

Fixed Points in Epistemic Game Theory

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ABSTRACT. The epistemic conditions of “rationality and common belief of rationality” and “rationality and common assumption of rationality” in a game are characterized by the solution concepts of a “best-response set” (Pearce [26, 1984]) and a “self-admissible set” (Brandenburger, Friedenberg, and Keisler [13, 2008]), respectively. We characterize each solution concept as the set of fixed points of a map on the lattice of rectangular subsets of the product of the strategy sets. Of note is that both maps we use are non-monotone.

1. Introduction

Topological fixed-point arguments have a long tradition in game theory. von Neumann’s [28, 1928] proof of his famous Minimax Theorem made use of a topological fixed-point argument. Nash’s existence proof for his equilibrium concept is also a topological fixed-point argument—using Brouwer’s Theorem ([14, 1910]) in [23, 1950] and [25, 1951] and Kakutani’s Theorem ([17, 1941]) in [24, 1950]. Subsequently, the Brouwer and Kakutani theorems became the standard tools in existence arguments in game theory.

Order-theoretic fixed-point arguments also have a role in game theory. These arguments are prominent in epistemic game theory (EGT). The purpose of this note is to explain the role of order-theoretic fixed point arguments in EGT.

This note focuses on two solution concepts that arise from the epistemic game theory approach: best-response sets and self-admissible sets. The treatment is in finite games. (The definitions of these concepts will be laid out later.) We will characterize these solution concepts as arising from fixed-points of certain non-monotone maps.

There is an important stream of papers that treats iterated dominance concepts in general infinite games. See, e.g., Apt [2, 2007a], [3, 2007b], [4, 2007c], and Apt and Zvesper [5, 2007]. The main focus in this stream of papers is on the case of monotonicity, but non-monotonic maps also arise.

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Mathematical logic, theoretical computer science, and lattice theory are other areas where order-theoretic fixed point methods are commonly employed. See Moschovakis [22, 1974], Libkin [18, 2010], Abramsky and Jung [1, 1994], and Davey and Priestley [15, 2002] for standard presentations. The order-theoretic maps most often used there are monotonic, but non-monotonic maps also arise. A notable instance of the non-monotone case is Martin’s [19, 2000] theory of measurements on domains. It would be very interesting to investigate whether what is known in other areas about the non-monotone case could be applied to EGT. We leave this to future work.

2. Epistemic Game Theory

We now give a very brief sketch of the EGT approach. The classical description of the strategic situation consists of a game matrix or game tree. The idea is that players may face uncertainty about how others play the game. Under the EGT approach, this uncertainty is part of the description of the strategic situation. That is, the description consists of a game (matrix or tree) and beliefs about the play of the game. These “beliefs about the play of the game” are, in fact, the so-called “hierarchies about the play of the game,” i.e., what each player thinks about “the strategies other players select,” what each player thinks about “what others think about ‘the strategies others select,’” etc.¹

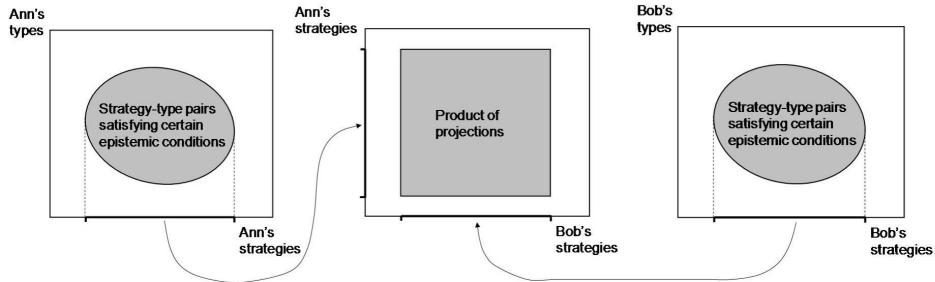


Figure 2.1

Harsanyi [16, 1967] introduced a **type structure** model to describe these hierarchies. Formally, this adds **types** for each player, where the type describes what that player thinks, think other players think, The Appendix provides a formal treatment of this step, in a baseline case. (See Siniscalchi [27, 2008] for a survey of the relevant literature.)

Figure 2.1 depicts the set-up in the two-player case. The starting point is the strategy-type pairs for Ann and Bob, given by the left-hand and right-hand panels. We then impose epistemic conditions. These correspond to the shaded sets in the left-hand and right-hand panels. The idea is that the epistemic conditions correspond to “strategic reasoning.” For example, we might consider those strategy-type pairs for Ann that satisfy the condition that Ann is **rational**—i.e., that her strategy is optimal for her given what, according to her type, she thinks Bob will play. (Ann’s remaining strategy-type pairs satisfy the condition that she is irrational.)

¹This discussion presumes that the structure of the game (e.g., payoff functions) is “transparent” among the players. If it is not, then the description consists of the game and beliefs about both the structure of the game and the play of the game.

Below, we will be more precise about what “thinks” means.) A basic epistemic condition is that Ann is rational, thinks Bob is rational, and so on. In this case, we take the shaded set in the left-hand panel to be the strategy-type pairs for Ann that satisfy this condition. Likewise, we take the shaded set in the right-hand panel to be the strategy-type pairs for Bob that are rational, think Ann is rational, and so on.

The **characterization** question is whether we can identify the strategies that can be played under such epistemic conditions, by looking only at the game (matrix or tree). That is, can we identify the projections of the shaded sets into the strategy sets, as depicted in the middle panel, without reference to the type structure model? There are several such characterization results in EGT. They differ according to how the terms “rationality” and “thinks” are formalized. The different formalizations, in turn, reflect different concepts of “strategic reasoning” (i.e., different epistemic conditions) that we the analysts can impose on games.

The remainder of this section formalizes the epistemic descriptions and the epistemic conditions. Section 4 returns to the question of characterization.

Fix a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$, where S^a and S^b are finite strategy sets for Ann and Bob, and $\pi^a : S^a \times S^b \rightarrow \mathbb{R}$ and $\pi^b : S^a \times S^b \rightarrow \mathbb{R}$ are their payoff functions.²

Begin with the most basic formalization. Append Polish spaces T^a and T^b of types for Ann and Bob. Here, a type t^a for Ann is associated with a probability measure on the Borel subsets of $S^b \times T^b$. A strategy-type pair $(s^a, t^a) \in S^a \times T^a$ for Ann is **rational** if s^a maximizes her expected payoff, under the marginal on S^b of the probability measure associated with t^a . In this case, “thinks” means “belief.” Say Ann **believes** $E^b \subseteq S^b \times T^b$ if E^b is Borel and the probability measure associated with type t^a assigns probability 1 to E^b . These and subsequent definitions have counterparts with Ann and Bob interchanged.

A second formalization follows Brandenburger, Friedenberg, and Keisler [13, 2008]. Again, append Polish spaces T^a and T^b of types for Ann and Bob. Here, a type t^a for Ann is associated with a lexicographic probability system on the Borel subsets of $S^b \times T^b$. A lexicographic probability system (Blume, Brandenburger, and Dekel [10, 1991]) is a finite sequence of mutually singular probability measures. It is to be thought of as a sequence of hypotheses—a primary hypothesis, a secondary hypothesis, . . .,—held by Ann about Bob’s strategy and type. A strategy-type pair $(s^a, t^a) \in S^a \times T^a$ for Ann is (**lexicographically**) **rational** if s^a lexicographically maximizes her sequence of expected payoffs, calculated under the marginals on S^b of the sequence of probability measures associated with t^a . In this case, “thinks” means “assumption.” If T^b is finite, say Ann **assumes** E^b if each point in E^b receives positive probability under an earlier probability measure in the sequence than does any point not in E^b . Alternatively put, the event E^b is “infinitely more likely” than the event not- E^b . See [13, 2008, Section 5] for a general treatment (i.e., in the case of infinite T^b).

For the purposes of this note, we will need one key property of both “belief” and “assumption.” Return to the general term “thinks,” in order to subsume both cases. Define a thinking operator C^a from the family of Borel subsets of $S^b \times T^b$

²We restrict attention to two-player games for notational simplicity, but our analysis immediately extends to games with three or more players.

to T^a by

$$C^a(E^b) = \{t^a \in T^a : t^a \text{ thinks } E^b \text{ is true}\}.$$

AXIOM 2.1 (Conjunction). *Fix a type $t^a \in T^a$ and Borel sets E_1^b, E_2^b, \dots in $S^b \times T^b$. Suppose, for each m , that $t^a \in C^a(E_m^b)$. Then $t^a \in C^a(\bigcap_m E_m^b)$.*

In words, this says that if Ann thinks that each event E_m^b is true, then she thinks the joint event $\bigcap_m E_m^b$ is true. It is immediate from the rules of probability that “belief” satisfies this conjunction property. For the case of “assumption,” see ([13, 2008, Property 6.3]).

Given Borel sets $E^a \subseteq S^a \times T^a$ and $E^b \subseteq S^b \times T^b$, define $E_1^a = E^a$, $E_1^b = E^b$, and for $m \geq 1$,

$$E_{m+1}^a = E_m^a \cap [S^a \times C^a(E_m^b)].$$

(For this to be well-defined, the sets E_m^a and E_m^b must be Borel.)

DEFINITION 2.1. *The event that $E^a \times E^b$ is true and commonly thought is*

$$\bigcap_{m=1}^{\infty} E_m^a \times \bigcap_{m=1}^{\infty} E_m^b.$$

In the situations we study in this note, $E^a \times E^b$ is the event that Ann and Bob are rational. Thus, the event $\bigcap_{m=1}^{\infty} E_m^a \times \bigcap_{m=1}^{\infty} E_m^b$ is either the event that there is **rationality and common belief of rationality (RCBR)** or the event that there is **rationality and common assumption of rationality (RCAR)**.

3. Epistemic Fixed Points

Given two Polish spaces P^a, P^b , let $\mathcal{B}(P^a, P^b)$ be the set of all rectangles $E^a \times E^b$ where E^a is a Borel subset of P^a and E^b is a Borel subset of P^b . Proposition 3.2 below will show that each of the events RCBR and RCAR is a fixed point of a mapping Γ from $\mathcal{B}(S^a \times T^a, S^b \times T^b)$ to itself. This suggests that we should be able to describe the strategies playable under RCBR or RCAR via fixed points of mappings from $\mathcal{B}(S^a, S^b)$ to itself. Section 5 will show that this is indeed the case. In sum: This section is about epistemic fixed points, i.e., fixed points in $\mathcal{B}(S^a \times T^a, S^b \times T^b)$. Section 5 will be about fixed points in the game matrix, i.e., fixed points in the smaller space $\mathcal{B}(S^a, S^b)$ where the type structure plays no part.

Define a mapping Γ from $\mathcal{B}(S^a \times T^a, S^b \times T^b)$ to itself, as follows. Given $E^a \times E^b \in \mathcal{B}(S^a \times T^a, S^b \times T^b)$, set

$$\Gamma(E^a \times E^b) = (E^a \times E^b) \cap ([S^a \times C^a(E^b)] \times [S^b \times C^b(E^a)]).$$

Note that the mapping Γ depends on C^a and C^b . In words, Γ maps an event $E^a \times E^b$ to the event that $E^a \times E^b$ is true, and Ann and Bob think their respective components of $E^a \times E^b$ are true. The next lemma is immediate:

LEMMA 3.1. *The event $E^a \times E^b$ is a fixed point of Γ , i.e. $\Gamma(E^a \times E^b) = E^a \times E^b$, if and only if*

$$\begin{aligned} E^a &\subseteq S^a \times C^a(E^b), \\ E^b &\subseteq S^b \times C^b(E^a). \end{aligned}$$

When C^a stands for belief, Lemma 3.1 says the fixed points of Γ are the so-called “self-evident events.” (The concept of a self-evident event is due to Monderer-Samet [21, 1989].) Lemma 3.1 is used in the proofs of the following two propositions.

PROPOSITION 3.2. *Fix $E^a \times E^b \in \mathcal{B}(S^a \times T^a, S^b \times T^b)$. The event*

$$\bigcap_{m=1}^{\infty} E_m^a \times \bigcap_{m=1}^{\infty} E_m^b$$

is a fixed point of Γ .

PROOF. Using the definitions,

$$\begin{aligned} \bigcap_{m=1}^{\infty} E_m^a &= E_1^a \cap \bigcap_{m=1}^{\infty} [S^a \times C^a(E_m^b)] \subseteq \bigcap_{m=1}^{\infty} [S^a \times C^a(E_m^b)] = \\ &S^a \times \bigcap_{m=1}^{\infty} C^a(E_m^b) \subseteq S^a \times C^a\left(\bigcap_{m=1}^{\infty} E_m^b\right), \end{aligned}$$

where the last inclusion relies on conjunction (Axiom 2.1). \square

Proposition 3.2 says that the event RCBR (resp. RCAR) is a fixed point of the map Γ when C^a, C^b correspond to belief (resp. assumption). We also have:

PROPOSITION 3.3. *Suppose $E^a \times E^b \in \mathcal{B}(S^a \times T^a, S^b \times T^b)$ is a fixed point of Γ . Then $E_m^a = E^a$ and $E_m^b = E^b$ for all m .*

PROOF. This is immediate for $m = 1$, so suppose it is true for m . We have

$$E_{m+1}^a = E_m^a \cap [S^a \times C^a(E_m^b)] = E^a \cap [S^a \times C^a(E^b)],$$

using the induction hypothesis. But since $E^a \times E^b$ is a fixed point,

$$E^a \cap [S^a \times C^a(E^b)] = E^a,$$

and so $E_{m+1}^a = E^a$, as required. \square

An important reference on fixed points on epistemic structures is Barwise [7, 1988].

4. Best-Response Sets and Self-Admissible Sets

We now turn to fixed-point characterizations of the strategies playable under RCBR and RCAR. We undertake these characterizations in two steps. First, we review the existing strategic characterizations of RCBR and RCAR. These are couched in terms of best-response sets (Pearce [26, 1984]) and self-admissible sets (Brandenburger, Friedenberg, and Keisler [13, 2008]). Then, we point to fixed-point characterizations of these sets.

Throughout, we treat RCBR and RCAR in parallel rather than in sequence. The reason is that certain mathematical techniques that are typically associated with self-admissible sets (and so RCAR) will be useful in our fixed-point characterization of best-response sets.³

We begin with some preliminary definitions and lemmas. Given a finite set Ω , let $\mathcal{M}(\Omega)$ denote the set of all probability measures on Ω . Write $\text{Supp } \sigma$ for the support of $\sigma \in \mathcal{M}(X)$.

The definitions to come all have counterparts with a and b reversed. Extend π^a to $\mathcal{M}(S^a) \times \mathcal{M}(S^b)$ by taking $\pi^a(\sigma^a, \sigma^b)$, for $\sigma^a \in \mathcal{M}(S^a)$, $\sigma^b \in \mathcal{M}(S^b)$, to be the expectation of π^a under $\sigma^a \otimes \sigma^b$:

$$\pi^a(\sigma^a, \sigma^b) = \sum_{(s^a, s^b) \in S^a \times S^b} \sigma^a(s^a) \sigma^b(s^b) \pi^a(s^a, s^b).$$

³We are grateful to a referee for asking us to emphasize this point.

DEFINITION 4.1. Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$, and $\sigma^b \in \mathcal{M}(S^b)$. A strategy $s^a \in X$ is **σ^b -justifiable with respect to $X \times Y$** if $\sigma^b(Y) = 1$ and $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in X$. Say s^a is **justifiable with respect to $X \times Y$** if s^a is σ^b -justifiable with respect to $X \times Y$ for some σ^b .

DEFINITION 4.2. Fix $Q^a \times Q^b \subseteq S^a \times S^b$. The set $Q^a \times Q^b$ is a **best-response set (BRS)** if for each $s^a \in Q^a$ there is a $\sigma^b \in \mathcal{M}(S^b)$ such that:

- (i) s^a is σ^b -justifiable with respect to $S^a \times Q^b$;
- (ii) if r^a is also σ^b -justifiable with respect to $S^a \times Q^b$, then $r^a \in Q^a$;

and likewise for each $s^b \in Q^b$.

The original definition of a BRS is due to Pearce [26, 1984]. Definition 4.2 is from Battigalli and Friedenberg [8, 2011]. It differs from Pearce's definition in two ways: players choose only pure (not mixed) strategies, and condition (ii) is new.⁴ Condition (ii) is important in the epistemic characterization; see the statement at the beginning of Section 5.

DEFINITION 4.3. Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$. A strategy $s^a \in X$ is **strongly dominated with respect to $X \times Y$** if there is a $\sigma^a \in \mathcal{M}(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for every $s^b \in Y$. Otherwise, say s^a is **undominated with respect to $X \times Y$** .

The following equivalence is standard. (Necessity is immediate. Sufficiency is proved via the supporting hyperplane theorem.)

LEMMA 4.4. Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$. A strategy $s^a \in X$ is justifiable with respect to $X \times Y$ if and only if it is undominated with respect to $X \times Y$.

The next definition picks out a notable BRS.

DEFINITION 4.5. Set $S_0^i = S^i$ for $i = a, b$, and define inductively

$$S_{m+1}^i = \{s^i \in S_m^i : s^i \text{ is undominated with respect to } S_m^a \times S_m^b\}$$

A strategy $s^i \in S_m^i$ is called **m -undominated**. A strategy $s^i \in \bigcap_{m=0}^{\infty} S_m^i$ is called **iteratively undominated (IU)**.

By finiteness, there is a (first) number M such that $\bigcap_{m=0}^{\infty} S_m^i = S_M^i \neq \emptyset$ for $i = a, b$. It is easy to check that the IU set is a BRS and every BRS is contained in the IU set.

We now repeat this development for self-admissible sets.

DEFINITION 4.6. Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$. A strategy $s^a \in X$ is **weakly dominated with respect to $X \times Y$** if there is a $\sigma^a \in \mathcal{M}(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Y$, and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Y$. Otherwise, say s^a is **admissible with respect to $X \times Y$** .

DEFINITION 4.7. Fix $Y \subseteq S^b$ with $Y \neq \emptyset$. Say r^a **supports s^a with respect to Y** if there is a $\sigma^a \in \mathcal{M}(S^a)$ with $r^a \in \text{Supp } \sigma^a$ and $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for all $s^b \in Y^b$. If r^a supports s^a with respect to S^b , say simply r^a **supports s^a** .

DEFINITION 4.8 (Brandenburger, Friedenberg, and Keisler [13, 2008]). Fix $Q^a \times Q^b \subseteq S^a \times S^b$. The set $Q^a \times Q^b$ is a **self-admissible set (SAS)** if:

⁴David Pearce (private communication) told one of us that he was aware of this condition, but to keep things simple did not include it in his definition in [26, 1984].

- (i) each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$;
- (ii) each $s^a \in Q^a$ is admissible with respect to $S^a \times Q^b$;
- (iii) if $r^a \in S^a$ supports some $s^a \in Q^a$, then $r^a \in Q^a$;

and likewise for each $s^b \in Q^b$.

The next equivalence is a special case of a classic result in convex analysis due to Arrow, Barankin, and Blackwell [6, 1953].

LEMMA 4.9. *Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$. A strategy $s^a \in X$ is admissible with respect to $X \times Y$ if and only if there is a $\sigma^b \in \mathcal{M}(S^b)$, with $\text{Supp } \sigma^b = Y$, such that s^a is σ^b -justifiable with respect to $X \times Y$.*

The next lemma rewrites Definition 4.8 in a way that brings out the comparison with BRS's.

LEMMA 4.10. *A set $Q^a \times Q^b$ is an SAS if and only if:*

- (i) *for each $s^a \in Q^a$, there is a $\sigma^b \in \mathcal{M}(S^b)$, with $\text{Supp } \sigma^b = S^b$, such that:*
 - s^a is σ^b -justifiable with respect to $S^a \times S^b$,
 - if $r^a \in S^a$ is also σ^b -justifiable with respect to $S^a \times S^b$, then $r^a \in Q^a$;
- (ii) *for each $s^a \in Q^a$, there is a $\rho^b \in \mathcal{M}(S^b)$, with $\text{Supp } \rho^b = Q^b$, such that s^a is ρ^b -justifiable with respect to $S^a \times Q^b$;*

and likewise for each $s^b \in Q^b$.

To show that an SAS satisfies the conditions in this Lemma, use Lemma D.4 in Brandenburger, Friedenberg, and Keisler [13, 2008]. For the converse, use Lemma D.2 in Brandenburger, Friedenberg, and Keisler [13, 2008].

The next definition picks out a notable SAS.

DEFINITION 4.11. *Set $\bar{S}_0^i = S^i$ for $i = a, b$, and define inductively*

$$\bar{S}_{m+1}^i = \{s^i \in \bar{S}_m^i : s^i \text{ is admissible with respect to } \bar{S}_m^a \times \bar{S}_m^b\}.$$

A strategy $s^i \in \bar{S}_m^i$ is called m -admissible. A strategy $s^i \in \bigcap_{m=0}^{\infty} \bar{S}_m^i$ is called iteratively admissible (IA).

By finiteness, there is a (first) number N such that $\bigcap_{m=0}^{\infty} \bar{S}_m^i = \bar{S}_N^i \neq \emptyset$ for $i = a, b$. The IA set is an SAS (Brandenburger and Friedenberg [12, 2010, Proposition 5.1]). But, unlike the case with IU and BRS's, it need not be the case that every SAS is contained in the IA set. See Example 5.9 to come.

5. Fixed-Point Characterizations

BRS's characterize the epistemic condition of RCBR: Fix a game $\langle S^a, S^b, \pi^a, \pi^b \rangle$ and an associated type structure, where each type is mapped to a (single) probability measure. Define "believes" and "rationality" as before. Then, the projection to $S^a \times S^b$ of the RCBR event (which lies in $S^a \times T^a \times S^b \times T^b$) constitutes a BRS of the game. Conversely, every BRS of $\langle S^a, S^b, \pi^a, \pi^b \rangle$ arises in this way, for a suitable choice of type structure. This follows from Battigalli and Friedenberg [8, 2011, Theorem 5.1].

Likewise, SAS's characterize the epistemic conditions of RCAR: Fix a game $\langle S^a, S^b, \pi^a, \pi^b \rangle$ and an associated type structure, where each type is now mapped to a lexicographic probability system. Define "assumes" and "(lexicographic) rationality" as before. Then, the projection to $S^a \times S^b$ of the RCAR event constitutes

an SAS of the game. Conversely, every SAS of $\langle S^a, S^b, \pi^a, \pi^b \rangle$ arises in this way, for a suitable choice of type structure. This is Theorem 8.1 in Brandenburger, Friedenberg, and Keisler [13, 2008].

So, to deliver on our promised fixed-point characterizations of RCBR and RCAR in terms of strategies played, it remains to provide fixed-point characterizations of BRS's and SAS's.

Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$. The next two lemmas are standard.

LEMMA 5.1. *A strategy $s^a \in X$ is σ^b -justifiable with respect to $X \times Y$ if and only if s^a is admissible with respect to $X \times \text{Supp } \sigma^b$.*

LEMMA 5.2. *Fix $s^a \in X$ and $\sigma^b \in \mathcal{M}(S^b)$ such that $\pi^a(s^a, \sigma^b) \geq \pi^a(q^a, \sigma^b)$ for all $q^a \in X$. If r^a supports s^a , then $\pi^a(r^a, \sigma^b) \geq \pi^a(q^a, \sigma^b)$ for all $q^a \in X$.*

LEMMA 5.3. *Suppose $s^a \in S^a$ is undominated (resp. admissible) with respect to $X \times Y$. If r^a supports s^a , then r^a is undominated (resp. admissible) with respect to $X \times Y$.*

PROOF. By Lemma 4.4 (resp. Lemma 4.9), there is a $\sigma^b \in \mathcal{M}(S^b)$, with $\sigma^b(Y) = 1$ (resp. $\text{Supp } \sigma^b = Y^b$) such that $\pi^a(s^a, \sigma^b) \geq \pi^a(q^a, \sigma^b)$ for all $q^a \in X$. By Lemma 5.2, r^a is then undominated (resp. admissible) with respect to $X \times Y$. \square

Given $\sigma^b \in \mathcal{M}(S^b)$, write $\mathcal{J}(\sigma^b)$ for the set of strategies $s^a \in S^a$ that are σ^b -justifiable.

DEFINITION 5.4. *Say that σ^b **minimally justifies s^a with respect to $X \times Y$** if s^a is σ^b -justifiable with respect to $X \times Y$ and, for each $\rho^b \in \mathcal{M}(S^b)$ such that s^a is ρ^b -justifiable with respect to $X \times Y$, we have $\mathcal{J}(\sigma^b) \subseteq \mathcal{J}(\rho^b)$.*

LEMMA 5.5. *If s^a is justifiable with respect to $S^a \times Y$, there is a σ^b that minimally justifies s^a with respect to $S^a \times Y$.*

PROOF. Suppose s^a is justifiable with respect to $S^a \times Y$. Then, by Lemma 5.1, there is $\emptyset \neq Z_k \subseteq Y$ such that s^a is admissible with respect to $S^a \times Z_k$. Let Z be the union of all such Z_k . Then, s^a is admissible with respect to $S^a \times Z$. To see this, suppose not, i.e., suppose there is $\sigma^a \in \mathcal{M}(S^a)$ with $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Z$, and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Z$. Then, we can find some $Z_k \subseteq Z$ such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Z_k$, and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Z_k$. This contradicts the fact that s^a is admissible with respect to each $S^a \times Z_k$.

We have established that s^a is admissible with respect to $S^a \times Z$. By Lemma D.4 in Brandenburger, Friedenberg, and Keisler [13, 2008], there is then a $\sigma^b \in \mathcal{M}(S^b)$, with $\text{Supp } \sigma^b = Z$, such that $\mathcal{J}(\sigma^b)$ is the set of strategies that support s^a with respect to $S^a \times Z$. We will show that σ^b minimally justifies s^a with respect to $S^a \times Y$.

Fix $\rho^b \in \mathcal{M}(S^b)$ that justifies s^a with respect to $S^a \times Y$. We show that $\mathcal{J}(\sigma^b) \subseteq \mathcal{J}(\rho^b)$. Fix $r^a \in \mathcal{J}(\sigma^b)$. Then r^a supports s^a with respect to $S^a \times Z$. Note that $\text{Supp } \rho^b \subseteq Z$, so that r^a also supports s^a with respect to $S^a \times \text{Supp } \rho^b$. It follows from Lemma 5.2 that $r^a \in \mathcal{J}(\rho^b)$ as required. \square

Now consider the complete lattice $\Lambda = \mathcal{B}(S^a, S^b)$. (The join of two subsets is the component-by-component union. The meet is the intersection.) We will define a map $\Phi : \Lambda \rightarrow \Lambda$ so that the fixed points of Φ are the BRS's. Specifically, for $Q^a \times Q^b \in \mathcal{B}(S^a, S^b)$, set $(s^a, s^b) \in \Phi(Q^a \times Q^b)$ if either: (a) $s^a \in Q^a$ and s^a is justifiable with respect to $S^a \times Q^b$, or (b) $s^a \in \mathcal{J}(\sigma^b)$ for some σ^b that minimally justifies some $r^a \in Q^a$ with respect to $S^a \times Q^b$. (The analogous conditions must hold for s^b .)

PROPOSITION 5.6. *If $Q^a \times Q^b$ is a BRS, then it is a fixed point of Φ , i.e., $\Phi(Q^a \times Q^b) = Q^a \times Q^b$. Conversely, if $Q^a \times Q^b$ is a fixed point of Φ , then it is a BRS.*

PROOF. Fix a BRS $Q^a \times Q^b$. We will show that it is a fixed point of Φ . First, fix $(s^a, s^b) \in \Phi(Q^a \times Q^b)$. We will show that $(s^a, s^b) \in Q^a \times Q^b$. Indeed, suppose that $s^a \notin Q^a$. Then there is an $r^a \in Q^a$ such that r^a is justifiable with respect to $S^a \times Q^b$ and, for any σ^b such that r^a is σ^b -justifiable with respect to $S^a \times Q^b$, $s^a \in \mathcal{J}(\sigma^b)$. It follows from condition (ii) of the definition of a BRS that $s^a \in Q^a$, a contradiction. Likewise, we reach a contradiction if we suppose that $s^b \notin Q^b$, so we conclude that $\Phi(Q^a \times Q^b) \subseteq Q^a \times Q^b$. Next, fix $(s^a, s^b) \in Q^a \times Q^b$. By condition (i) of the definition of a BRS and condition (a) of the definition of Φ , we get $(s^a, s^b) \in \Phi(Q^a \times Q^b)$, establishing that $Q^a \times Q^b \subseteq \Phi(Q^a \times Q^b)$.

For the converse, suppose $Q^a \times Q^b = \Phi(Q^a \times Q^b)$. Fix $s^a \in Q^a$ and note that, by condition (a) of the definition of Φ , there is a $\sigma^b \in \mathcal{M}(S^b)$ that justifies s^a with respect to $S^a \times Q^b$. By Lemma 5.5, we can choose σ^b to minimally justify s^a with respect to $S^a \times Q^b$. Then, by condition (b) of the definition of Φ , we get that s^a satisfies conditions (i)-(ii) of a BRS (using the measure σ^b). We can make the same argument for each $s^b \in Q^b$. This establishes that $Q^a \times Q^b$ is a BRS. \square

EXAMPLE 5.7. *The map Φ is not monotone (increasing). Consider the game in Figure 5.1. We have $\Phi(\{(U, L)\}) = \{U, D\} \times \{L, R\}$ but $\Phi(\{U\} \times \{L, R\}) = \{U\} \times \{L, R\}$.*

| | | | |
|-----|---|------|------|
| | | Bob | |
| | | L | R |
| Ann | U | 1, 1 | 1, 1 |
| | D | 1, 1 | 0, 0 |

Figure 5.1

Next, we define a map $\Psi : \Lambda \rightarrow \Lambda$ so that the fixed points of Ψ are the SAS's. Specifically, set $(s^a, s^b) \in \Psi(Q^a \times Q^b)$ if either: (a) $s^a \in Q^a$ and satisfies conditions (i)-(ii) of the definition of an SAS; or (b) s^a supports an $r^a \in Q^a$ that satisfies these conditions. (The analogous conditions must hold for s^b .)

Here is the analog to Proposition 5.6:

PROPOSITION 5.8. *If $Q^a \times Q^b$ is an SAS, then it is a fixed point of Ψ . Conversely, if $Q^a \times Q^b$ is a fixed point of Ψ , then it is an SAS.*

PROOF. Fix an SAS $Q^a \times Q^b$. If $s^a \in Q^a$, then s^a satisfies condition (a) for Ψ . Likewise for s^b . Thus $Q^a \times Q^b \subseteq \Psi(Q^a \times Q^b)$. Next, fix $(s^a, s^b) \in \Psi(Q^a \times Q^b)$. We need to show that $s^a \in Q^a$. If $s^a \notin Q^a$ then s^a supports r^a for some $r^a \in Q^a$. But then condition (iii) of an SAS implies $s^a \in Q^a$, a contradiction.

For the converse, fix $(s^a, s^b) \in Q^a \times Q^b = \Psi(Q^a \times Q^b)$. If s^a satisfies condition (a) for Ψ , then it satisfies conditions (i) and (ii) of an SAS. Next suppose s^a fails condition (a) for Ψ , i.e., is inadmissible with respect to $S^a \times S^b$, or $S^a \times Q^b$, or both. But then s^a must satisfy condition (b) for Ψ , i.e. s^a supports r^a for some $r^a \in Q^a$ satisfying conditions (i) and (ii) of an SAS. By Lemma 5.3, s^a is then admissible with respect to both $S^a \times S^b$ and $S^a \times Q^b$, a contradiction. Finally, suppose q^a supports s^a . We just saw that s^a satisfies condition (a) for Ψ , so q^a satisfies condition (b) for Ψ . Thus $q^a \in \text{proj}_{S^a} \Psi(Q^a \times Q^b) = Q^a$. This establishes condition (iii) of an SAS. \square

EXAMPLE 5.9. Like Φ , the map Ψ is non-monotone. Consider the game in Figure 5.2. We have $\Psi(\{U\} \times \{L, R\}) = \{U\} \times \{L, R\}$ but $\Psi(\{U, D\} \times \{L, R\}) = \{(U, R)\}$. Note also that the fixed points of Ψ are $\{U\} \times \{L, R\}$, $\{(U, R)\}$, and $\{(M, L)\}$. The SAS $\{(M, L)\}$ is the IA set. We see that, different from BRS vs. IU, the SAS's need not be contained in the IA set.

| | | | |
|-------|--|------|------|
| | | Bob | |
| | | L | R |
| U | | 2, 2 | 2, 2 |
| Ann M | | 3, 1 | 0, 0 |
| D | | 0, 0 | 1, 3 |

Figure 5.2

Appendix A. From Types to Hierarchies

In this appendix, we focus on the basic formalization, where a type of Ann is associated with a probability measure on the Borel subsets on the strategies and types of Bob. We show how, in this case, types naturally induce hierarchies of beliefs.⁵ For this construction, it may not be immediately clear how to generalize from the two-player case, so we prefer to treat the n -player case explicitly.

Given a Polish space Ω , write $\mathcal{B}(\Omega)$ for the Borel σ -algebra on Ω . Also, extend our earlier notation to write $\mathcal{M}(\Omega)$ for the space of all Borel probability measures on Ω , where $\mathcal{M}(\Omega)$ is endowed with the topology of weak convergence (and so is again Polish). Given sets X^1, \dots, X^n , write $X^{-i} = \prod_{j \neq i} X^j$.

⁵This presentation is repeated from Brandenburger and Friedenberg [11, 2008, Section 8], which itself closely follows Mertens-Zamir [20, 1985, Section 2] and Battigalli-Siniscalchi [9, 2002, Section 3].

Fix an n -player strategic-form game $\langle S^1, \dots, S^n; \pi^1, \dots, \pi^n \rangle$, where S^i is the finite set of strategies for player i and $\pi^i : S \rightarrow \mathbb{R}$ is i 's payoff function. An (S^1, \dots, S^n) -based **type structure** is a structure

$$\langle S^1, \dots, S^n; T^1, \dots, T^n; \lambda^1, \dots, \lambda^n \rangle,$$

where each T^i is a Polish space and each $\lambda^i : T^i \rightarrow \mathcal{M}(S^{-i} \times T^{-i})$ is continuous. Members of T^i are called **types** for player i .

Associated with each type t^i for each player i in a type structure is a hierarchy of beliefs about the strategies chosen. To see this, inductively define sets Y_m^i , by setting $Y_1^i = S^{-i}$ and

$$Y_{m+1}^i = Y_m^i \times \prod_{j \neq i} \mathcal{M}(Y_m^j).$$

Define continuous maps $\rho_m^i : S^{-i} \times T^{-i} \rightarrow Y_m^i$ inductively by

$$\begin{aligned} \rho_1^i(s^{-i}, t^{-i}) &= s^{-i}, \\ \rho_{m+1}^i(s^{-i}, t^{-i}) &= (\rho_m^i(s^{-i}, t^{-i}), (\delta_m^j(t^j))_{j \neq i}), \end{aligned}$$

where $\delta_m^j = \rho_m^j \circ \lambda^j$ and, for each $\mu \in \mathcal{M}(S^{-j} \times T^{-j})$, $\rho_m^j(\mu)$ is the image measure under ρ_m^j .

Standard arguments (see [11, 2008, Appendix B] for details) show that these maps are indeed continuous, and so are well-defined. Define a continuous map $\delta^i : T^i \rightarrow \prod_{m=1}^{\infty} \mathcal{M}(Y_m^i)$ by $\delta^i(t^i) = (\delta_1^i(t^i), \delta_2^i(t^i), \dots)$. (Again, see [11, 2008, Appendix B] for details.) Then $\delta^i(t^i)$ is the hierarchy of beliefs about strategies induced by type t^i .

References

- [1] Abramsky, S., and A. Jung, "Domain Theory," in Abramsky, S., D. Gabbay, and T. Maibaum (eds.), *Handbook of Logic in Computer Science*, III, Oxford University Press, 1994, 1–168.
- [2] Apt, K., "Epistemic Analysis of Strategic Games with Arbitrary Strategy Sets," in Samet, D. (ed.), *Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge (TARK XI)*, Presses univ. de Louvain, 2007, 22–38.
- [3] Apt, K., "Relative Strength of Strategy Elimination Procedures," *Economics Bulletin*, 3, 2007, 1–9.
- [4] Apt, K., "The Many Faces of Rationalizability," *The B.E. Journal of Theoretical Economics (Topics)*, 7, 2007, Article 18.
- [5] Apt, K., and J. Zvesper, "Common Beliefs and Public Announcements in Strategic Games with Arbitrary Strategy Sets," 2007. Available at <http://arxiv.org/abs/0710.3536>.
- [6] Arrow, K., E. Barankin, and D. Blackwell, "Admissible Points of Convex Sets," in Kuhn, H., and A. Tucker (eds.), *Contributions to the Theory of Games*, Volume II, Princeton University Press, 1953, 87–91.
- [7] Barwise, J., "Three Views of Common Knowledge," in Vardi, M., (ed.), *Proceedings of the Second Conference on Theoretical Aspects of Reasoning about Knowledge*, Morgan Kaufmann, 1988, 365–379.
- [8] Battigalli, P., and A. Friedenberg, "Forward Induction Reasoning Revisited," 2011. Forthcoming in *Theoretical Economics*.
- [9] Battigalli, P., and M. Siniscalchi, "Strong Belief and Forward-Induction Reasoning," *Journal of Economic Theory*, 106, 2002, 356–391.
- [10] Blume, L., A. Brandenburger, and E. Dekel, "Lexicographic Probabilities and Choice under Uncertainty," *Econometrica*, 59, 1991, 61–79.
- [11] Brandenburger, A., and A. Friedenberg, "Intrinsic Correlation in Games," *Journal of Economic Theory*, 141, 2008, 28–67.
- [12] Brandenburger, A., and A. Friedenberg, "Self-Admissible Sets," *Journal of Economic Theory*, 145, 2010, 785–811.

- [13] Brandenburger, A., A. Friedenberg, and H.J. Keisler, “Admissibility in Games,” *Econometrica*, 76, 2008, 307–352.
- [14] Brouwer, L., “Über eindeutige stetige Transformationen von Flächen in Sich,” *Mathematische Annalen*, 67, 1910, 176–180.
- [15] Davey, B., and H. Priestley, *Introduction to Lattices and Order*, 2nd edition, Cambridge University Press, 2002.
- [16] Harsanyi, J.C., “Games with Incomplete Information Played by ‘Bayesian’ Players, I-III,” *Management Science*, 1967, 159–182.
- [17] Kakutani, S., “A Generalization of Brouwer’s Fixed Point Theorem,” *Duke Mathematical Journal*, 8, 1941, 457–459.
- [18] Libkin, L. *Elements of Finite Model Theory*, Tttxts in Theoretical Computer Science, 2010.
- [19] Martin, K., “The Measurement Process in Domain Theory,” in *Proceedings of the 27th International Colloquium on Automata, Languages, and Programming*, July 2000, 116–126.
- [20] Mertens, J.-F., and S. Zamir, “Formulation of Bayesian Analysis for Games with Incomplete Information,” *International Journal of Game Theory*, 14, 1985, 1–29.
- [21] Monderer, D., and D. Samet, “Approximating Common Knowledge with Common Beliefs,” *Games and Economic Behavior*, 1, 1989, 170–190.
- [22] Moschovakis, Y. *Elementary Induction on Abstract Structures*, North-Holland Elsevier, 1974.
- [23] Nash, J., “Non-Cooperative Games,” doctoral dissertation, Princeton University, 1950.
- [24] Nash, J., “Equilibrium Points in n -Person Games,” *Proceedings of the National Academy of Sciences*, 36, 1950, 48–49.
- [25] Nash, J., “Non-Cooperative Games,” *Annals of Mathematics*, 54, 1951, 286–295.
- [26] Pearce, D., “Rational Strategic Behavior and the Problem of Perfection,” *Econometrica*, 52, 1984, 1029–1050.
- [27] Siniscalchi, M., “Epistemic Game Theory: Beliefs and Types,” in Durlauf, S., and L. Blume (eds.), *The New Palgrave Dictionary of Economics*, 2nd edition, Palgrave Macmillan, 2008.
- [28] Von Neumann, J., “Zur Theorie der Gesellschaftsspiele,” *Mathematische Annalen*, 100, 1928, 295–320.

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