *Suppose M is μ-measurable.*

(i) *For each k, Y^k is μ-measurable, with* $\mu^*(Y^k) = 1$.

(ii) $\bigcap_{k=0}^{\infty} Y^k \in \mathcal{B}(\Omega;\mu)$ *with* $\mu^*(\bigcap_{k=0}^{\infty} Y^k) = 1$.

(iii) *For each K, if* $(i,x) \in \bigcap_{k=0}^{K} Y^k$, *then* $F(i,x) = x_0 \cdots x_K F(S^{K+1}(i,x))$.

Proof. Begin with Part (i). The proof is by induction on k. For $k = 0$, this is immediate from Lemma 1.11. Assume the lemma is true for k. Since Y^k is μ-measurable, there exists $E, F \in \mathcal{B}(\Omega)$ with $E \subseteq Y^k \subseteq F$ and $\mu(E) = \mu(F)$. Since S is a Borel measure preserving mapping, $S^{-1}(E), S^{-1}(F) \in \mathcal{B}(\Omega)$ with $S^{-1}(E) \subseteq S^{-1}(Y^k) \subseteq S^{-1}(F)$ and $\mu(S^{-1}(E)) = \mu(S^{-1}(F))$. Thus, $Y^{k+1} = S^{-1}(Y^k) \in \mathcal{B}(\Omega;\mu)$. Moreover, since S is μ^*-preserving and $\mu^*(Y^k) = 1$, $\mu^*(Y^{k+1}) = 1$.

Part (ii) is an immediate consequence of Part (i). Turn then to Part (iii). The proof is by induction on K. The case of $K = 0$ follows immediate from Lemma 1.11. Assume the lemma is true for K and we will show it also holds for $K+1$.

Fix $(i,x) \in \bigcap_{k=0}^{K+1} Y^k$. Then, $(i,x) \in Y^{K+1}$ and so $S^{K+1}(i,x) \in Y^0$. Applying Lemma 1.11(ii),

$$F(S^{K+1}(i,x)) = [S^{K+1}(i,x)]_0 F(S^{K+2}(i,x)) = x_{K+1}F(S^{K+2}(i,x)).$$

But, $(i,x) \in \bigcap_{k=0}^{K} Y^k$ and the induction hypothesis also give that, $F(i,x) = x_0 \cdots x_K F(S^{K+1}(i,x))$, i.e.,

$$F(S^{K+1}(i,x)) = \frac{F(i,x)}{x_0 \cdots x_K}.$$

Putting these two facts together gives

$$\frac{F(i,x)}{x_0 \cdots x_K} = x_{K+1} F(S^{K+2}(i,x)),$$

establishing that

$$F(i,x) = x_0 \cdots x_K x_{K+1} F(S^{K+2}(i,x)),$$

as desired. ■

Lemma 1.13. *If $(i,x) \in \bigcap_{k=0}^{\infty} Y^k$ and $|N(x,x')| < \infty$, then $(i,x') \in \bigcap_{k}^{\infty} Y^k$.*

Proof. We will show that for each K and each $(i,x) \in \cap_{k=0}^{K} Y^k$, $(i,x') \in \cap_{k=0}^{K} Y^k$ provided $|N(x,x')| < \infty$. The proof is by induction on K.

Begin with $K = 0$. Fix $(i,x) \in Y^0$ and some x' with $|N(x,x')| < \infty$. Write $S(i,x) = (-i,y)$ and $S(i,x') = (-i,y')$ and note that $|N(y,y')| < \infty$. If $(i,x') \notin Y^0 = S^{-1}(\Omega \backslash \overline{M})$, then $(i,y') \in \overline{M}$. From this and the fact that $|N(y,y')| < \infty$, it follows that $S(i,x) = (i,y) \in \overline{M}$, i.e., $(i,x) \notin S^{-1}(\Omega \backslash \overline{M}) = Y^0$.

Assume the claim holds for K. Fix some $(i,x) \in \cap_{k=0}^{K+1} Y^k$ and some x' with $|N(x,x')| < \infty$. Note that $S(i,x) = (-i,y) \in \bigcap_{k=0}^{K} Y^k$, since $(i,x) \in \bigcap_{k=0}^{K+1} Y^k$. Since $S(i,x') = (i,y')$ has $|N(y,y')| < \infty$, it follows from the induction hypothesis $(i,y') \in \bigcap_{k=0}^{K} Y^k$. From this, $(i,x') \in \bigcap_{k=0}^{K} Y^{k+1}$, as required. ■

Lemma 1.14. *Suppose M is μ-measurable. Then, for each i, there exists some $Y_i \in \mathcal{B}(X;\nu)$ with $\nu^*(Y_i) = 1$ so that, for all $x \in Y_i$ and all x' with $|N(x,x')| < \infty$,*

$$F(i,x) = (-1)^{|N(x,x')|} F(i,x').$$

Proof. Take Y_i to be the i-section of the set $\bigcap_{k=0}^{\infty} Y^k$, i.e., the set $\{x \in X : (i,x) \in \bigcap_{k=0}^{\infty} Y^k\}$. Since $\mathcal{B}(\Omega;\mu) = 2^{\{1,2\}} \times \mathcal{B}(X;\nu)$ is a product sigma-algebra that contains $\bigcap_{k=0}^{\infty} Y^k$, the i-section Y_i is also in $\mathcal{B}(\Omega;\mu) = 2^{\{1,2\}} \times \mathcal{B}(X;\nu)$. (See Aliprantis and Border (2007, Lemma 4.45).) Fix $x \in Y_i$ and some x' with $|N(x,x')| < \infty$; in particular, let $K = \max\{n : x_n \neq x'_n\}$. By Lemma 1.13, $x' \in Y_i$. So, by Lemma 1.12,

$$F(i,x) = x_0 \cdots x_K F(S^{K+1}(i,x))$$

and

$$F(i,x') = x'_0 \cdots x'_K F(S^{K+1}(i,x')).$$

Now notice that, in fact, $S^{K+1}(i,x) = S^{K+1}(i,x')$, so

$$\frac{F(i,x)}{F(i,x')} = \frac{x_0 \cdots x_K}{x'_0 \cdots x'_K}.$$

From this, $F(i,x) = (-1)^{|N(x,x')|} F(i,x')$, as desired. ■

Proof of Lemma 1.3. If l is μ-measurable, then M is μ measurable. Thus, the claim follows from Lemma 1.14. ■

Proof of Lemma 1.4 Suppose there exists some ν-measurable $Y \subseteq X$ with $\mu^*(Y) = 1$ so that, for all $x \in Y$,

$$f(x) = (-1)^{|N(x,x')|} f(x') \tag{2}$$

whenever $|N(x,x')| < \infty$. We will show that f cannot be ν-measurable.

It will be useful to introduce terminology: Say q is a finite permutation on $\mathbb{N}_0$ if $q : \mathbb{N}_0 \to \mathbb{N}_0$ is a bijective mapping where, there exists K so that $q(k) = k$ for all $k \geq K$. Say Q is a finite permutation on X if $Q : X \to X$ so that, there exists a finite permutation q on $\mathbb{N}_0$ where, for each $x = (x_0, x_1, \ldots) \in X$, $Q(x) = y = (y_0, y_1, \ldots)$ where $y_k = x_{q^{-1}(k)}$. If this condition holds, we will say that Q is derived from the permutation q. We will make use of the following property:

Property 1.1. *If Q is a finite permutation of X, then, for each $x \in X$, $|N(x, Q(x))|$ is even.*

Consider the set

$$Z = (X \backslash Y) \cup \{x \in X : |N(x,x')| < \infty \text{ for some } x' \in X \backslash Y\}.$$

There are several things to note. First, if $x \in Z$ and $|N(x,y)| < \infty$, then $y \in Z$. Second, $X \backslash Y$ is a set of ν^*-measure zero. So, by Lemma 1.5, Z is ν-measurable with $\nu^*(Z) = 0$. Define $\overline{Y} = X \backslash Z \subseteq Y$ and note that it is a ν-measurable set with $\nu^*(\overline{Y}) = 1$.

A set $A \subseteq X$ is called *symmetric* if, for each finite permutation of X, viz. Q, $Q(A) = A$.

Lemma 1.15. *The sets $f^{-1}(\{1\}) \cap \overline{Y}$ and $f^{-1}(\{-1\}) \cap \overline{Y}$ are symmetric.*

Proof. Fix a finite permutation Q of X and let $r \in \{1, -1\}$.

First we show that $Q(f^{-1}(\{r\}) \cap \overline{Y}) \subseteq f^{-1}(\{r\}) \cap \overline{Y}$. For this, fix $x \in Q(f^{-1}(\{r\}) \cap \overline{Y})$. So, there exists some $y \in f^{-1}(\{r\}) \cap \overline{Y}$ with $Q(y) = x$. By definition $f(y) = r$ and $y \in \overline{Y} \subseteq Y$. So, $f(y) = f(y')$ for all y' with $|N(y,y')|$ finite and even. Notice that $Q(y)$ has $|N(y, Q(y))|$ finite and even. It follows that $f(Q(y)) = r$, i.e., $x = Q(y) \in f^{-1}(\{r\})$. Moreover, $Q(y) \in \overline{Y}$: If not, then $Q(y) \in Z$ and so, using the fact that $|N(y, Q(y))|$ is finite, $y \in Z$, a contradiction. Thus, $x = Q(y) \in f^{-1}(\{r\}) \cap \overline{Y}$, as required.

Next we show that $f^{-1}(\{r\}) \cap \overline{Y} \subseteq Q(f^{-1}(\{r\}) \cap \overline{Y})$. For this, fix $x \in f^{-1}(\{r\}) \cap \overline{Y}$ and write $x = (x_0, x_1, \ldots)$. Construct $y = (y_0, y_1, \ldots)$ so that $y_k = x_{q(k)}$, where q is the permutation on $\mathbb{N}_0$ that induces Q. Thus, $Q(y) = x$. We will show that $y \in f^{-1}(\{r\}) \cap \overline{Y}$, from which it follows that $x \in Q(f^{-1}(\{r\}) \cap \overline{Y})$.

To see that $y \in f^{-1}(\{r\}) \cap \overline{Y}$: Since $f(x) = r$ and $x \in \overline{Y} \subseteq Y$, for any x' with $|N(x,x')|$ finite and even $f(x) = f(x')$. Since $|N(x,y)|$ finite and even $f(y) = r$, i.e., $y \in f^{-1}(\{r\})$. Moreover, $y \in \overline{Y}$: If not, then $y \in Z$ and so, using the fact that $|N(y, Q(y))| = |N(y,x)|$ is finite, $x \in Z$, a contradiction. ■

It will be convenient to set $Y^+ = f^{-1}(\{1\}) \cap \overline{Y}$ and $Y^- = f^{-1}(\{-1\}) \cap \overline{Y}$. Note, $Y^+ \cap Y^- = \emptyset$ and $Y^+ \cup Y^- = \overline{Y}$. So, if these sets are ν-measurable, $\nu^*(Y^+) + \nu^*(Y^-) = 1$.

Lemma 1.16. *If f is ν-measurable, then $\nu^*(Y^+), \nu^*(Y^-) \in \{0,1\}$.*

Proof. Since f is ν-measurable, the sets $Y^+ = f^{-1}(\{1\}) \cap \overline{Y}$ and $Y^- = f^{-1}(\{-1\}) \cap \overline{Y}$ are ν-measurable. Moreover, by Lemma 1.15, they are symmetric. So, by the Hewitt-Savage Theorem (see, e.g., Dudley, 2002, Theorem 8.4.6), $\nu^*(Y^+), \nu^*(Y^-) \in \{0,1\}$. ■

Proof of Lemma 1.4. Throughout, assume f is ν-measurable. Introduce a mapping $g : X \to X$ so that $g(x) = y$, where $x_0 \neq y_0$ but, for each $k \geq 1$, $x_k = y_k$. Note that, for each $x \in f^{-1}(\{r\}) \cap \overline{Y} \subseteq Y$, $f(g(x)) = -r$, since $f(x) = r$ and $f(x) = -1^1 f(g(x))$. Thus $g(Y^+) \cap \overline{Y} \subseteq Y^-$ and $g(Y^-) \cap \overline{Y} \subseteq Y^+$. It is straightforward that $g(Y^+)$ and $g(Y^-)$ are ν-measurable since Y^+ and Y^- are ν-measurable. Using the fact that $\nu^*(g(Y^+) \cap \overline{Y}) = \nu^*(g(Y^+))$ and $\nu^*(g(Y^-) \cap \overline{Y}) = \nu^*(g(Y^-))$, it follows that

- $\nu^*(Y^+) + \nu^*(g(Y^+)) + \nu^*(Y^- \backslash (g(Y^+) \cap \overline{Y})) = 1$ and
- $\nu^*(Y^-) + \nu^*(g(Y^-)) + \nu^*(Y^+ \backslash (g(Y^-) \cap \overline{Y})) = 1$.

But, then, by construction of the measure ν^*, $\nu^*(Y^+) = \nu^*(g(Y^+))$ and $\nu^*(Y^-) = \nu^*(g(Y^-))$. So,

- $2\nu^*(Y^+) + \nu^*(Y^- \backslash (g(Y^+) \cap \overline{Y})) = 1$ and
- $2\nu^*(Y^-) + \nu^*(Y^+ \backslash (g(Y^-) \cap \overline{Y})) = 1$.

The first bullet plus Lemma 1.16 implies that $\nu^*(Y^+) = 0$ and the second bullet plus Lemma 1.16 implies that $\nu^*(Y^-) = 0$. But this contradicts the fact that $\nu^*(Y^+) + \nu^*(Y^-) = 1$. ■

2 Correlated Rationalizability

We begin by extending the definitions in Dekel, Fudenberg and Morris (2007) and Battigalli, DiTillio, Grillo and Penta (2011) to arbitrary non-finite games.[2]

Fix an Θ-based Bayesian game $(\Gamma, \mathcal{T})$. Set $R_i^0 = C_i \times T_i$. Assume that R_i^m has been defined for each i. Then $(c_i, t_i) \in R_i^{m+1}$ if there exists a measure $\mu \in \Delta(\Theta \times C_{-i} \times T_{-i})$ so that

(R-i) $\int_{\Theta \times C_{-i}} \pi_i(\theta, c_i, c_{-i}) d\mathrm{marg}_{\Theta \times C_{-i}} \mu \geq \int_{\Theta \times C_{-i}} \pi_i(\theta, d_i, c_{-i}) d\mathrm{marg}_{\Theta \times C_{-i}} \mu$, for each $d_i \in C_i$,

(R-ii) $\mu(\Theta \times R_{-i}^m) = 1$; and

(R-iii) $\beta_i(t_i) = \mathrm{marg}_{\Theta \times T_{-i}} \mu$.

Call $R^m = \prod_{i \in I} R_i^m$ the set of **m-rationalizable** choice-type pairs. Call $R_i = \bigcap_m R_i^m$ the set of **i-rationalizable** choice-type pairs and $R = \prod_{i \in I} R_i$ the set of **rationalizable** choice-type pairs.

Remark 2.1. Suppose, for each m, $R_1^m, \ldots, R_I^m$ are non-empty and measurable. Then, for each m, $R^{m+1} \subseteq R^m$.

[2] No epistemic justification is given for these definitions.

Lemma 2.1. *Suppose $\prod_{i\in I} R_i \neq \emptyset$. Then, for each i and each m, R_i^m are measurable.*

Proof. Suppose, for some player $j \neq i$ and some m, R_j^m is not measurable. Then, there is no $\mu \in \Delta(\Theta \times C_{-i} \times T_{-i})$ with $\mu(\Theta \times R_{-i}^m) = 1$. As such, $R_i^{m+1} = \emptyset$ and so $R^i = \emptyset$. ■

Fix a Θ-based Bayesian games $(\Gamma, \mathcal{T})$ and $(\Gamma, \mathcal{T}^*)$. Write $R^m = \prod_{i\in I} R_i^m$ (resp. $R = \prod_{i\in I} R_i$) for the set of m-rationalizable (resp. rationalizable) choice-type pairs in $(\Gamma, \mathcal{T})$. Write $R^{m,*} = \prod_{i\in I} R_i^{m,*}$ (resp. $R^* = \prod_{i\in I} R_i^*$) for the set of m-rationalizable (resp. rationalizable) choice-type pairs in $(\Gamma, \mathcal{T}^*)$.

Definition 2.1. *Fix a Θ-based game Γ and let $\mathcal{T}$ and $\mathcal{T}^*$ be two Θ-based type structures, so that $\mathcal{T}$ can be embedded into $\mathcal{T}^*$*

*(i) $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the **Rationalizable Extension Property for Γ** if, for every injective type morphism$(h_1, \ldots, h_{|I|})$ from $\mathcal{T}$ to $\mathcal{T}^*$, $(c_i, t_i) \in R_i$ implies $(c_i, h_i(t_i)) \in R_i^*$.*

*(ii) $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the **Rationalizable Pull-Back Property for Γ** if, for every injective type morphism$(h_1, \ldots, h_{|I|})$ from $\mathcal{T}$ to $\mathcal{T}^*$, $(c_i, h_i(t_i)) \in R_i^*$ implies $(c_i, t_i) \in R_i$.*

Proposition 2.1. *Fix Θ-based Bayesian games $(\Gamma, \mathcal{T})$ and $(\Gamma, \mathcal{T}^*)$ so that $\mathcal{T}$ can be embedded into $\mathcal{T}^*$. If, for each i, $R_i, R_i^* \neq \emptyset$, then $\langle \mathcal{T}, \mathcal{T}^* \rangle$ satisfies the Rationalizable Extension and Pull-Back Properties for Γ.*

Proposition 2.1 will follow from Lemmata 2.1-2.2.

Lemma 2.2. *Fix a Θ-based game Γ and Θ-based structures $\mathcal{T}$ and $\mathcal{T}^*$, where $\mathcal{T}$ can be properly embedded into $\mathcal{T}^*$ via $(h_1, \ldots, h_I)$. Suppose, for each m, $R_1^m, \ldots, R_I^m$ and $R_1^{m,*}, \ldots, R_I^{m,*}$ are measurable. Then, for each $i = 1, \ldots, I$,*

(i) $(c_i, t_i) \in R_i^m$ implies $(c_i, h_i(t_i)) \in R_i^{m,*}$;

(ii) $(c_i, t_i) \in [C_i \times T_i] \setminus R_i^m$ implies $(c_i, h_i(t_i)) \in [C_i \times T_i^*] \setminus R_i^{m,*}$.

Proof. The proof is by induction on m. For $m = 0$ the result is immediate. Assume the result holds for m. We will show that it also holds for $m + 1$.

Begin with part (i): Fix $(c_i, t_i) \in R_i^{m+1}$. Then, we can find a measure $\mu \in \Delta(\Theta \times C_{-i} \times T_{-i})$ satisfying (R-i)-(R-iii). Extend h_{-i} to $\overrightarrow{h}_{-i} : \Theta \times C_{-i} \times T_{-i} \to \Theta \times C_{-i} \times T_{-i}^*$, by setting each $\overrightarrow{h}_{-i}(\theta, c_{-i}, t_{-i}) = (\theta, c_{-i}, h_{-i}(t_{-i}))$. Let μ^* be the image measure of μ under $\overrightarrow{h}_{-i}$. We will show that μ^* satisfies analogs of (R-i)-(R-iii), relative to the structure $\mathcal{T}^*$, i.e.,

(BFK*-i) $\int_{\Theta\times C_{-i}} \pi_i(\theta, c_i, c_{-i}) d\text{marg}_{\Theta\times C_i} \mu^* \geq \int_{\Theta\times C_{-i}} \pi_i(\theta, d_i, c_{-i}) d\text{marg}_{\Theta\times C_{-i}} \mu^*$, for each $d_i \in C_i$;

(BFK*-ii) $\mu^*(\Theta \times R_{-i}^{m,*}) = 1$; and

(BFK*-iii) $\beta_i^*(h_i(t_i)) = \text{marg}_{\Theta\times T_{-i}^*} \mu^*$.

This will establish that $(c_i, h_i(t_i)) \in R_i^{m+1,*}$.

Condition (R*-i) follows from (R-i) and the fact that $\text{marg}_{\Theta \times C_{-i}} \mu = \text{marg}_{\Theta \times C_{-i}} \mu_*$. For condition (R*-ii) note that

$$\mu^*(\Theta \times R_{-i}^{m,*}) = \mu((\overrightarrow{h}_{-i})^{-1}(\Theta \times R_{-i}^{m,*})) = \mu(\Theta \times R_{-i}^{m}) = 1,$$

where the first equality the fact that $R_{-i}^{m,*}$ is measurable and the definition of an image measure, the second equality follows from parts (i)-(ii) of the induction hypothesis, and the third equality follows from (R-ii). For condition (R*-iii), fix some event $E^* \subseteq \Theta \times T_{-i}^*$ and note that

$$\begin{aligned}
\beta_i^*(h_i(t_i))(E^*) &= \beta_i(t_i)((\text{id} \times h_{-i})^{-1}(E^*)) \\
&= \text{marg}_{\Theta \times T_{-i}} \mu((\text{id} \times h_{-i})^{-1}(E^*)) \\
&= \mu(C_{-i} \times (\text{id} \times h_{-i})^{-1}(E^*)) \\
&= \mu((\overrightarrow{h}_{-i})^{-1}(C_{-i} \times E^*)) \\
&= \mu^*(C_{-i} \times E^*) \\
&= \text{marg}_{\Theta \times T_{-i}^*} \mu^*(E^*),
\end{aligned}$$

where the first line follows from the definition of a type morphism, the second line follows from condition (R-iii), and the fourth and fifth lines follow from construction.

Now we turn to part (ii). Suppose $(c_i, h_i(t_i)) \in R_i^{m+1,*}$. Then, we can find a measure μ^* satisfying conditions (R*-i)-(R*-iii). Let $\overrightarrow{g}_{-i} : \Theta \times C_{-i} \times h_{-i}(T_{-i}) \to \Theta \times C_{-i} \times T_{-i}$ be a map, with each $\overrightarrow{g}_{-i}(\theta, c_{-i}, h_{-i}(t_{-i})) = (\theta, c_{-i}, t_{-i})$. Recall that h_{-i} is an embedding. As such, it is injective and so $\overrightarrow{g}_{-i}$ is well-defined. Likewise, h_{-i} is bimeasurable, and so $\overrightarrow{g}_{-i}$ is the product of measurable maps and so measurable. Using (R*-ii), $\mu^*(\Theta \times C_{-i} \times h_{-i}(T_{-i})) = 1$. So, the image measure of μ^* under $\overrightarrow{g}_{-i}$ is well-defined. Write μ for this measure. We need to show that μ satisfies conditions (R-i)-(R-iii).

Conditions (R-i)-(R-ii) are shown by repeating the arguments (R*-i)-(R*-ii) above. We focus on condition (R-iii). Fix some event $E \subseteq \Theta \times T_{-i}$ and recall that

$$\begin{aligned}
\mu(C_{-i} \times E) &= \mu^*((\overrightarrow{g}_{-i})^{-1}(C_{-i} \times E)) \\
&= \mu^*(C_{-i} \times (\text{id} \times h_{-i})(E)) \\
&= \text{marg}_{\Theta \times T_{-i}^*} \mu^*((\text{id} \times h_{-i})(E)) \\
&= \beta_i^*(h_i(t_i))((\text{id} \times h_{-i})(E)) \\
&= \beta_i(t_i)((\text{id} \times h_{-i})^{-1}((\text{id} \times h_{-i})(E))) \\
&= \beta_i(t_i)(E),
\end{aligned}$$

where the first line follows from the fact that $\overrightarrow{g}_{-i}$ is measurable, the fourth line follows from condition (R*-iii), the fifth line follows from the definition of a type morphism, and the last line uses the fact that h_{-i} is injective. This establishes (R-iii). ■

References

Aliprantis, C.D. and K.C. Border. 2007. *Infinite dimensional analysis: a hitchhiker's guide*. Springer Verlag.

Battigalli, P., A. DiTillio, E. Grillo and A. Penta. 2011. "Interactive Epistemology and Solution Concepts for Games of Incomplete Information." *The B.E. Journal of Theoretical Economics* 11.

Dekel, E., D. Fudenberg and S. Morris. 2007. "Interim correlated rationalizability." *Theoretical Economics* 2(1):15–40.

Dudley, R.M. 2002. *Real analysis and probability*. Vol. 74.

Hellman, Z. 2014. "A Game with No Approximate Bayesian Equilibrium." *Journal of Economic Theory* 153:138–151.

Levy, J. 2012. "A Discounted Stochastic Game with No Stationary Nash Equilibrium: The Case of Determinstic Transitions." *Center for the Study of Rationality Discussion Paper* (596).

Purves, R. 1966. "Bimeasurable functions." *Fundam. Math.* 58:149–157.