Intrinsic Correlation in Games[1]

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Abstract

Correlations arise naturally in non-cooperative games, e.g., in the equivalence between undominated and optimal strategies in games with more than two players. But the non-cooperative assumption is that players do not coordinate their strategy choices, so where do these correlations come from? The epistemic view of games gives an answer. Under this view, the players' hierarchies of beliefs (beliefs, beliefs about beliefs, ...) about the strategies played in the game are part of the description of a game. This gives a source of correlation: A player believes other players' strategy choices are correlated, because he believes their hierarchies of beliefs are correlated. We refer to this kind of correlation as "intrinsic," since it comes from variables–viz., the hierarchies of beliefs–that are part of the game. We compare the intrinsic route with the "extrinsic" route taken by Aumann [2, 1974], which adds signals to the original game.
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1 Introduction

Correlation is basic to game theory. For example, consider the equivalence between undominated strategies–strategies that are not strongly dominated–and strategies that are optimal under some measure on the strategy profiles of the other players. As is well known, for this to hold in games with more than two players, the measure may need to be dependent (i.e., correlated).

In Figure 1.1, Ann chooses the row, Bob chooses the column, Charlie chooses the matrix, and the payoffs are written in the order Ann then Bob then Charlie. The strategy Y is optimal for Charlie, under a measure that puts probability $\frac{1}{2}$ on (U, L) and probability $\frac{1}{2}$ on (D, R). It is therefore undominated. But there is no product measure under which Y is optimal.[1]

	L	R
U	1,1,3	1,0,3
D	0,1,0	0,0,0

X

	L	R
U	1,1,2	0,0,0
D	0,0,0	1,1,2

Y

	L	R
U	1,1,0	1,0,0
D	0,1,3	0,0,3

Z

Figure 1.1

Where does a correlated assessment–such as probability $\frac{1}{2}$ on (U, L) and probability $\frac{1}{2}$ on (D, R)–come from? After all, the non-cooperative assumption is that players do not coordinate their strategy choices. Alternatively put, there is no physical correlation across players.

The epistemic approach to game theory suggests an answer. Under the epistemic approach, a complete description of a game includes specifying not only the players' strategy sets and payoff functions, but also their hierarchies of beliefs (beliefs, beliefs about beliefs, ...) about the strategies played in the game. This gives a source of correlation. A player can think that other players' strategy choices are correlated, because he thinks what they believe about the game is correlated. For example, in Figure 1.1, Charlie might assign: (i) probability $\frac{1}{2}$ to "Ann's having hierarchy of beliefs h^a and playing U, and Bob's having hierarchy of beliefs h^b and playing L;" and (ii) probability $\frac{1}{2}$ to "Ann's having hierarchy of beliefs $\tilde{h}^a$ and playing D, and Bob's having hierarchy of beliefs $\tilde{h}^b$ and playing R."

This paper formalizes this line of argument. There are two requirements. (Here, and throughout, we will use the shorthand "hierarchy of beliefs" for "hierarchy of beliefs about the strategies played in the game." Figure 5.1 gives a numerical example of such hierarchies.)

(a) We impose a **conditional independence** (**CI**) requirement on hierarchies of beliefs. Conditional on Ann's and Bob's hierarchies of beliefs, Charlie assesses Ann's and Bob's strategies are independent. (But this needn't hold unconditionally!)

[1] Let p be the probability on U, and q be the probability on L. It is straightforward to check that $\max\{3p, 3(1-p)\} > 2pq + 2(1-p)(1-q)$.

(b) We impose a **sufficiency** (**SUFF**) requirement on hierarchies of beliefs. Conditional on Ann's hierarchy of beliefs, Charlie's assessment about Ann's strategy doesn't change if she[2] learns Bob's hierarchy of beliefs.

Both requirements are immediately satisfied in the example above. We show later that under CI and SUFF, if a player has an independent assessment about the other players' hierarchies of beliefs, he must have an independent assessment about their strategy choices (Proposition 9.1). Equivalently, if a player has a correlated assessment about the other players' strategy choices, he must have a correlated assessment about their hierarchies of beliefs. This is exactly our view of correlation: The correlation in strategies is 'non-physical' as per non-cooperative game theory, and comes from correlation in what the players believe about the play of the game.

What strategies can be played under our view of correlation? Our restrictions limit how Charlie (for example) thinks Ann's and Bob's strategies and hierarchies of beliefs are related. For these restrictions to matter, we need to be in a setting where there is a connection between strategies and hierarchies. Rationality is the natural such connection–a strategy may be optimal under some (first-order) beliefs but not under others. This leads to a third requirement:

(c) We impose **rationality and common belief of rationality** (**RCBR**). That is, each player is rational, assigns probability 1 to the event the other players are rational, and so on. This is the usual 'baseline' epistemic condition on a game.

In sum, this paper asks: What strategies can be played in a game under the requirements of CI, SUFF, and RCBR?

At a broad level, our analysis of correlation in games is analogous to the Bayesian or subjective view of coin tossing. To rule out physical correlation across tosses, one subscribes to an assessment that is conditionally independent given an extra variable not included in the original description–in this case, the parameter or 'bias' of the coin. For the game setting, there are also extra variables–in this paper, they are the players' hierarchies of beliefs about the strategies played in the game. We ask for conditional independence relative to these variables. We also impose a sufficiency condition to ensure that we 'attach' the correct extra variable to the correct player.

2 Intrinsic vs. Extrinsic Correlation

We will refer to correlation of the kind we study in this paper as **intrinsic correlation**, because the correlations come from variables–viz., hierarchies of beliefs–that are part of the description of the game. (At least, they are part of the game under the epistemic approach to game theory.)

The existing route, initiated by Aumann [2], can be termed **extrinsic correlation**. This is because it adds to the given game payoff-irrelevant moves by Nature (often called "signals,"

[2] In this paper, Charlie is female.

"sunspots," or similar). Figure 2.1 is a typical scenario. A coin is tossed. Ann and Bob observe the outcome and choose strategies as shown. Charlie doesn't observe the outcome, and assigns probabilities $\frac{1}{2}$ to $Heads$ and $\frac{1}{2}$ to $Tails$. Thus, Charlie has a correlated assessment about Ann's and Bob's strategies, because Ann and Bob get a correlated signal and choose strategies according to the realization of the signal.

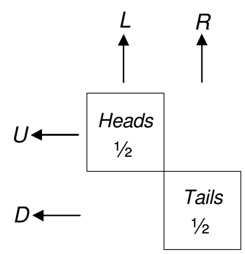

Figure 2.1

What is the relationship between intrinsic and extrinsic correlation? We will answer at both the formal and the conceptual level.

Aumann [2], [3] gave two definitions: objective and subjective correlated equilibrium, according as there is or is not a common prior. The idea of a common prior is irrelevant to this project, so we will focus on the subjective case.

The following equivalence gives the relationship to intrinsic correlation. Fix a game G. The set of strategy profiles that can be played in some subjective correlated equilibrium of G is the same as the set of correlated rationalizable profiles in G. (See Brandenburger-Dekel [7]. The **correlated rationalizable** strategies are those that survive iterated elimination of strategies that are never best replies–or, equivalently, are dominated.) It is well known that the condition of RCBR in a game is characterized by correlated rationalizability. (This result goes back to [7] and Tan-Werlang [23]. Proposition 10.1 is a statement in the set-up of this paper.) It follows that intrinsic correlation is a weakly stronger theory than extrinsic correlation: Any strategy profile allowed under intrinsic correlation is certainly allowed under extrinsic correlation.

But–surprisingly, in our view–the inclusion can be strict. The main result of the paper (Theorem 11.1) exhibits a game G in which there is a correlated rationalizable strategy that cannot be played under intrinsic correlation. The two theories of correlation are different.

We also record the relationship between intrinsic correlation and the Bernheim [6] and Pearce [20] concept of independent rationalizability. (The **independent rationalizable** strategies are those that survive iterated elimination of strategies that are never best replies under product measures.) Proposition G1 says that any independent rationalizable profile is allowed under intrinsic correlation. This isn't quite obvious. Under independent rationalizability, each player assesses the other players' strategy choices as independent. But, as is well known in probability theory, independence doesn't imply conditional independence. So, demonstrating this inclusion takes some work. The inclusion is strict: The game of Figure 1.1 is an example. Strategy Y can be played under intrinsic correlation (see Footnote 5 for the details), but it is easy to check that only X is independent rationalizable. (So, the expected payoffs to Charlie are also different.)

Intrinsic correlation is a distinct theory of how games are played. Refer to Figure 2.2.

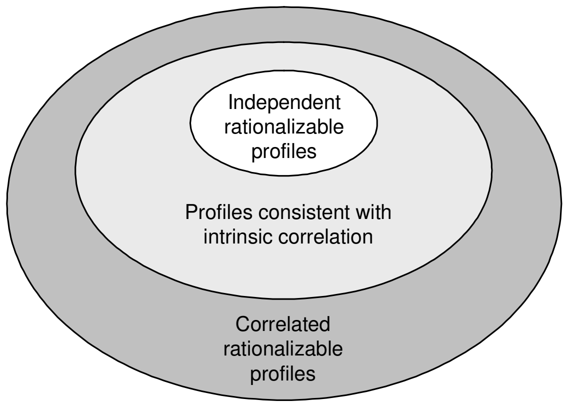

Figure 2.2

3 Comparison

We now ask about the conceptual meaning of intrinsic correlation vs. the other routes in Figure 2.2.

Bernheim [6, p.1014] argued for independent rationalizability, on the grounds that "in a non-cooperative framework ... the choices of any two agents are by definition independent events; they cannot affect each other." Likewise, Pearce [20, p.1035] distinguishes independent rationalizability from the situation where "a player's opponents could *coordinate* their randomized strategic actions."

The epistemic view of games reaches a different conclusion. Even if players choose strategies independently, correlation is possible because some aspect of who the players are (their hierarchies of beliefs) may be correlated. This line of reasoning–the intrinsic route–is really just an adaptation to game theory of the usual idea of common-cause correlation.

The Bernheim-Pearce analysis now appears as a special case. Suppose each player thinks that the other players' hierarchies of beliefs are uncorrelated. Then, under CI and SUFF, our Proposition 9.1 implies that each player thinks the other players' strategy choices are uncorrelated. If we add RCBR, we get exactly independent rationalizability.[3]

[3] So, as a by-product, we also get new epistemic conditions for independent rationalizability. Existing treatments (e.g., Tan-Werlang [23]) directly assume that each player assesses the other players' choices are independent. Our results show that this can be deduced from more basic conditions on the hierarchies of beliefs. See Corollary G1 for a formal statement.

Why do we prefer the case of correlated hierarchies of beliefs? To us, such correlation is in line with Savage's Small-Worlds idea in decision theory ([22, pp.82-91]). The given game is only a part of a larger whole. In particular, there is a 'context' or 'history' to the game. Who the players are then becomes a shorthand for their prior experiences. That is, the players' prior experiences (partially) determine their characteristics–including their beliefs, beliefs about beliefs, etc. It seems natural that these experiences could be (though don't have to be) correlated, in which case the players' hierarchies of beliefs could be correlated, too.

Next, the distinction between the intrinsic and extrinsic (Aumann) routes. We establish they are formally different. (This is our main result, Theorem 11.1.) But how do they differ conceptually?

The answer is that Aumann takes the next step and actually changes the game to include its 'context.' The signals in his analysis are what the players see before they play the given game–and these signals are added to the game. (Recall that the correlated equilibria of a game are the Nash equilibria of the extended game.) So, the starting point for Aumann is also the Savage Small-Worlds idea.[4] But Aumann's and our analysis then differ in whether or not the 'context' of the game is made part of the game.

We should point out a subtlety: Our approach certainly allows for signals beyond the game as given. Signals might give a story for why the players have the beliefs they have. But the signals would remain in the background and are not brought into the game itself.

Summarizing, both the intrinsic and the extrinsic routes to correlation in games get correlation by recognizing the larger context in which a game is played. The difference is whether we analyze the game as originally given or the larger game.

Logically, the intrinsic route comes before the extrinsic route (per Figure 2.2). Historically, the extrinsic route came first. There is a simple 'mechanical' reason for this. Correlation requires additional variables. Nowadays, with the epistemic view of games well established, these variables are immediately at hand–they are the players' hierarchies of beliefs. But Aumann, writing prior to the epistemic program ([2]), had to look beyond the game to find extra variables. This is why the outer part of Figure 2.2 (extrinsic correlation) came before the middle part (intrinsic correlation). In a literal way, we see this paper as filling in the existing picture of correlation in games.

4 Organization of the Paper

In Sections 5-7, we give a heuristic treatment of our theory of correlation. The formal presentation and proofs are in Sections 8-11 and accompanying appendices. (The formal treatment turns out to be quite involved and lengthy.) Section 12 concludes. The heuristic treatment can be read either before or in parallel with the formal treatment.

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[4] We thank a referee for clarifying the connection to Savage [22].

5 Type Structures

Return to the game in Figure 1.1. Even if the matrix itself is 'transparent' to the players, each player may be uncertain about the strategies chosen by the other players, about what the other players believe about the strategies chosen, and so on. A key feature of the epistemic approach is incorporating this kind of uncertainty into the description of a game situation.

Type t^a

$S^c \times T^c$ \ $S^b \times T^b$	(L, t^b)	(R, t^b)	(L, u^b)	(R, u^b)
(Z, t^c)	0	0	0	0
(Y, t^c)	1	0	0	0
(X, t^c)	0	0	0	0

Type u^a

$S^c \times T^c$ \ $S^b \times T^b$	(L, t^b)	(R, t^b)	(L, u^b)	(R, u^b)
(Z, t^c)	0	0	0	0
(Y, t^c)	0	0	0	1
(X, t^c)	0	0	0	0

Type t^b

$S^c \times T^c$ \ $S^a \times T^a$	(U, t^a)	(D, t^a)	(U, u^a)	(D, u^a)
(Z, t^c)	0	0	0	0
(Y, t^c)	1	0	0	0
(X, t^c)	0	0	0	0

Type u^b

$S^c \times T^c$ \ $S^a \times T^a$	(U, t^a)	(D, t^a)	(U, u^a)	(D, u^a)
(Z, t^c)	0	0	0	0
(Y, t^c)	0	0	0	1
(X, t^c)	0	0	0	0

Type t^c

$S^b \times T^b$ \ $S^a \times T^a$	(U, t^a)	(D, t^a)	(U, u^a)	(D, u^a)
(R, u^b)	0	0	0	½
(L, u^b)	0	0	0	0
(R, t^b)	0	0	0	0
(L, t^b)	½	0	0	0

Figure 5.1

Figure 5.1 is an example of a **type structure** which, when added to a game like Figure 1.1, gives what can be called an epistemic game. The type structure shown describes the players' possible hierarchies of beliefs about the strategies chosen, as follows. There are two **types** for Ann, viz. t^a and u^a, two types for Bob, viz. t^b and u^b, and one type for Charlie, viz. t^c. Each type is associated with a probability measure on the strategies and types of other players. In the usual way, each type induces a **hierarchy of beliefs** about the strategies in the game.

Thus, type t^c assigns probability $\frac{1}{2}$ to (U, t^a, L, t^b) and probability $\frac{1}{2}$ to (D, u^a, R, u^b), and so has a first-order belief that assigns: (i) probability $\frac{1}{2}$ to "Ann's playing U and Bob's playing L;" and (ii) probability $\frac{1}{2}$ to "Ann's playing D and Bob's playing R." Type t^a has a first-order belief that assigns probability 1 to (L, Y), and type u^a has a first-order belief that assigns probability 1 to (R, Y). Type t^b has a first-order belief that assigns probability 1 to (U, Y), and type u^b has a first-order belief that assigns probability 1 to (D,Y). So, type t^c has a second-order belief that assigns: (i) probability $\frac{1}{2}$ to "Ann's playing U and assigning probability 1 to (L,Y)," and "Bob's playing L and assigning probability 1 to (U,Y);" and (ii) probability $\frac{1}{2}$ to "Ann's playing D and assigning probability 1 to (R,Y)," and "Bob's playing R and assigning probability 1 to (D,Y)." And so on.

Notice that Charlie's type t^c has a correlated assessment about the strategies Ann and Bob choose. Charlie also has a correlated assessment about Ann's and Bob's hierarchies of beliefs. To see this, note that Ann's types t^a and u^a (resp. Bob's types t^b and u^b) are associated with distinct hierarchies. (Associating different hierarchies with different types keeps the example simple. We don't impose this condition in our formal treatment.)

Next, we want to capture our idea that Charlie's correlated assessment about Ann's and Bob's strategies comes from the fact that she is uncertain about their hierarchies of beliefs. Indeed, if this is the source of the correlation, then, if Charlie's uncertainty about Ann's and Bob's hierarchies is resolved, she should have an independent assessment about their strategies. This leads to the CI condition from Section 1.

But CI alone doesn't fully capture the meaning of correlation via hierarchies of beliefs. We need to take account of the information in the game. Ann's information is her own hierarchy, not Bob's. So, Bob's hierarchy should provide information about Ann's choice of strategy only insofar as it provides information about her hierarchy of beliefs. It follows that if Charlie's uncertainty about Ann's hierarchy is resolved, she should not change her assessment about Ann's strategy if she learns Bob's hierarchy. (And vice versa with Ann and Bob interchanged.) This leads to the SUFF condition from Section 1.

Let us state CI and SUFF in the context of the example:

(a) **Conditional Independence (CI)** The measure associated with each type assesses the other players' strategy choices as independent, conditional on their hierarchies of beliefs. Consider, for example, Charlie's type t^c and the associated measure–which we will denote $\lambda^c(t^c)$. We have

$$\lambda^c(t^c)(U,L|t^a,t^b) = \lambda^c(t^c)(U|t^a,t^b) \times \lambda^c(t^c)(L|t^a,t^b).$$

(Here and below, the conditioning is to be thought of as on the hierarchies. This again uses the fact that for Ann and Bob, different types induce different hierarchies.) The corresponding equality holds when conditioning on (the hierarchies associated with) u^a and u^b. So, Charlie's type t^c assesses Ann's and Bob's strategies as independent conditional on their hierarchies.

(b) **Sufficiency (SUFF)** We have

$$\lambda^c(t^c)(U|t^a,t^b) = \lambda^c(t^c)(U|t^a),$$

and similarly for D. If Charlie knows Ann's hierarchy of beliefs, and comes to learn Bob's hierarchy of beliefs too, this won't change her (Charlie's) assessment of Ann's choice. Likewise

$$\lambda^c(t^c)(L|t^a, t^b) = \lambda^c(t^c)(L|t^b),$$

and similarly for R. Ann's hierarchy of beliefs is sufficient for the assessment Charlie's type t^c makes about her choice of strategy. Bob's hierarchy of beliefs is sufficient in the same way.

We repeat that we show later (Proposition 9.1): *Under CI and SUFF, if a player has an independent assessment about the other players' hierarchies of beliefs, he must have an independent assessment about their strategy choices.* (In Appendix A, we show that the CI and SUFF conditions are independent, and neither can be dropped in this result.) CI and SUFF give us correlation in strategies from correlation in hierarchies of beliefs (about the strategies). This is our theory of intrinsic correlation.[5]

6 The Main Result

What is the prediction of intrinsic correlation? Figure 2.2 gave the relationship to independent and correlated rationalizability. Here, we give a heuristic treatment of our main finding–that the second inclusion is strict. There are correlated rationalizable strategies that cannot be played under intrinsic correlation. In fact, we show:

(i) *There is a game G and a correlated rationalizable strategy in G that cannot be played under RCBR and CI.*

(ii) *There is a game G' and a correlated rationalizable strategy in G' that cannot be played under RCBR and SUFF.*

We now give a sketch of the proofs. (The formal proofs are quite involved and are given in Section 11.)

For statement (i), consider the game in Figure 6.1. Here, all strategies are correlated rationalizable. Yet we will argue that if Y is consistent with RCBR then Charlie's type cannot satisfy CI.

Fix an associated type structure. First note the following three facts:

(a) If strategy-type pairs (U, t^a) and (M, u^a) are rational, then t^a and u^a must each assign probability 1 to the strategies (L, Y).

[5] The type structure of Figure 5.1 proves the claim in Section 2 that Y can be played under intrinsic correlation. All types satisfy CI and SUFF. (We checked for Charlie's type t^c above. The checks for Ann's types t^a, u^a and Bob's types t^b, u^b are immediate, since each of these types is associated with a degenerate measure.) For Ann, the strategy-type pairs (U, t^a) and (D, u^a) are rational: Strategy U (resp. D) maximizes her expected payoff under the marginal on $S^b \times S^c$ of the measure associated with t^a (resp. u^a). Similarly, (L, t^b) and (R, u^b) are rational for Bob; and (Y, t^c) is rational for Charlie. Also, each type for each player assigns positive probability only to rational strategy-type pairs for the other players. That is, each player believes the other players are rational. By induction, each of these strategy-type pairs is therefore consistent with RCBR. In particular, Charlie can play Y.

(b) If $\left(L, t^{b}\right)$ and $\left(C, u^{b}\right)$ are rational, then t^{b} and u^{b} must each assign probability 1 to the strategies (U, Y).

(c) If (Y, t^{c}) and (Y, u^{c}) are rational, then t^{c} and u^{c} must each assign probability $\frac{1}{2}$ to (U, L) and probability $\frac{1}{2}$ to (M, C).

	L	C	R
U	0,0,2	0,0,2	0,1,2
M	0,0,0	0,0,0	0,1,0
D	1,0,2	1,0,2	1,1,2

X

	L	C	R
U	1,1,1	0,1,0	0,1,0
M	1,0,0	0,0,1	0,1,0
D	1,0,0	1,0,0	1,1,0

Y

	L	C	R
U	0,0,0	0,0,0	0,1,0
M	0,0,2	0,0,2	0,1,2
D	1,0,2	1,0,2	1,1,2

Z

Figure 6.1

In words, U and M are each optimal only if Ann assigns probability 1 to (L, Y). Likewise, L and C are each optimal only if Bob assigns probability 1 to (U, Y). Finally, Y is optimal only if Charlie assigns probability $\frac{1}{2}$ to (U, L) and probability $\frac{1}{2}$ to (M, C).

It follows that if (U, t^{a}) and (M, u^{a}) are rational, then the hierarchies of beliefs associated with t^{a} and u^{a} must agree at the first level. Likewise for $\left(L, t^{b}\right)$ and $\left(C, u^{b}\right)$, and for (Y, t^{c}) and (Y, u^{c}).

Next, suppose that (U, t^{a}) and (M, u^{a}) are rational and t^{a} and u^{a} believe the other players are rational. We already know that t^{a} and u^{a} must each assign probability 1 to (L, Y). Given this, they must also assign probability 1 to: (i) Bob's assigning probability 1 to (U, Y); and (ii) Charlie's assigning probability $\frac{1}{2}$ to (U, L) and probability $\frac{1}{2}$ to (M, C). This follows from (b) and (c) above. Thus, the hierarchies of beliefs associated with t^{a} and u^{a} must agree up to the second level. And so on, inductively.

This gives:

(a′) If (U, t^{a}) and (M, u^{a}) are consistent with RCBR, then t^{a} and u^{a} must induce the same hierarchy of beliefs.

(b′) If $\left(L, t^{b}\right)$ and $\left(C, u^{b}\right)$ are consistent with RCBR, then t^{b} and u^{b} must induce the same hierarchy of beliefs.

Let h^{a} (resp. h^{b}) be this hierarchy of beliefs for Ann (resp. Bob). Now refer to Figure 6.2. We know that for Y to be consistent with even rationality (a fortiori, RCBR), Charlie must assign probability $\frac{1}{2}$ to each of (U, L) and (M, C). Also, Charlie must assign probability 1 to the event "RCBR with respect to Ann and Bob." (This uses a conjunction property of belief.) Taken together with (a′) and (b′) above, this means that Charlie must assign probability $\frac{1}{2}$ to each of the two indicated points on the (h^a, h^b)-plane. But CI requires Charlie's conditional measure, conditioned on any such horizontal plane, to be a product measure. So we have a contradiction, completing the argument for statement (i).

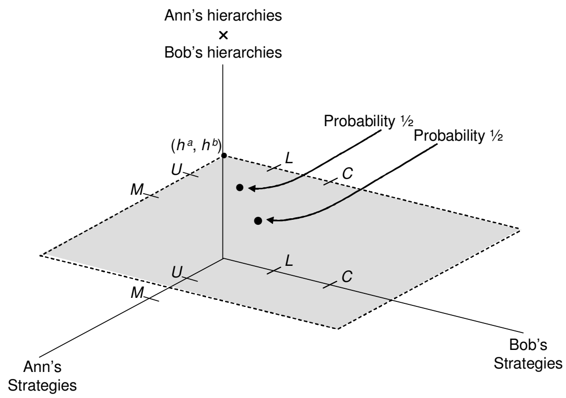

Figure 6.2

The argument for statement (ii) is very similar. This time, consider the game in Figure 6.3. (This is the same as Figure 6.1, except for swapping Bob's payoffs in (U, C, Y) and (M, C, Y).) Again, all strategies are correlated rationalizable. We will see that if Y is consistent with RCBR then Charlie's type cannot satisfy SUFF.

X

	L	C	R
U	0,0,2	0,0,2	0,1,2
M	0,0,0	0,0,0	0,1,0
D	1,0,2	1,0,2	1,1,2

Y

	L	C	R
U	1,1,1	0,0,0	0,1,0
M	1,0,0	0,1,1	0,1,0
D	1,0,0	1,0,0	1,1,0

Z

	L	C	R
U	0,0,0	0,0,0	0,1,0
M	0,0,2	0,0,2	0,1,2
D	1,0,2	1,0,2	1,1,2

Figure 6.3

Fix an associated type structure, and note the following four facts:

(a) If (U, t^a) and (M, u^a) are rational, then t^a and u^a must each assign probability 1 to the strategies (L, Y).

(b_L) If (L, t^b) and (L, u^b) are rational, then t^b and u^b must each assign probability 1 to the strategies (U, Y).

(b_C) If (C, v^b) and (C, w^b) are rational, then v^b and w^b must each assign probability 1 to the strategies (M, Y).

(c) If (Y, t^c) and (Y, u^c) are rational, then t^c and u^c must each assign probability $\frac{1}{2}$ to (U, L) and probability $\frac{1}{2}$ to (M, C).

It follows that if (U, t^a) and (M, u^a) are rational, then the hierarchies of beliefs associated with t^a and u^a must agree at the first level. Likewise for (L, t^b) and (L, u^b), for (C, v^b) and (C, w^b), and for (Y, t^c) and (Y, u^c).

Using a similar argument to above:

(a′) If (U, t^a) and (M, u^a) are consistent with RCBR, then t^a and u^a must be associated with the same hierarchy of beliefs.

(b'_L) If (L, t^b) and (L, u^b) are consistent with RCBR, then t^b and u^b must be associated with the same hierarchy of beliefs.

(b'_C) If (C, v^b) and (C, w^b) are consistent with RCBR, then v^b and w^b must be associated with the same hierarchy of beliefs.

Write h^a for the hierarchy associated with requirement (a′). Write h^b_L (resp. h^b_C) for the hierarchy associated with requirement (b'_L) (resp. (b'_C)). Note, by (b_L)-(b_C), the hierarchies h^b_L and h^b_C are necessarily distinct.

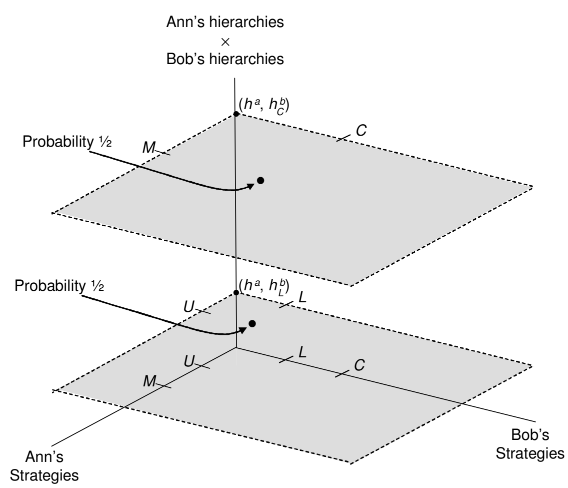

Figure 6.4

Now refer to Figure 6.4. Charlie must assign probability $\frac{1}{2}$ to each of (U, L) and (M, C), and must assign probability 1 to the event "RCBR with respect to Ann and Bob." Together with (a′)-

(b'_C), this means that Charlie must assign probability $\frac{1}{2}$ to the point (U, L) on the $\left(h^a, h^b_L\right)$-plane and probability $\frac{1}{2}$ to the point (M, C) on the $\left(h^a, h^b_C\right)$-plane. Notice, Charlie assigns: (i) probability $\frac{1}{2}$ to U, conditional on Ann's hierarchy h^a; and (ii) probability 1 to U, conditional on Ann's hierarchy h^a and Bob's hierarchy h^b_L. This contradicts SUFF, completing the argument for statement (ii).

We have shown that there is a gap between the middle and outer sets in Figure 2.2. The raises the question: How 'big' is this gap? In Appendix H, we give a sufficient condition on a game under which the two sets coincide, and also discuss genericity properties.[6]

Finally, we note an open question. Fix a game G. We say a strategy in G can be played under intrinsic correlation if it can be played under RCBR, CI, and SUFF. But these conditions refer to some type structure. Obviously, it would be desirable to have a (type-free) characterization of the strategies that can be played. Unfortunately, we don't have one–and leave this to future work.

7 Comparison contd.

Heads

	L	C	R		L	C	R		L	C	R
U	0,0,2	0,0,2	0,1,2	U	1,1,1	0,1,0	0,1,0	U	0,0,0	0,0,0	0,1,0
M	0,0,0	0,0,0	0,1,0	M	1,0,0	0,0,1	0,1,0	M	0,0,2	0,0,2	0,1,2
D	1,0,2	1,0,2	1,1,2	D	1,0,0	1,0,0	1,1,0	D	1,0,2	1,0,2	1,1,2
		X				Y				Z	

Tails

	L	C	R		L	C	R		L	C	R
U	0,0,2	0,0,2	0,1,2	U	1,1,1	0,1,0	0,1,0	U	0,0,0	0,0,0	0,1,0
M	0,0,0	0,0,0	0,1,0	M	1,0,0	0,0,1	0,1,0	M	0,0,2	0,0,2	0,1,2
D	1,0,2	1,0,2	1,1,2	D	1,0,0	1,0,0	1,1,0	D	1,0,2	1,0,2	1,1,2
		X				Y				Z	

Figure 7.1

Let us return to the comparison of intrinsic and extrinsic correlation–now in the specific case of the key game of Figure 6.1. We give an explicit construction–in the style of Aumann [2]–under which Y can be played under RCBR. In Figure 7.1, which adds a coin toss to the original game, Ann, Bob, Charlie, and Nature move simultaneously. Figure 7.2 is an associated type structure, where each player's type is now associated with a measure on the strategies and types for the other players–and the outcome *Heads* vs. *Tails* of the coin toss.

[6] It would be equally interesting to find conditions under which the inner and middle sets in Figure 2.2 coincide. We don't have any results on this.

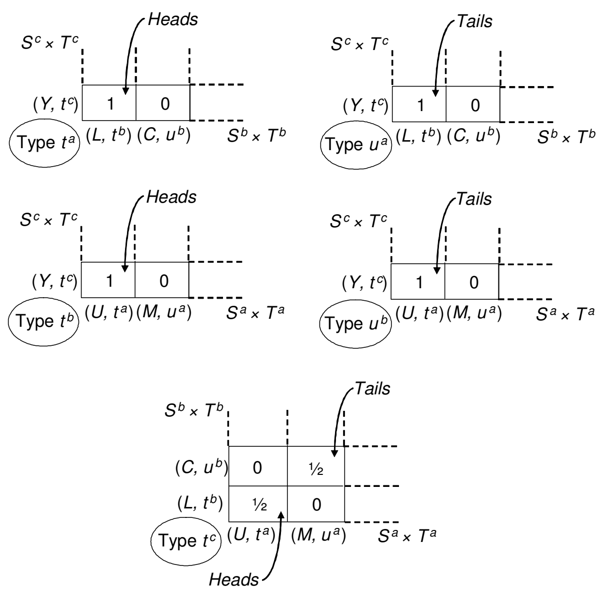

Figure 7.2

There are two types for Ann, t^a and u^a, two types for Bob, t^b and u^b, and one type for Charlie t^c. Ann's type t^a assigns probability 1 to $\left(L, t^b, Y, t^c, Heads\right)$, while her type u^a assigns probability 1 to $\left(L, t^b, Y, t^c, Tails\right)$; and similarly for Bob's types t^b and u^b, as shown. Charlie's (unique) type t^c assigns probability $\frac{1}{2}$ to $(U, t^a, L, t^b, Heads)$ and probability $\frac{1}{2}$ to $(D, u^a, R, u^b, Tails)$. Notice that each type assigns positive probability only to rational strategy-type pairs of the other players. So, by induction, RCBR holds at the state $\left(U, t^a, L, t^b, Y, t^c, Heads\right)$, for example.

The construction also satisfies CI–defined for the game with a coin toss. Ann's types t^a and u^a induce different hierarchies of beliefs over what she is uncertain about–viz., the strategies played and the outcome of the coin toss. Likewise for Bob's types t^b and u^b. So, when she conditions on Ann's and Bob's hierarchies of beliefs (now about strategies and the coin toss), Charlie gets a degenerate (marginal) measure on their strategies–probability 1 on (U, L) or probability 1 on (M, C). CI is immediately satisfied. It is easy to check that SUFF–defined for the game with a coin toss–is likewise satisfied.

The conclusion is that in the game with a coin toss, Charlie can now play Y under the conditions of RCBR, CI, and SUFF–unlike what we found in Section 6. This construction is general: *Starting with any game, if we add a (nondegenerate) coin toss to the game, then we can construct a type structure where: (i) each type satisfies CI and SUFF; and (ii) for each correlated rationalizable profile there is a state at which there is RCBR and the profile is played.*[7]

Of course, the reason for the different conclusion is that when we add a coin toss (or any other

[7] See Appendix I for the proof, and also a discussion of other sources of extrinsic correlation.

external signal), we change the game. We are no longer analyzing the original game. Intrinsic correlation doesn't allow this.

The distinction between variables internal vs. external to the game also arises in the literature on "interim rationalizability." (See Battigalli-Siniscalchi [5], Ely-Peski [10] and Dekel-Fudenberg-Morris [8].) An important issue in this literature is the presence or absence of redundant types–i.e., two or more types for a player that induce the same hierarchy of beliefs (Mertens-Zamir [17, Definition 2.4]). Conceptually, redundant types arise if the analysis is done on a partial state space–in this case, we are to think that there are signals external to the game as given. (Liu [13] gives a formal treatment of signals for the case of Bayesian equilibrium.)

We define and study the intrinsic route. For us, then, signals affect the analysis via the players' hierarchies of beliefs about the strategies chosen. Go back to the statements of CI and SUFF in Section 5 and note that the conditioning is on hierarchies and not types.

8 Formal Presentation

We now begin the formal treatment. Given a Polish space Ω, write $\mathcal{B}(\Omega)$ for the Borel σ-algebra on Ω. Also, write $\mathcal{M}(\Omega)$ for the space of all Borel probability measures on Ω, where $\mathcal{M}(\Omega)$ is endowed with the topology of weak convergence (and so is again Polish).

Given sets $X^1, \ldots, X^n$, write $X = \prod_{i=1}^n X^i$, $X^{-i} = \prod_{j \neq i} X^j$, and $X^{-i-j} = \prod_{k \neq i,j} X^k$. (Throughout, we adopt the convention that if X is a product set and $X^j = \emptyset$ then $X^i = \emptyset$ for all i.) An n-**player strategic-form game** is given by $G = \langle S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \rangle$, where S^i is player i's finite strategy set and $\pi^i : S \to \mathbb{R}$ is i's payoff function. Extend π^i to $\mathcal{M}(S)$ in the usual way.

Definition 8.1 *Fix an n-player strategic-form game G. An $(S^1, \ldots, S^n)$-**based type structure** is a structure*

$$\Phi = \langle S^1, \ldots, S^n; T^1, \ldots, T^n; \lambda^1, \ldots, \lambda^n \rangle,$$

*where each T^i is a Polish space and each $\lambda^i : T^i \to \mathcal{M}(S^{-i} \times T^{-i})$ is continuous. Members of T^i are called **types** of player i. Members of $S \times T$ are called **states**.*

Associated with each type t^i for each player i in a type structure Φ is a hierarchy of beliefs about the strategies played.[8] To see this, inductively define sets Y_m^i, by setting $Y_1^i = S^{-i}$ and

$$Y_{m+1}^i = Y_m^i \times \prod_{j \neq i} \mathcal{M}(Y_m^j).$$

Now define continuous maps $\rho_m^i : S^{-i} \times T^{-i} \to Y_m^i$ inductively by

$$\rho_1^i \left(s^{-i}, t^{-i}\right) = s^{-i},$$
$$\rho_{m+1}^i \left(s^{-i}, t^{-i}\right) = (\rho_m^i \left(s^{-i}, t^{-i}\right), (\delta_m^j \left(t^j\right))_{j \neq i}),$$

[8] The formulation below closely follows Mertens-Zamir [17, Section 2] and Battigalli-Siniscalchi [4, Section 3].

where $\delta_m^j = \underline{\rho}_m^j \circ \lambda^j$ and, for each $\mu \in \mathcal{M}\left(S^{-j} \times T^{-j}\right)$, $\underline{\rho}_m^j(\mu)$ is the image measure under ρ_m^j. (Appendix B shows that these maps are indeed continuous and so are well-defined.) Define a continuous map $\delta^i : T^i \to \prod_{m=1}^{\infty} \mathcal{M}\left(Y_m^i\right)$ by $\delta^i\left(t^i\right) = \left(\delta_1^i\left(t^i\right), \delta_2^i\left(t^i\right), \ldots\right)$. (Again, see Appendix B.) In words, $\delta^i\left(t^i\right)$ is simply the hierarchy of beliefs (about strategies) induced by type t^i.

For each player i, define a map $\delta^{-i} : T^{-i} \to \prod_{j \neq i} \prod_{m=1}^{\infty} \mathcal{M}(Y_m^j)$ by

$$\delta^{-i}\left(t^1, \ldots, t^{i-1}, t^{i+1}, \ldots, t^n\right) = \left(\delta^1\left(t^1\right), \ldots, \delta^{i-1}\left(t^{i-1}\right), \delta^{i+1}\left(t^{i+1}\right), \ldots, \delta^n\left(t^n\right)\right).$$

Since each δ^j is continuous, δ^{-i} is continuous.

Later, we will use the following definitions:

Definition 8.2 *The map δ^i is **bimeasurable** if, for each Borel subset E of T^i, the image $\delta^i(E)$ is a Borel subset of $\prod_{m=1}^{\infty} \mathcal{M}\left(Y_m^i\right)$. The type structure Φ is **bimeasurable** if, for each i, the map δ^i is bimeasurable.*

Applying a theorem due to Purves [21] (see also Mauldin [15]), the map δ^i is bimeasurable if and only if the set

$$\{y \in \prod_{m=1}^{\infty} \mathcal{M}\left(Y_m^i\right) : \left(\delta^i\right)^{-1}(y) \text{ is uncountable}\}$$

is countable.

Recall that types $t^i, u^i \in T^i$ are **redundant** (Mertens-Zamir [17, Definition 2.4]) if they induce the same hierarchies of beliefs, i.e. $\delta^i\left(t^i\right) = \delta^i\left(u^i\right)$. So, a type structure is bimeasurable if and only if, for each player i, there are (at most) a countable number of uncountable redundancies. In particular, a non-redundant type structure, i.e., a structure where each type induces a distinct hierarchy of beliefs, is bimeasurable. Any belief-closed subset (Mertens-Zamir [17, Definition 2.15]) of the universal type structure is bimeasurable.

9 CI and SUFF Formalized

Fix a type structure Φ, and a player $i = 1, \ldots, n$. For each $j \neq i$, define random variables $\overrightarrow{s}_i^j$ and $\overrightarrow{t}_i^j$ on $S^{-i} \times T^{-i}$ by $\overrightarrow{s}_i^j = \text{proj}_{S^j}$ and $\overrightarrow{t}_i^j = \text{proj}_{T^j}$. (Here proj denotes the projection map.) Let $\overrightarrow{s}_i$ and $\overrightarrow{t}_i$ be random variables on $S^{-i} \times T^{-i}$ with $\overrightarrow{s}_i = \text{proj}_{S^{-i}}$ and $\overrightarrow{t}_i = \text{proj}_{T^{-i}}$. Also, define the composite maps $\eta_i^j = \delta^j \circ \overrightarrow{t}_i^j$ and $\eta^{-i} = \delta^{-i} \circ \overrightarrow{t}_i$.

Write $\sigma(\overrightarrow{s}_i^j)$ (resp. $\sigma(\overrightarrow{s}_i)$, $\sigma(\eta_i^j)$, $\sigma(\eta^{-i})$) for the σ-algebra on $S^{-i} \times T^{-i}$ generated by $\overrightarrow{s}_i^j$ (resp. $\overrightarrow{s}_i$, η_i^j, η^{-i}). Similarly, let $\sigma(\overrightarrow{s}_i^j : j \neq i)$ (resp. $\sigma(\eta_i^j : j \neq i)$) be the σ-algebra on $S^{-i} \times T^{-i}$ generated by the random variables $\overrightarrow{s}_i^1, \ldots, \overrightarrow{s}_i^{i-1}, \overrightarrow{s}_i^{i+1}, \ldots, \overrightarrow{s}_i^n$ (resp. $\eta_i^1, \ldots, \eta_i^{i-1}, \eta_i^{i+1}, \ldots, \eta_i^n$). Note that $\sigma(\overrightarrow{s}_i^j : j \neq i) = \sigma(\overrightarrow{s}_i)$ and $\sigma(\eta_i^j : j \neq i) = \sigma(\eta^{-i})$. (See Dellacherie-Meyer [9, p.9].)

Fix a measure $\lambda^i(t^i) \in \mathcal{M}\left(S^{-i} \times T^{-i}\right)$, an event $E \in \mathcal{B}\left(S^{-i} \times T^{-i}\right)$, and a sub σ-algebra $\mathcal{S}$ of $\mathcal{B}\left(S^{-i} \times T^{-i}\right)$. Write $\lambda^i(t^i)(E||\mathcal{S}) : S^{-i} \times T^{-i} \to \mathbb{R}$ for a version of conditional probability of E given $\mathcal{S}$.

Definition 9.1 *The random variables* $\overrightarrow{s}_i^1, \ldots, \overrightarrow{s}_i^{i-1}, \overrightarrow{s}_i^{i+1}, \ldots, \overrightarrow{s}_i^n$ *are* $\lambda^i(t^i)$***-conditionally independent given the random variable*** η^{-i} *if, for all* $j \neq i$ *and* $E^j \in \sigma(\overrightarrow{s}_i^j)$,

$$\lambda^i\left(t^i\right)\left(\bigcap_{j\neq i}E^j||\sigma(\eta^{-i})\right) = \prod_{j\neq i}\lambda^i\left(t^i\right)\left(E^j||\sigma(\eta^{-i})\right) \quad a.s.$$

Say the type t^i *satisfies* ***conditional independence (CI)*** *if* $\overrightarrow{s}_i^1, \ldots, \overrightarrow{s}_i^{i-1}, \overrightarrow{s}_i^{i+1}, \ldots, \overrightarrow{s}_i^n$ *are* $\lambda^i(t^i)$*-conditionally independent given* η^{-i}.

Definition 9.2 *The random variable* η_i^j *is* $\lambda^i(t^i)$***-sufficient for the random variable*** $\overrightarrow{s}_i^j$ *if, for each* $j \neq i$ *and* $E^j \in \sigma(\overrightarrow{s}_i^j)$,

$$\lambda^i\left(t^i\right)\left(E^j||\sigma(\eta^{-i})\right) = \lambda^i\left(t^i\right)\left(E^j||\sigma(\eta_i^j)\right) \quad a.s.$$

Say the type t^i *satisfies* ***sufficiency (SUFF)*** *if, for each* $j \neq i$, η_i^j *is* $\lambda^i(t^i)$*-sufficient for* $\overrightarrow{s}_i^j$.

In words, Definition 9.1 says that a type t^i satisfies CI if, conditional on knowing the hierarchies of beliefs for the other players j, type t^i's assessment of their strategies is independent. For SUFF, suppose type t^i knows player j's hierarchy of beliefs, and comes to learn the hierarchies of beliefs for the players other than j. Type t^i satisfies SUFF if this new information doesn't change t^i's assessment of j's strategy.

The next result assumes the type structure is bimeasurable (Definition 8.2). It says that under CI and SUFF, if a type t^i for player i assesses other players' hierarchies as independent, then t^i assesses their strategies as independent.[9] Equivalently, if t^i assesses other players' strategies as correlated, then t^i must assess their hierarchies as correlated. So, taken together, CI and SUFF capture our concept of correlation via the players' hierarchies of beliefs.

Proposition 9.1 *Suppose the type structure* Φ *is bimeasurable, and consider a type* t^i *that satisfies CI and SUFF. If the random variables* $\eta_i^1, \ldots, \eta_i^{i-1}, \eta_i^{i+1}, \ldots, \eta_i^n$ *are* $\lambda^i(t^i)$*-independent, then the random variables* $\overrightarrow{s}_i^1, \ldots, \overrightarrow{s}_i^{i-1}, \overrightarrow{s}_i^{i+1}, \ldots, \overrightarrow{s}_i^n$ *are* $\lambda^i(t^i)$*-independent.*

10 RCBR Formalized

Definition 10.1 *Say* $\left(s^i, t^i\right) \in S^i \times T^i$ *is* ***rational*** *if*

$$\sum_{s^{-i}\in S^{-i}} \pi^i\left(s^i, s^{-i}\right) \mathrm{marg}_{S^{-i}}\, \lambda^i\left(t^i\right)\left(s^{-i}\right) \geq \sum_{s^{-i}\in S^{-i}} \pi^i\left(r^i, s^{-i}\right) \mathrm{marg}_{S^{-i}}\, \lambda^i\left(t^i\right)\left(s^{-i}\right)$$

for every $r^i \in S^i$. *Let* R_1^i *be the set of all rational pairs* (s^i, t^i).

Definition 10.2 *Say* $E \subseteq S^{-i} \times T^{-i}$ *is* ***believed under*** $\lambda^i\left(t^i\right)$ *if* E *is Borel and* $\lambda^i\left(t^i\right)(E) = 1$. *Let*

$$B^i(E) = \{t^i \in T^i : E \textit{ is believed under } \lambda^i(t^i)\}.$$

[9] See the appendices for proofs not given in the text.

For $m > 1$, define R^i_m inductively by

$$R^i_{m+1} = R^i_m \cap [S^i \times B^i \left(R^{-i}_m\right)].$$

Definition 10.3 *If* $(s^1, t^1, \ldots, s^n, t^n) \in R_{m+1}$*, say there is* ***rationality and*** m***th-order belief of rationality (RmBR)*** *at this state. If* $(s^1, t^1, \ldots, s^n, t^n) \in \bigcap_{m=1}^{\infty} R_m$*, say there is* ***rationality and common belief of rationality (RCBR)*** *at this state.*

Next, define sets S^i_m inductively by $S^i_0 = S^i$, and

$$S^i_{m+1} = \{s^i \in S^i_m : \text{there exists } \mu \in \mathcal{M}\left(S^{-i}\right), \text{ with } \mu(S^{-i}_m) = 1, \text{ such that} \\ \pi^i\left(s^i, \mu\right) \geq \pi^i\left(r^i, \mu\right) \text{ for every } r^i \in S^i_m\}.$$

Note there is an M such that $\bigcap_{m=0}^{\infty} S^i_m = S^i_M \neq \emptyset$, for all i. A strategy $s^i \in S^i_M$ (resp. strategy profile $s \in S_M$) is called **correlated rationalizable** (Brandenburger-Dekel [7]).

We will make use of the following concept adapted from Pearce [20]:

Definition 10.4 *Fix a game* $G = \left\langle S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \right\rangle$ *and subsets* $Q^i \subseteq S^i$*, for* $i = 1, \ldots, n$*. The set* $\prod_{i=1}^n Q^i$ *is a* ***best-response set (BRS)*** *if, for every* i *and each* $s^i \in Q^i$*, there is a* $\mu \in \mathcal{M}(S^{-i})$ *with* $\mu(Q^{-i}) = 1$*, such that* $\pi^i\left(s^i, \mu\right) \geq \pi^i\left(r^i, \mu\right)$ *for every* $r^i \in S^i$*.*

Standard facts about BRS's are: (i) the set S_M of correlated rationalizable profiles is a BRS; and (ii) every BRS is contained in S_M.

Next is a statement of the result that RCBR is characterized by the correlated rationalizable strategies:[10]

Proposition 10.1 *Consider a game* $G = \left\langle S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \right\rangle$*.*

(i) Fix a type structure $\left\langle S^1, \ldots, S^n; T^1, \ldots, T^n; \lambda^1, \ldots, \lambda^n \right\rangle$*, and suppose there is RCBR at the state* $\left(s^1, t^1, \ldots, s^n, t^n\right)$*. Then the strategy profile* $\left(s^1, \ldots, s^n\right)$ *is correlated rationalizable in* G*.*

(ii) There is a type structure $\left\langle S^1, \ldots, S^n; T^1, \ldots, T^n; \lambda^1, \ldots, \lambda^n \right\rangle$ *such that, for each correlated rationalizable strategy profile* $\left(s^1, \ldots, s^n\right)$*, there is a state* $\left(s^1, t^1, \ldots, s^n, t^n\right)$ *at which there is RCBR.*

11 Main Result Formalized

Here we give a formal treatment of statements (i) and (ii) in Section 6.

[10] Brandenburger-Dekel [7] and Tan-Werlang [23] show related results. Proposition 2.1 in [7] demonstrates an equivalence between common knowledge of rationality and correlated rationalizability. Theorem 5.1 in [23] shows that (in a universal structure) RmBR yields strategy profiles that survive $(m+1)$ rounds of correlated rationalizability. ([23] also states a converse (Theorem 5.3) and references the proof to the unpublished version of the paper.)

Theorem 11.1

(i) *There is a game G and a correlated rationalizable strategy s^i of G, such that the following holds: For any type structure Φ, there does not exist a state at which each type satisfies CI, RCBR holds, and s^i is played.*

(ii) *There is a game G' and a correlated rationalizable strategy s^i of G', such that the following holds: For any type structure Φ, there does not exist a state at which each type satisfies SUFF, RCBR holds, and s^i is played.*

Begin with part (i). A suitable game G was given in Figure 6.1.

Lemma 11.1 *For the game G:*

(i) *the strategy U (resp. M) is optimal under $\mu \in \mathcal{M}\left(S^b \times S^c\right)$ if and only if $\mu(L, Y) = 1$;*

(ii) *the strategy L (resp. C) is optimal under $\mu \in \mathcal{M}(S^a \times S^c)$ if and only if $\mu(U, Y) = 1$;*

(iii) *the strategy Y is optimal under $\mu \in \mathcal{M}\left(S^a \times S^b\right)$ if and only if $\mu(U, L) = \mu(M, C) = \frac{1}{2}$ (moreover, this measure is not independent).*

Proof. Parts (i) and (ii) are immediate.

For part (iii), note that Y is optimal under $\mu \in \mathcal{M}\left(S^a \times S^b\right)$ if and only if

$$\mu(U, L) + \mu(M, C) \geq \max\{2(1 - \mu(M)), 2(1 - \mu(U))\},$$

where we write M for the set $\{M\} \times S^b$ and U for the set $\{U\} \times S^b$. Since $1 \geq \mu(U, L) + \mu(M, C)$, it follows that

$$1 \geq \max\{2(1 - \mu(M)), 2(1 - \mu(U))\},$$

or $\mu(M) \geq \frac{1}{2}$ and $\mu(U) \geq \frac{1}{2}$. Since M and U are disjoint, we get $\mu(M) = \frac{1}{2}$ and $\mu(U) = \frac{1}{2}$. From this, $\mu(U, L) + \mu(M, C) = 1$. But $\mu(U) \geq \mu(U, L)$ and $\mu(M) \geq \mu(M, C)$, and so $\mu(U, L) = \mu(M, C) = \frac{1}{2}$.

Finally, notice that μ is not independent, since $\frac{1}{2} = \mu(U, L) \neq \mu(U) \times \mu(L) = \frac{1}{2} \times \frac{1}{2}$. ■

Corollary 11.1 *The correlated rationalizable set in G is $\{U, M, D\} \times \{L, C, R\} \times \{X, Y, Z\}$.*

Proof. We've just established the optimality of U, M, L, C, and Y. The optimality of D, R, X, and Z is clear. So, $S_1 = S$. Then, by induction, $S_m = S$ for all m. ■

In particular then, correlated rationalizability allows Charlie to play Y, and to have an expected payoff of 1. By Proposition 10.1(ii), there is a type structure Φ and a state at which there is RCBR and Charlie plays Y. However, we will see below (Corollary 11.5), that this cannot happen when we add CI. Moreover, Charlie must then have an (expected) payoff of 2 not 1.

We introduce some notation. Fix a player i. For $j \neq i$, we write $\left[s^j\right]$ for the subset $\left\{s^j\right\} \times S^{-i-j} \times T^{-i}$ of $S^{-i} \times T^{-i}$. We also write $\left[t^j\right]$ for the subset $S^{-i} \times \left\{u^j \in T^j : \delta^j\left(u^j\right) = \delta^j\left(t^j\right)\right\} \times T^{-i-j}$ of $S^{-i} \times T^{-i}$. Note, $\left[t^j\right] = (\eta_i^j)^{-1}(\delta^j\left(t^j\right))$ and is measurable.

Corollary 11.2 *Fix a type structure Φ for G, with $(Y, t^c) \in R_1^c$. Then*

$$\lambda^c\left(t^c\right)(E) = \lambda^c\left(t^c\right)([U] \cap [L] \cap E) + \lambda^c\left(t^c\right)([M] \cap [C] \cap E)$$

for any event E in $S^a \times T^a \times S^b \times T^b$. Moreover, if $\lambda^c\left(t^c\right)(E) = 1$, then

$$\lambda^c\left(t^c\right)([U] \cap [L] \cap E) = \lambda^c\left(t^c\right)([M] \cap [C] \cap E) = \tfrac{1}{2}.$$

Proof. The first part is immediate from $\lambda^c\left(t^c\right)([U] \cap [L]) = \lambda^c\left(t^c\right)([M] \cap [C]) = \frac{1}{2}$ (Lemma 11.1), and the fact that $[U] \cap [M] = \emptyset$. The second part follows from $[U] \cap [L] \cap E \subseteq [U] \cap [L]$ and $[M] \cap [C] \cap E \subseteq [M] \cap [C]$. ■

Lemma 11.2 *Fix a type structure Φ for G. Suppose $(Y, t^c) \in \bigcap_m R_m^c$ where t^c satisfies CI. Then there are $(t^a, t^b), (u^a, u^b) \in T^a \times T^b$, with $\left(U, t^a, L, t^b\right), \left(M, u^a, C, u^b\right) \in \bigcap_m (R_m^a \times R_m^b)$ and either $\delta^a(t^a) \neq \delta^a(u^a)$ or $\delta^b(t^b) \neq \delta^b(u^b)$ (or both).*

The idea of the proof was given in Section 6 (see Figure 6.2 and the surrounding discussion).

Proof of Lemma 11.2. Fix Φ with $(Y, t^c) \in \bigcap_m R_m^c$. Then $\lambda^c\left(t^c\right)\left(R_m^a \times R_m^b\right) = 1$ for all m, so that $\lambda^c\left(t^c\right)\left(\bigcap_m (R_m^a \times R_m^b)\right) = 1$. Corollary 11.2 then gives

$$\lambda^c\left(t^c\right)([U] \cap [L] \cap \textstyle\bigcap_m (R_m^a \times R_m^b)) = \lambda^c\left(t^c\right)([M] \cap [C] \cap \bigcap_m (R_m^a \times R_m^b)) = \frac{1}{2}.$$

Suppose, for any $\left(U, t^a, L, t^b\right), \left(M, u^a, C, u^b\right) \in \bigcap_m (R_m^a \times R_m^b)$, $\delta^a\left(t^a\right) = \delta^a\left(u^a\right)$ and $\delta^b\left(t^b\right) = \delta^b\left(u^b\right)$. Then

$$[U] \cap [L] \cap \textstyle\bigcap_m (R_m^a \times R_m^b) \subseteq [U] \cap [L] \cap [t^a] \cap \left[t^b\right],$$
$$[M] \cap [C] \cap \textstyle\bigcap_m (R_m^a \times R_m^b) \subseteq [M] \cap [C] \cap [t^a] \cap \left[t^b\right].$$

So, $[U] \cap [M] = \emptyset$ implies $\lambda^c(t^c)([t^a] \cap \left[t^b\right]) = 1$. From this, Corollary 11.2 implies

$$\lambda^c\left(t^c\right)([U] \cap [L]) = \lambda^c\left(t^c\right)([U]) = \lambda^c\left(t^c\right)([L]) = \tfrac{1}{2}.$$

From this, the fact that $\lambda^c(t^c)([t^a] \cap \left[t^b\right]) = 1$, and Corollary E2 in Appendix E,

$$\lambda^c\left(t^c\right)([U] \cap [L] \,||\sigma(\eta^{-c})) = \tfrac{1}{2} \quad \text{a.s.}$$
$$\lambda^c\left(t^c\right)([U] \,||\sigma(\eta^{-c})) = \tfrac{1}{2} \quad \text{a.s.}$$
$$\lambda^c\left(t^c\right)([L] \,||\sigma(\eta^{-c})) = \tfrac{1}{2} \quad \text{a.s.}$$

Thus, in particular,

$$\lambda^c(t^c)([U]\,||\sigma(\eta^{-c})) \times \lambda^c(t^c)([L]\,||\sigma(\eta^{-c})) = \tfrac{1}{4} \quad \text{a.s.}$$

so that t^c doesn't satisfy CI. ■

Proposition 11.1 *Fix a game* $\langle S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \rangle$ *and a BRS* $\prod_{i=1}^n Q^i$ *satisfying: For every* i *and each* $s^i \in Q^i$, *there is a unique* $\mu(s^i) \in \mathcal{M}(S^{-i})$ *under which* s^i *is optimal. Fix also a type structure* Φ. *Then for every* i *and all* m *the following hold:*

(i) If $(s^{-i}, t^{-i}), (s^{-i}, u^{-i}) \in R_m^{-i} \cap (Q^{-i} \times T^{-i})$ *then* $\rho_m^i(s^{-i}, t^{-i}) = \rho_m^i(s^{-i}, u^{-i})$.

(ii) If $(s^i, t^i), (r^i, u^i) \in R_m^i \cap (Q^i \times T^i)$ *and* $\mu(s^i) = \mu(r^i)$ *then* $\delta_n^i(t^i) = \delta_n^i(u^i)$ *for all* $n \leq m$.

Again, the idea of the proof was given in Section 6.

Proof of Proposition 11.1. By induction on m.

Begin with $m = 1$: Part (i) is immediate from the fact that $\rho_1^i(s^{-i}, t^{-i}) = \rho_1^i(s^{-i}, u^{-i}) = s^{-i}$. For part (ii), fix $(s^i, t^i), (r^i, u^i) \in R_1^i \cap (Q^i \times T^i)$ with $\mu(s^i) = \mu(r^i)$. By definition, $\underline{\rho}_1^i(\lambda^i(t^i)) = \text{marg}_{S^{-i}} \lambda^i(t^i)$ and $\underline{\rho}_1^i(\lambda^i(u^i)) = \text{marg}_{S^{-i}} \lambda^i(u^i)$. Since $(s^i, t^i), (r^i, u^i) \in R_1^i$,

$$\text{marg}_{S^{-i}} \lambda^i(t^i) = \mu(s^i) = \mu(r^i) = \text{marg}_{S^{-i}} \lambda^i(u^i).$$

Now assume the lemma is true for m. Begin with part (i). Suppose $(s^{-i}, t^{-i}), (s^{-i}, u^{-i}) \in R_{m+1}^{-i} \cap (Q^{-i} \times T^{-i})$. The induction hypothesis applied to part (i) gives $\rho_m^i(s^{-i}, t^{-i}) = \rho_m^i(s^{-i}, u^{-i})$. Also, the induction hypothesis applied to part (ii) gives $\delta_m^j(t^j) = \delta_m^j(u^j)$ for each $j \neq i$. With this, $\rho_{m+1}^i(s^{-i}, t^{-i}) = \rho_{m+1}^i(s^{-i}, u^{-i})$, establishing part (i) for $(m+1)$.

Turn to part (ii). Suppose $(s^i, t^i), (r^i, u^i) \in R_{m+1}^i \cap (Q^i \times T^i)$ and $\mu(s^i) = \mu(r^i)$. Then $(s^i, t^i), (r^i, u^i) \in R_m^i \cap (Q^i \times T^i)$, and so the induction hypothesis applied to part (ii) gives $\delta_n^i(t^i) = \delta_n^i(u^i)$ for all $n \leq m$. As such, it suffices to show $\delta_{m+1}^i(t^i) = \delta_{m+1}^i(u^i)$.

Fix an event E in Y_{m+1}^i, and a point $(s^{-i}, t^{-i}) \in (\rho_{m+1}^i)^{-1}(E) \cap \text{Supp}\, \lambda^i(t^i)$. Then, for each $(s^{-i}, u^{-i}) \in \text{Supp}\, \lambda^i(t^i) \cup \text{Supp}\, \lambda^i(u^i)$, it must be that $(s^{-i}, u^{-i}) \in (\rho_{m+1}^i)^{-1}(E)$. To see this, first notice that, by Corollary D1 in Appendix D, $\text{Supp}\, \lambda^i(t^i) \cup \text{Supp}\, \lambda^i(u^i) \subseteq R_m^{-i}$. Also note that since $(s^i, t^i), (r^i, u^i) \in R_1^i$,

$$\text{marg}_{S^{-i}} \lambda^i(t^i) = \mu(s^i) = \mu(r^i) = \text{marg}_{S^{-i}} \lambda^i(u^i).$$

Since $\mu(s^i)(Q^{-i}) = 1$, it follows that $\text{Supp}\, \lambda^i(t^i) \cup \text{Supp}\, \lambda^i(u^i) \subseteq Q^{-i} \times T^{-i}$. So $(s^{-i}, t^{-i}), (s^{-i}, u^{-i}) \in R_m^{-i} \cap (Q^{-i} \times T^{-i})$. By part (i) of the induction hypothesis, $\rho_m^i(s^{-i}, t^{-i}) = \rho_m^i(s^{-i}, u^{-i})$. By part (ii) of the induction hypothesis, for each $j \neq i$, $\delta_m^j(t^j) = \delta_m^j(u^j)$. So, $\rho_{m+1}^i(s^{-i}, t^{-i}) = \rho_{m+1}^i(s^{-i}, u^{-i})$. From this it follows that $(s^{-i}, u^{-i}) \in (\rho_{m+1}^i)^{-1}(E)$, as required.

Using this, we can now write

$$\lambda^i\left(t^i\right)\left(\left(\rho^i_{m+1}\right)^{-1}(E)\right) = \lambda^i\left(t^i\right)\left(\left(\rho^i_{m+1}\right)^{-1}(E) \cap \operatorname{Supp}\lambda^i\left(t^i\right)\right) =$$

$$\sum_{s^{-i} \in \operatorname{proj}_{S^{-i}}\left(\rho^i_{m+1}\right)^{-1}(E)} \lambda^i\left(t^i\right)\left(\left\{s^{-i}\right\} \times \left\{t^{-i} : \left(s^{-i}, t^{-i}\right) \in \left(\rho^i_{m+1}\right)^{-1}(E) \cap \operatorname{Supp}\lambda^i\left(t^i\right)\right\}\right) =$$

$$\sum_{s^{-i} \in \operatorname{proj}_{S^{-i}}\left(\rho^i_{m+1}\right)^{-1}(E)} \lambda^i\left(t^i\right)\left(\left\{s^{-i}\right\} \times \left\{t^{-i} : \left(s^{-i}, t^{-i}\right) \in \operatorname{Supp}\lambda^i\left(t^i\right)\right\}\right) =$$

$$\sum_{s^{-i} \in \operatorname{proj}_{S^{-i}}\left(\rho^i_{m+1}\right)^{-1}(E)} \operatorname{marg}_{S^{-i}} \lambda^i\left(t^i\right)\left(s^{-i}\right).$$

A corresponding argument shows that

$$\lambda^i\left(u^i\right)\left(\left(\rho^i_{m+1}\right)^{-1}(E)\right) = \sum_{s^{-i} \in \operatorname{proj}_{S^{-i}}\left(\rho^i_{m+1}\right)^{-1}(E)} \operatorname{marg}_{S^{-i}} \lambda^i\left(u^i\right)\left(s^{-i}\right).$$

Now note

$$\operatorname{marg}_{S^{-i}} \lambda^i\left(t^i\right) = \mu\left(s^i\right) = \mu\left(r^i\right) = \operatorname{marg}_{S^{-i}} \lambda^i\left(u^i\right),$$

establishing $\delta^i_{m+1}\left(t^i\right) = \delta^i_{m+1}\left(u^i\right)$, as required. ■

Corollary 11.3 *Fix a game* $\left\langle S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \right\rangle$ *and a BRS* $\prod_{i=1}^n Q^i \subseteq S$ *satisfying: For every* i *and each* $s^i \in Q^i$, *there is a unique* $\mu(s^i) \in \mathcal{M}(S^{-i})$ *under which* s^i *is optimal. Fix also a type structure* Φ. *If* $(s^i, t^i), (r^i, u^i) \in \bigcap_m R^i_m \cap (Q^i \times T^i)$ *and* $\mu(s^i) = \mu(r^i)$, *then* $\delta^i(t^i) = \delta^i(u^i)$.

Proof. Suppose instead that $\delta^i\left(t^i\right) \neq \delta^i\left(u^i\right)$. Then there exists m such that $\delta^i_m\left(t^i\right) \neq \delta^i_m\left(u^i\right)$. Since $\left(s^i, t^i\right), \left(r^i, u^i\right) \in R^i_m$ this contradicts Proposition 11.1. ■

Corollary 11.4 *Let* $Q^a = \{U, M\}$, $Q^b = \{L, C\}$, $Q^c = \{Y\}$ *in the game* G. *Fix a type structure* Φ. *For each* i, *if* $(s^i, t^i), (r^i, u^i) \in \bigcap_m R^i_m \cap (Q^i \times T^i)$, *then* $\delta^i(t^i) = \delta^i(u^i)$.

Proof. Immediate from Corollaries 11.1 and 11.3. ■

Proof of Theorem 11.1(i). Fix a type structure Φ for G. Corollary 11.4 implies that if (U, t^a), $(M, u^a) \in \bigcap_m R^a_m$, then $\delta^a(t^a) = \delta^a(u^a)$. Likewise, if $\left(L, t^b\right)$, $\left(C, u^b\right) \in \bigcap_m R^b_m$, then $\delta^b(t^b) = \delta^b(u^b)$. With this, Lemma 11.2 implies that if $(Y, t^c) \in \bigcap_m R^c_m$, then t^c does not satisfy CI. But Y is a correlated rationalizable strategy, by Lemma 11.1. Setting $s^i = Y$ establishes the theorem. ■

Corollary 11.5 *Fix a type structure* Φ *for* G, *and a state* $(s^a, t^a, s^b, t^b, s^c, t^c)$ *at which there is RCBR. If* t^c *satisfies CI, then* $s^a = D$, $s^b = R$, *and* $s^c = X$ *or* Z.

To prove part (ii) of Theorem 11.1, we use the game G' in Figure 6.3.

Corollary 11.6 *Let $Q^a = \{U, M\}$, $Q^b = \{L, C\}$, $Q^c = \{Y\}$ in the game G'. For $i = 1, 3$, if $(s^i, t^i), (r^i, u^i) \in \bigcap_m R^i_m \cap (Q^i \times T^i)$, then $\delta^i(t^i) = \delta^i(u^i)$. For $i = 2$, if $(s^i, t^i), (r^i, u^i) \in \bigcap_m R^i_m \cap (Q^i \times T^i)$, then $\delta^i(t^i) = \delta^i(u^i)$ only if $s^i = r^i$.*

Proof. Same as for Corollary 11.4, except that while L is optimal only under the measure that assigns probability one to (U, Y), now C is optimal only under the measure that assigns probability one to (M, Y). So if $(L, t^b), (C, u^b) \in \bigcap_m R^b_m$, then $t^b \neq u^b$. ■

Proof of Theorem 11.1(ii). Fix a type structure Φ for G'. Suppose $(Y, t^c) \in \bigcap_m R^c_m$. Then $\lambda^c(t^c)\left(\bigcap_m (R^a_m \times R^b_m)\right) = 1$, so that Corollary 11.2 gives

$$\lambda^c(t^c)([U] \cap [L] \cap \textstyle\bigcap_m (R^a_m \times R^b_m)) = \lambda^c(t^c)([M] \cap [C] \cap \bigcap_m (R^a_m \times R^b_m)) = \frac{1}{2}.$$

Using Corollary 11.6, there are t^a, t^b, u^b, with $\delta^b(t^b) \neq \delta^b(u^b)$, such that

$$[U] \cap [L] \cap \textstyle\bigcap_m (R^a_m \times R^b_m) \subseteq [U] \cap [L] \cap [t^a] \cap [t^b],$$
$$[M] \cap [C] \cap \textstyle\bigcap_m (R^a_m \times R^b_m) \subseteq [M] \cap [C] \cap [t^a] \cap [u^b].$$

Paralleling the argument in the proof of Lemma 11.2, we then have

$$\lambda^c(t^c)([U] \cap [L] \cap [t^a] \cap [t^b]) = \tfrac{1}{2},$$
$$\lambda^c(t^c)([M] \cap [C] \cap [t^a] \cap [u^b]) = \tfrac{1}{2},$$

since $[U] \cap [M] = \emptyset$. Paralleling Corollary 11.2, we get that for any event E,

$$\lambda^c(t^c)(E) = \lambda^c(t^c)([U] \cap [L] \cap [t^a] \cap [t^b] \cap E) + \lambda^c(t^c)([M] \cap [C] \cap [t^a] \cap [u^b] \cap E).$$

Setting $E = [t^a]$, $[U] \cap [t^a]$, $[U] \cap [t^a] \cap [t^b]$, and $[t^a] \cap [t^b]$, yields respectively

$$\lambda^c(t^c)([t^a]) = 1,$$
$$\lambda^c(t^c)([U] \cap [t^a]) = \frac{1}{2},$$

$$\lambda^c(t^c)([U] \cap [t^a] \cap [t^b]) = \frac{1}{2},$$
$$\lambda^c(t^c)([t^a] \cap [t^b]) = \frac{1}{2}.$$

Corollary E1 in Appendix E gives, for any $(s^a, v^a, s^b, v^b) \in [t^a]$,

$$\lambda^c(t^c)([U] || \sigma(\eta^a_c))(s^a, v^a, s^b, v^b) = \tfrac{\lambda^c(t^c)([U] \cap [t^a])}{\lambda^c(t^c)([t^a])} = \tfrac{1}{2}.$$

The same corollary yields for any $\left(s^a, v^a, s^b, v^b\right) \in [t^a] \cap \left[t^b\right]$,

$$\lambda^c\left(t^c\right)\left([U]\,||\sigma(\eta^{-c})\right)\left(s^a, v^a, s^b, v^b\right) = \frac{\lambda^c(t^c)([U]\cap[t^a]\cap\left[t^b\right])}{\lambda^c(t^c)([t^a]\cap[t^b])} = 1.$$

Since $[t^a] \cap \left[t^b\right] \subseteq [t^a]$ and $\lambda^c\left(t^c\right)\left([t^a] \cap \left[t^b\right]\right) > 0$, this says t^c doesn't satisfy SUFF. ■

Appendix F considers an immediate implication of the results of this section, corresponding to the case where players reason only up to some finite number of levels.[11]

12 Conclusion

We have looked at three routes to correlations in games–independent rationalizability (no correlation!), intrinsic correlation (our route), and correlated rationalizability (extrinsic correlation). But there is also another route–related to physical correlation.

Return to the game of Figure 6.1. Suppose Charlie is sitting in her own "cubicle" (the term is from Kohlberg-Mertens [12, p.1005]). Charlie may nonetheless play Y because she thinks that Ann and Bob are coordinating their strategy choices–they jointly choose (U, L) or jointly choose (M, C). While Charlie makes her decision in her own cubicle, she does not think Ann and Bob do the same.

From the perspective of the analyst, this justification of Y is asymmetric. The analyst thinks of each player as a separate decision maker sitting in his own cubicle. But, at the same, the analyst allows each player to think others coordinate.

The intrinsic route avoids this asymmetry. Under this route, we can do non-cooperative game theory without physical correlation, by specifying the players' hierarchies of beliefs (about strategies) and doing the analysis relative to them. As we noted earlier, this leads naturally to the question of the characterization of the strategies that can be played under this analysis. We leave this as open.

[11] We thank Yossi Feinberg for prompting us to investigate this case.

Appendix A CI and SUFF contd.

The following two examples illustrate our CI and SUFF conditions.[12] In particular, they show that the conditions are independent, and neither can be dropped in Proposition 9.1.

Example A1 *Here, Charlie's type t^c satisfies CI and assesses Ann's and Bob's hierarchies as independent, but does not assess their strategies as independent. Let the type spaces be $T^a = \{t^a\}$, $T^b = \{t^b, u^b\}$, $T^c = \{t^c\}$. Suppose that t^b and u^b induce different hierarchies of beliefs for Bob (we don't need to specify the hierarchies). Figure A.1 depicts the measure associated with Charlie's type t^c.*

	L	R
U	½	0
D	0	0

(t^a, t^b)

	L	R
U	0	0
D	0	½

(t^a, u^b)

Figure A.1

We have:

$$\lambda^c(t^c)\left([U] \cap [L] \mid [t^a] \cap [t^b]\right) = 1 = 1 \times 1 = \lambda^c(t^c)\left([U] \mid [t^a] \cap [t^b]\right) \times \lambda^c(t^c)\left([L] \mid [t^a] \cap [t^b]\right).$$

Corresponding equalities hold for each of $\lambda^c(t^c)\left([D] \cap [L] \mid [t^a] \cap [t^b]\right)$, $\lambda^c(t^c)\left([U] \cap [R] \mid [t^a] \cap [t^b]\right)$, and $\lambda^c(t^c)\left([D] \cap [R] \mid [t^a] \cap [t^b]\right)$, and also for $\lambda^c(t^c)\left([\cdot] \cap [\cdot] \mid [t^a] \cap [u^b]\right)$, so that CI holds.

Independence over hierarchies is immediate since $\lambda^c(t^c)([t^a]) = 1$, so that

$$\lambda^c(t^c)([t^a] \cap [t^b]) = \lambda^c(t^c)([t^a]) \times \lambda^c(t^c)([t^b]),$$
$$\lambda^c(t^c)([t^a] \cap [u^b]) = \lambda^c(t^c)([t^a]) \times \lambda^c(t^c)([u^b]).$$

Yet we also have

$$\lambda^c(t^c)([U] \cap [L]) = \frac{1}{2} \neq \frac{1}{2} \times \frac{1}{2} = \lambda^c(t^c)([U]) \times \lambda^c(t^c)([L]),$$

so that independence over strategies is violated. (It is easy to check that SUFF is also violated, as it must be.)

Example A2 *Here, Charlie's type t^c satisfies SUFF and again assesses Ann's and Bob's hierarchies as independent, but again does not assess their strategies as independent. Let $T^a = \{t^a\}$, $T^b =$*

[12] We are grateful to Pierpaolo Battigalli for simplifying the previous versions of these examples.

$\{t^b, u^b\}$, $T^c = \{t^c\}$. *Suppose again that* t^b *and* u^b *induce different hierarchies of beliefs for Bob. Figure A.2 depicts the measure associated with Charlie's type* t^c.

	L	R
U	¼	0
D	0	¼

(t^a, t^b)

	L	R
U	¼	0
D	0	¼

(t^a, u^b)

Figure A.2

Notice that

$$\lambda^c(t^c)\left([U] \mid [t^a] \cap [t^b]\right) = \lambda^c(t^c)\left([U] \mid [t^a] \cap [u^b]\right) = \frac{1}{2} = \lambda^c(t^c)([U] \mid [t^a]),$$
$$\lambda^c(t^c)\left([D] \mid [t^a] \cap [t^b]\right) = \lambda^c(t^c)\left([D] \mid [t^a] \cap [u^b]\right) = \frac{1}{2} = \lambda^c(t^c)([D] \mid [t^a]).$$

The corresponding equalities with respect to Bob are immediate, since $\lambda^c(t^c)([t^a]) = 1$. *This establishes SUFF.*

Independence over hierarchies is immediate, as in Example A.1, since again $\lambda^c(t^c)([t^a]) = 1$. *Yet we also have*

$$\lambda^c(t^c)([U] \cap [L]) = \frac{1}{2} \neq \frac{1}{2} \times \frac{1}{2} = \lambda^c(t^c)([U]) \times \lambda^c(t^c)([L]),$$

so that independence over strategies is violated. (It is readily checked that CI is also violated.)

Appendix B Proofs for Section 8

Lemma B1 *Fix Polish spaces* A, B *and a continuous map* $f : A \to B$. *Let* $g : \mathcal{M}(A) \to \mathcal{M}(B)$ *be given by* $g(\mu) = \mu \circ f^{-1}$ *for each* $\mu \in \mathcal{M}(A)$. *Then* g *is continuous.*

Proof. We need to show that the inverse image of every closed set in $\mathcal{M}(B)$ is closed in $\mathcal{M}(A)$. Let E be a closed set in $\mathcal{M}(B)$, then we want that: Fix a sequence of measures μ_n in $g^{-1}(E)$, where μ_n converges weakly to μ (in $\mathcal{M}(A)$). Then $\mu \in g^{-1}(E)$. To show this, it suffices to show that $g(\mu_n)$ converges weakly to $g(\mu)$ (in $\mathcal{M}(B)$). If so, $g(\mu) \in E$ and so $\mu \in g^{-1}(E)$.

So: Fix an open set U in B. Then $f^{-1}(U)$ is open in A. By the Portmanteau Theorem, $\liminf \mu_n\left(f^{-1}(U)\right) \geq \mu\left(f^{-1}(U)\right)$. But this says that $\liminf g(\mu_n)(U) \geq g(\mu)(U)$, and so, by the Portmanteau Theorem again, $g(\mu_n)$ converges weakly to $g(\mu)$. ■

Proposition B1 *The maps* $\rho^i_m : S^{-i} \times T^{-i} \to Y^i_m$ *and* $\delta^i_m : T^i \to \mathcal{M}\left(Y^i_m\right)$ *are continuous.*

Proof. First note that $\rho^i_1 = \text{proj}_{S^{-i}}$, and so is certainly continuous. So $\underline{\rho}^i_1$ is continuous, by Lemma B1, and thus δ^i_1 is continuous.

Assume ρ^i_m and δ^i_m are continuous, for all i. Fix a rectangular open set $U \times \prod_{j \neq i} V^j \subseteq Y^i_{m+1} = Y^i_m \times \prod_{j \neq i} \mathcal{M}(Y^j_m)$. Notice

$$\left(\rho^i_{m+1}\right)^{-1}\left(U \times \prod_{j \neq i} V^j\right) = \left(\rho^i_m\right)^{-1}(U) \cap \bigcap_{j \neq i} [S^{-i} \times (\delta^j_m)^{-1}(V^j)].$$

Thus $\left(\rho^i_{m+1}\right)^{-1}\left(U \times \prod_{j \neq i} V^j\right)$ is open since each set on the right-hand side is open. Since the rectangular sets form a basis, this shows that ρ^i_{m+1} is continuous. Again, by Lemma B1, for each i, $\underline{\rho}^i_{m+1}$ is then continuous, and so each δ^i_{m+1} is continuous. ■

Since each δ^i_m is continuous, it follows that the map δ^i is continuous. (See, e.g., Munkres [19, Theorem 8.5].)

Appendix C Proofs for Section 9

Let $X^1, \ldots, X^m$ be finite sets, $Y^1, \ldots, Y^m, Z^1, \ldots, Z^m$ be Polish spaces, and set

$$\Omega = \prod_j X^j \times \prod_j Y^j.$$

For each j, define a measurable maps $f^j : \Omega \to X^j$ and $g^j : \Omega \to Y^j$ by $f^j = \text{proj}_{X^j}$ and $g^j = \text{proj}_{Y^j}$. Also, for each j, let $h^j : Y^j \to Z^j$ be a measurable map. Define the product maps $g : \Omega \to \prod_j Y^j$ and $h : \prod_j Y^j \to \prod_j Z^j$ by $g(\omega) = \left(g^1(\omega), \ldots, g^m(\omega)\right)$ and $h(y) = (h^1\left(y^1\right), \ldots, h^m(y^m))$. Note that these maps are measurable.

Fix a probability measure μ on Ω, an event E in Ω, and a sub σ-algebra $\mathcal{S}$ on Ω. Write $\mu(E \| \mathcal{S}) : \Omega \to \mathbb{R}$ for (a version of) the conditional probability of E given $\mathcal{S}$.

Say the random variables $f^1, \ldots, f^m$ are μ**-conditionally independent given** $h \circ g$ if, for all $j = 1, \ldots, m$ and $E^j \in \sigma(f^j)$,

$$\mu(\bigcap_j E^j \| \sigma(h \circ g)) = \prod_j \mu(E^j \| \sigma(h \circ g)) \quad \text{a.s.} \tag{C1}$$

Say the random variable $h^j \circ g^j$ is μ**-sufficient for** f^j if, for each $E^j \in \sigma(f^j)$,

$$\mu(E^j \| \sigma(h \circ g)) = \mu(E^j \| \sigma(h^j \circ g^j)) \quad \text{a.s.} \tag{C2}$$

Proposition C1 *Fix bimeasurable maps* $h^1, \ldots, h^m$. *Suppose the random variables* $f^1, \ldots, f^m$ *are* μ*-conditionally independent given* $h \circ g$*, and that, for each* $j = 1, \ldots, m$*, the random variable* $h^j \circ g^j$ *is* μ*-sufficient for* f^j. *If* $h^1 \circ g^1, \ldots, h^m \circ g^m$ *are* μ*-independent, then* $f^1, \ldots, f^m$ *are* μ*-independent.*

For the proof, we will let $\nu = \mu \circ (h \circ g)^{-1}$ (resp. $\nu^j = \mu \circ (h^j \circ g^j)^{-1}$) be the image measure of μ under $h \circ g$ (resp. $h^j \circ g^j$).

Lemma C1 *Fix events F^j in Z^j, for all j. Then*

$$(h \circ g)^{-1} \left(\prod_j F^j\right) = \bigcap_j (h^j \circ g^j)^{-1} \left(F^j\right).$$

Proof. If $\omega \in (h \circ g)^{-1} \left(\prod_j F^j\right)$, then $h\left(g\left(\omega\right)\right) \in \prod_j F^j$, i.e. $h^j\left(g^j\left(\omega\right)\right) \in F^j$ for all j. Thus $\omega \in \bigcap_j (h^j \circ g^j)^{-1} \left(F^j\right)$. Conversely, if $\omega \in \bigcap_j (h^j \circ g^j)^{-1} \left(F^j\right)$, then $h^j(g^j(\omega)) \in F^j$ for all j, i.e. $h\left(g\left(\omega\right)\right) \in \prod_j F^j$. Thus $\omega \in (h \circ g)^{-1} \left(\prod_j F^j\right)$. ■

Lemma C2 *If $h^1 \circ g^1, \ldots, h^m \circ g^m$ are μ-independent, then ν is a product measure.*

Proof. Fix events F^j in Z^j, for all j. Then

$$\begin{aligned}
\nu(\textstyle\prod_j F^j) &= \mu((h \circ g)^{-1} (\textstyle\prod_j F^j)) \\
&= \mu(\textstyle\bigcap_j (h^j \circ g^j)^{-1} \left(F^j\right)) \\
&= \textstyle\prod_j \mu((h^j \circ g^j)^{-1} \left(F^j\right)) \\
&= \textstyle\prod_j \mu((h \circ g)^{-1}(F^j \times \prod_{k \neq j} Z^k)) \\
&= \textstyle\prod_j \nu(F^j \times \prod_{k \neq j} Z^k),
\end{aligned}$$

where the first and fifth lines are from the definition of ν, the second and fourth lines come from Lemma C1, and the third line follows from the fact that $h^1 \circ g^1, \ldots, h^m \circ g^m$ are μ-independent. ■

Lemma C3 *$\nu^j = \operatorname{marg}_{Z^j} \nu$.*

Proof. Fix an event F^j in Z^j. Then

$$\begin{aligned}
\nu^j(F^j) &= \mu((h^j \circ g^j)^{-1}(F^j)) \\
&= \mu((h \circ g)^{-1}(F^j \times \textstyle\prod_{k \neq j} Z^k)) = \nu(F^j \times \textstyle\prod_{k \neq j} Z^k),
\end{aligned}$$

where the second line follows from Lemma C1. ■

Below, we will sometimes write $\phi^j = \mu(E \| \sigma(h^j \circ g^j))$.

Lemma C4 *Fix an event E in Ω, and $\omega, \tilde{\omega} \in \Omega$. If $\operatorname{proj}_{Y^j} \omega = \operatorname{proj}_{Y^j} \tilde{\omega}$, then*

$$\mu(E \| \sigma(h^j \circ g^j))_{\omega} = \mu(E \| \sigma(h^j \circ g^j))_{\tilde{\omega}}.$$

Proof. Since ϕ^j is $\sigma(h^j \circ g^j)$-measurable, for any number r we have $\left(\phi^j\right)^{-1} (\{r\})$ is contained in $\sigma(h^j \circ g^j)$. So, there exists some event G in Z^j with $(h^j \circ g^j)^{-1} (G) = \left(\phi^j\right)^{-1} (\{r\})$ (Aliprantis-Border [1, Lemma 4.22]). By construction, $(h^j \circ g^j)^{-1} (G) = \prod_k X^k \times F^j \times \prod_{k \neq j} Y^k$ for some event F^j in Y^j. ■

Lemma C5 *Fix $j = 1, \ldots, m$ and some $E^j \in \sigma\left(f^j\right)$. If h^j is a bimeasurable map then there is a measurable map $\psi^j : Z^j \to \mathbb{R}$ with $\psi^j \circ h^j \circ g^j = \mu(E^j \| \sigma(h^j \circ g^j))$.*

Proof. By Kechris [11, Theorem 12.2], it suffices to show that there is a measurable map $\psi^j : h^j\left(Y^j\right) \to \mathbb{R}$ with $\psi^j \circ h^j \circ g^j = \phi^j$. By Lemma C4, such a map is well defined. We show it is measurable.

Fix an event G in $\mathbb{R}$. Then $\left(\phi^j\right)^{-1}(G)$ is measurable in $\prod_k X^k \times \prod_k Y^k$. Following the argument in the proof of Lemma C4, $\left(\phi^j\right)^{-1}(G)$ must take the form $\prod_k X^k \times F^j \times \prod_{k \neq j} Y^k$ for some event F^j in Y^j. Then $g^j((\phi^j)^{-1}(G)) = F^j$ is measurable. Now $h^j\left(F^j\right)$ is measurable in Z^j, since h^j is bimeasurable. Since $h^j\left(F^j\right) \subseteq h^j\left(Y^j\right)$, $h^j\left(F^j\right)$ is Borel in $h^j\left(Y^j\right)$ (Aliprantis-Border [1, Lemma 4.19]). Now note that $h^j\left(F^j\right) = \left(\psi^j\right)^{-1}\left(G^j\right)$, so that $\left(\psi^j\right)^{-1}\left(G^j\right)$ is indeed Borel in $h^j\left(Y^j\right)$. ■

Proof of Proposition C1. Assume $g^1, \ldots, g^m$ are μ-independent. By Lemma C2, ν is a product measure. We will use this fact below.

For each $j = 1, \ldots, m$, fix $E^j \in \sigma\left(f^j\right)$. Then, by definition of conditional probability,

$$\mu(\textstyle\bigcap_j E^j) = \int_\Omega \mu(\bigcap_j E^j \| \sigma(h \circ g))_\omega \, \mathrm{d}\mu(\omega).$$

Using conditional independence (equation C1) and sufficiency (equation C2),

$$\mu(\textstyle\bigcap_j E^j) = \int_\Omega \prod_j \mu(E^j \| \sigma(h^j \circ g^j))_\omega \, \mathrm{d}\mu(\omega). \tag{C3}$$

For each $j = 1, \ldots, m$, use Lemma C5 to define a measurable map $\psi^j : Z^j \to \mathbb{R}$ with $\psi^j \circ h^j \circ g^j = \mu(E^j \| \sigma(h^j \circ g^j))$. Also define $\psi : \prod_j Z^j \to \mathbb{R}$ by $\psi\left(z^1, \ldots, z^m\right) = \prod_j \psi^j\left(z^j\right)$, which is again measurable. Note, $\psi \circ h \circ g = \prod_j \phi^j$.

Using the properties above,

$$\begin{aligned}\mu(\textstyle\bigcap_j E^j) &= \int_\Omega \prod_j \mu(E^j \| \sigma(g^j))_\omega \, \mathrm{d}\mu(\omega) \\ &= \int_{Z^1 \times \cdots \times Z^m} \prod_j \psi^j\left(z^j\right) \, \mathrm{d}\nu(z^1, \ldots, z^m),\end{aligned}$$

where the first line is equation C3 and the second line is a change of variables (Aliprantis-Border [1, Theorem 12.46]). Now use the fact that ν is a product measure, and Fubini's Theorem, to get

$$\mu(\textstyle\bigcap_j E^j) = \int_{Z^1 \times \cdots \times Z^m} \prod_j \psi^j\left(z^j\right) \, \mathrm{d}\nu(z^1, \ldots, z^m) = \prod_j \int_{Z^j} \psi^j\left(z^j\right) \, \mathrm{d}\,\mathrm{marg}_{Z^j}\, \nu(z^1, \ldots, z^m).$$

Using this and Lemma C3,

$$\mu(\textstyle\bigcap_j E^j) = \prod_j \int_{Z^j} \psi^j\left(z^j\right) \, \mathrm{d}\nu^j(z^j). \tag{C4}$$

Now note that, by definition of conditional probability, we also have that, for each $j = 1, \ldots, m$,

$$\mu(E^j) = \int_\Omega \mu(E^j \| \sigma(h^j \circ g^j))_\omega \, \mathrm{d}\mu(\omega).$$

Using this fact and another change of variables,

$$\mu(E^j) = \int_\Omega \mu(E^j \| \sigma(h^j \circ g^j))_\omega \, \mathrm{d}\mu(\omega) = \int_{Z^j} \psi^j \left(z^j\right) \, \mathrm{d}\nu^j(z^j). \tag{C5}$$

So, by equations C4 and C5,

$$\mu(\textstyle\bigcap_j E^j) = \prod_j \mu(E^j),$$

as required. ■

Proposition 9.1 is an immediate corollary of Proposition C1. Set $X^j = S^j$, $Y^j = T^j$, and $Z^j = \prod_{m=1}^{\infty} \mathcal{M}(Y^j_m)$, $f^j = \overrightarrow{s}^j_i$, $g^j = \overrightarrow{t}^j_i$, and $h^j = \delta^j$.

Appendix D Proofs for Section 10

Lemma D1 *Let E be a closed subset of a Polish space X, and $\mathcal{M}(X; E)$ be the set of $\mu \in \mathcal{M}(X)$ with $\mu(E) = 1$. Then $\mathcal{M}(X; E)$ is closed.*

Proof. Take a sequence μ_n of measures in $\mathcal{M}(X; E)$, with $\mu_n \to \mu$. It follows from the Portmanteau Theorem that $\limsup \mu_n(E) \leq \mu(E)$. Since $\limsup \mu_n(E) = 1$ for all n, $\mu(E) = 1$ and so $\mu \in \mathcal{M}(X; E)$ as desired. ■

Lemma D2 *The set R^i_m is closed for each i and m.*

Proof. By induction on m.

$m = 1$: Let $E\left(s^i\right)$ be the set of $\mu \in \mathcal{M}\left(S^{-i} \times T^{-i}\right)$ such that s^i is optimal under μ. It suffices to show the sets $E\left(s^i\right)$ are closed. If so, since λ^i is continuous, $(\lambda^i)^{-1}(E(s^i))$ is closed. The set R^i_1 is the (finite) union over all sets $\left\{s^i\right\} \times (\lambda^i)^{-1}(E(s^i))$; so, R^i_1 is closed.

First, notice that for each $s^{-i} \in S^{-i}$, the set $\left\{s^{-i}\right\} \times T^{-i}$ is clopen. It follows that

$$\mathrm{cl}\left(\left\{s^{-i}\right\} \times T^{-i}\right) \setminus \mathrm{int}\left(\left\{s^{-i}\right\} \times T^{-i}\right) = \left(\left\{s^{-i}\right\} \times T^{-i}\right) \setminus \left(\left\{s^{-i}\right\} \times T^{-i}\right) = \emptyset,$$

and so, for each $\mu \in \mathcal{M}\left(S^{-i} \times T^{-i}\right)$, $\mathrm{cl}\left(\left\{s^{-i}\right\} \times T^{-i}\right) \setminus \mathrm{int}\left(\left\{s^{-i}\right\} \times T^{-i}\right)$ is μ-null.

Now, take a sequence μ_n of measures in $E\left(s^i\right)$, with $\mu_n \to \mu$. The Portmanteau Theorem, together with the fact that each $\mathrm{cl}\left(\left\{s^{-i}\right\} \times T^{-i}\right) \setminus \mathrm{int}\left(\left\{s^{-i}\right\} \times T^{-i}\right)$ is μ-null, implies that $\mu_n\left(\left\{s^{-i}\right\} \times T^{-i}\right) \to \mu\left(\left\{s^{-i}\right\} \times T^{-i}\right)$.

For each $r^i \in S^i$ and integer n, define

$$x_n(r^i) = \textstyle\sum_{s^{-i} \in S^{-i}} \left[\pi^i\left(s^i, s^{-i}\right) - \pi^i\left(r^i, s^{-i}\right)\right] \mathrm{marg}_{S^{-i}} \mu_n\left(s^{-i}\right).$$

Note that $x_n(r^i) \geq 0$, and $x_n(r^i) \to x(r^i)$ where

$$x(r^i) = \sum_{s^{-i} \in S^{-i}} \left[\pi^i\left(s^i, s^{-i}\right) - \pi^i\left(r^i, s^{-i}\right)\right] \operatorname{marg}_{S^{-i}} \mu\left(s^{-i}\right).$$

Since each $x_n(r^i) \geq 0$, $x(r^i) \geq 0$. With this, $\mu \in E\left(s^i\right)$ as desired.

$m \geq 2$: Assume the lemma holds for m. Then, using the induction hypothesis, it suffices to show that $S^i \times B^i\left(R_m^{-i}\right)$ is closed, i.e., that $B^i\left(R_m^{-i}\right)$ is closed. The induction hypothesis gives that R_m^{-i} is closed. So, by Lemma D1, $\mathcal{M}\left(S^{-i} \times T^{-i}; R_m^{-i}\right)$ is closed in $\mathcal{M}\left(S^{-i} \times T^{-i}\right)$. Since λ^i is continuous, $B^i\left(R_m^{-i}\right)$ is closed. ■

We note the following:

Corollary D1 *If* $t^i \in B^i\left(R_m^{-i}\right)$ *then* $\operatorname{Supp}\lambda^i\left(t^i\right) \subseteq R_m^{-i}$. *Similarly, if* $t^i \in B^i\left(\bigcap_m R_m^{-i}\right)$ *then* $\operatorname{Supp}\lambda^i\left(t^i\right) \subseteq \bigcap_m R_m^{-i}$.

Proof of Proposition 10.1. Begin with part (i), and fix a type structure. We will show that the set $\operatorname{proj}_S \bigcap_m R_m$ is a BRS. From this it follows that, for each $\left(s^1, t^1, \ldots, s^n, t^n\right) \in \bigcap_m R_m$, $\left(s^1, \ldots, s^n\right)$ is correlated rationalizable. To see that $\operatorname{proj}_S \bigcap_m R_m$ is a BRS, fix $\left(s^i, t^i\right) \in \bigcap_m R_m^i$. Certainly s^i is optimal under $\operatorname{marg}_{S^{-i}} \lambda^i\left(t^i\right)$, since $\left(s^i, t^i\right) \in R_1^i$. Also, for all m, $\lambda^i\left(t^i\right)\left(R_m^{-i}\right) = 1$, and so $\lambda^i\left(t^i\right)\left(\bigcap_m R_m^{-i}\right) = 1$. From this, $\lambda^i\left(t^i\right)(\operatorname{proj}_{S^{-i}}(\bigcap_m R_m^{-i}) \times T^{-i}) = 1$, or

$$\operatorname{marg}_{S^{-i}} \lambda^i\left(t^i\right)(\operatorname{proj}_{S^{-i}} \bigcap_m R_m^{-i}) = 1,$$

as required.

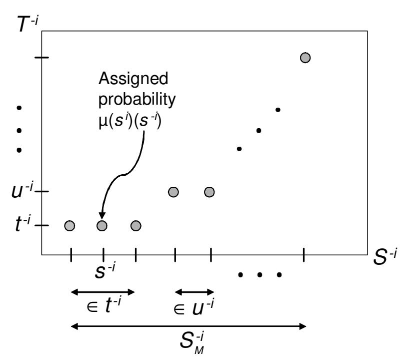

Figure D.1

Now part (ii). Construct a type structure as follows. For each i and $s^i \in S_M^i$, there is a measure $\mu\left(s^i\right) \in \mathcal{M}\left(S^{-i}\right)$, with $\mu\left(s^i\right)\left(S_M^{-i}\right) = 1$, under which s^i is optimal. Fix such a measure $\mu\left(s^i\right)$ and define an equivalence relation $\sim^i$ on S_M^i, where $r^i \sim^i s^i$ if and only if $\mu\left(r^i\right) = \mu\left(s^i\right)$. Set $T^i = S_M^i / \sim^i$ (the quotient space). For $t^i \in T^i$, construct the measure $\lambda^i(t^i)$ on $S^{-i} \times T^{-i}$ as follows.

Pick an $s^i \in t^i$ and set

$$\lambda^i(t^i)(s^{-i},t^{-i}) = \begin{cases} \mu\left(s^i\right)\left(s^{-i}\right) & \text{if } s^j \in t^j \text{ for all } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

(The definition is clearly independent of which $s^i \in t^i$ we choose.) Figure D.1 depicts the construction of $\lambda^i(t^i)$.

We will show that $S_M \subseteq \text{proj}_S \bigcap_m R_m$. By construction, $\lambda\left(t^i\right)\left(\left\{\left(s^{-i},t^{-i}\right) : s^{-i} \in S_M^{-i} \text{ and } s^{-i} \in t^{-i}\right\}\right) = 1$ for each $t^i \in T^i$. So, it suffices to show that, for all i and all m, if $s^i \in S_M^i$ and $s^i \in t^i$, then $\left(s^i,t^i\right) \in R_m^i$. For $m = 1$ this is immediate. Assume this is true for m. Then certainly $\lambda^i\left(t^i\right)\left(R_m^{-i}\right) = 1$ for all $t^i \in T^i$. Therefore $\left(s^i,t^i\right) \in R_{m+1}^i$ when $s^i \in S_M^i$ and $s^i \in t^i$, as desired. ■

Appendix E Proofs for Section 11

Fix a probability space $(\Omega, \mathcal{F}, \mu)$, and a measure space $(X, \mathcal{B}(X))$ where X is Polish. Note that each singleton is contained in $\mathcal{B}(X)$. Let $f : \Omega \to X$ be a random variable. Also, fix $E \in \mathcal{F}$, and let $g : \Omega \to \mathbb{R}$ be a version of the conditional probability of E given $\sigma(f)$.

Lemma E1 *If $\omega, \omega' \in f^{-1}(\{x\})$ then $g(\omega) = g(\omega')$.*

Proof. Fix $\omega \in f^{-1}(\{x\})$. Since $\{g(\omega)\}$ is closed and g is $\sigma(f)$-measurable, $g^{-1}(\{g(\omega)\}) \in \sigma(f)$. Also note that $\omega \in f^{-1}(\{x\}) \cap g^{-1}(\{g(\omega)\})$. From this, there is an event $G \in \mathcal{B}(X)$ with $x \in G$ and $f^{-1}(G) = g^{-1}(\{g(\omega)\})$ (Aliprantis-Border [1, Lemma 4.22]). Using this, $\omega \in f^{-1}(\{x\}) \cap f^{-1}(G)$, from which it follows that $x \in G$. So, $f^{-1}(\{x\}) \subseteq f^{-1}(G) = g^{-1}(\{g(\omega)\})$, as required. ■

Let $\overline{g}$ be the constant value of g on $f^{-1}(\{x\})$.

Corollary E1 $\mu\left(E \cap f^{-1}(\{x\})\right) = \overline{g} \times \mu\left(f^{-1}(\{x\})\right)$.

Proof. Using Lemma E1 we have

$$\mu\left(E \cap f^{-1}(\{x\})\right) = \int_{f^{-1}(\{x\})} g(\omega)\, d\mu(\omega) = \overline{g} \times \mu\left(f^{-1}(\{x\})\right),$$

as required. ■

Corollary E2 *If $\mu\left(f^{-1}(\{x\})\right) = 1$ then $g = \mu(E)$ a.s.*

Proof. By Corollary E1 we have $\mu(E) = \overline{g}$, where $\overline{g}$ is the value of g on the probability-1 set $f^{-1}(\{x\})$. ■

Appendix F A Finite-Levels Result

Suppose it is given that player i only reasons to m levels. In this case, the relevant variable associated with player i is his hierarchy of beliefs up to m levels.

To formalize this, begin by noticing that, if $\delta_m^i(t^i) = \delta_m^i(u^i)$ then $\delta_n^i(t^i) = \delta_n^i(u^i)$ for all $n \leq m$. Define composite maps $\eta_{i,m}^j = \delta_m^j \circ \overrightarrow{t}_i^j$ and $\eta_m^{-i} = \delta_m^{-i} \circ \overrightarrow{t}_i$.

Definition F1 *The random variables* $\overrightarrow{s}_i^1, \ldots, \overrightarrow{s}_i^{i-1}, \overrightarrow{s}_i^{i+1}, \ldots, \overrightarrow{s}_i^n$ *are* $\lambda^i(t^i)$***-conditionally independent given the random variable*** η_m^{-i} *if, for all* $j \neq i$ *and* $E^j \in \sigma(\overrightarrow{s}_i^j)$,

$$\lambda^i\left(t^i\right)\left(\bigcap_{j\neq i} E^j || \sigma(\eta_m^{-i})\right) = \prod_{j\neq i} \lambda^i\left(t^i\right)\left(E^j || \sigma(\eta_m^{-i})\right) \quad a.s.$$

Say the type t^i *satisfies* m***-conditional independence (***m***-CI)*** *if* $\overrightarrow{s}_i^1, \ldots, \overrightarrow{s}_i^{i-1}, \overrightarrow{s}_i^{i+1}, \ldots, \overrightarrow{s}_i^n$ *are* $\lambda^i(t^i)$*-conditionally independent given* η_m^{-i}.

Definition F2 *The random variable* $\eta_{i,m}^j$ *is* $\lambda^i(t^i)$***-sufficient for the random variable*** $\overrightarrow{s}_i^j$ *if, for each* $j \neq i$ *and* $E^j \in \sigma(\overrightarrow{s}_i^j)$,

$$\lambda^i\left(t^i\right)\left(E^j || \sigma(\eta_m^{-i})\right) = \lambda^i\left(t^i\right)\left(E^j || \sigma(\eta_{i,m}^j)\right) \quad a.s.$$

Say the type t^i *satisfies* m***-sufficiency (***m***-SUFF)*** *if, for each* $j \neq i$, $\eta_{i,m}^j$ *is* $\lambda^i(t^i)$*-sufficient for* $\overrightarrow{s}_i^j$.

We then get the following corollary to Theorem 11.1:

Corollary F1 *Fix* $m \geq 1$.

(i) There is a game G*, a player* i *in* G*, and a correlated rationalizable strategy* s^i *of* i*, such that the following holds: For any type structure* Φ*, there is no* $t^i \in T^i$ *such that* $(s^i, t^i) \in R_{m+1}^i$ *and* t^i *satisfies m-CI.*

(ii) There is a game G'*, a player* i *in* G'*, and a correlated rationalizable strategy* s^i *of* i*, such that the following holds: For any type structure* Φ*, there is no* $t^i \in T^i$ *such that* $(s^i, t^i) \in R_{m+1}^i$ *and* t^i *satisfies m-SUFF.*

This follows from taking player i to be Charlie in the game of Figure 6.1. or 6.3, the strategy s^i to be her choice Y, and repeating the steps of the proof of Theorem 11.1.

Appendix G Independent Rationalizability

Here we give a proof of the relationship stated in Section 2, and repeated below.

Proposition G1 *Fix a game $G = \left\langle S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \right\rangle$. There is an associated type structure $\left\langle S^1, \ldots, S^n; T^1, \ldots, T^n; \lambda^1, \ldots, \lambda^n \right\rangle$ such that each type satisfies CI and SUFF, and for each independent rationalizable strategy profile $\left(s^1, \ldots, s^n\right)$, there is a state $\left(s^1, t^1, \ldots, s^n, t^n\right)$ at which RCBR holds.*

First a definition: A set $\prod_{i=1}^n Q^i \subseteq S$ is an **independent best-response set** (**IBRS**) (cf. Pearce [20, 1984]) if, for each i and every $s^i \in Q^i$, there is a $\mu \in \prod_{j \neq i} \mathcal{M}\left(S^j\right)$ with $\mu(Q^{-i}) = 1$, under which s^i is optimal. It is well known that the set of independent rationalizable profiles is an IBRS, and every IBRS is contained in the independent rationalizable set.

To prove Proposition G1, we follow exactly the proof of Proposition 10.1(ii), in Appendix D. Throughout, simply replace the set of player i's correlated rationalizable strategies with the set of i's independent rationalizable strategies. We have to show, in addition, that the type structure Φ constructed there satisfies CI and SUFF.

Using the IBRS property, for each independent rationalizable strategy s^i, there is a product measure $\mu(s^i)$ on S^{-i}, which assigns probability 1 to the independent rationalizable strategies of players $j \neq i$ and under which s^i is optimal. For $s^i \in t^i$, construct the measure $\lambda^i(t^i)$ as before.

We now give the intuition for why CI and SUFF hold, and then the formal proof. For each $j \neq i$, fix an independent rationalizable strategy s^j for player j. Consider the hierarchy of beliefs for j induced by the measure $\lambda^j(t^j)$ for $s^j \in t^j$. CI requires that the conditional of $\lambda^i(t^i)$, conditioned on the event that each player $j \neq i$ has the hierarchy induced by $\lambda^j(t^j)$, be a product measure. But this conditional comes from $\mu(s^i)$, conditioned on a certain rectangular subset of strategies for players $j \neq i$. (For each $j \neq i$, consider the other strategies r^j of player j with measures $\mu\left(r^j\right) = \mu\left(s^j\right)$. Take the product of these subsets.) Since $\mu(s^i)$ is a product measure, so is its conditional on any rectangular subset. The same argument establishes SUFF.

Recall from the text that $\left[t^j\right]$ is the subset $S^{-i} \times \left\{u^j \in T^j : \delta^j\left(u^j\right) = \delta^j\left(t^j\right)\right\} \times T^{-i-j}$ of $S^{-i} \times T^{-i}$.

Proof of Proposition G1. Follow the proof of Proposition 10.1(ii), in Appendix D. Throughout, simply replace S^i_M with the set of i's independent rationalizable strategies. Then $S^i_M \subseteq \operatorname{proj}_{S^i} \bigcap_m R^i_m$. We have to show, in addition, that each $t^i \in T^i$ satisfies CI and SUFF.

To do this, we will make use of a property the construction satisfies. Specifically, for each $t^i \in T^i$, $\delta^i\left(t^i\right) = \delta^i\left(u^i\right)$ only if $t^i = u^i$. To see this, fix $t^i \neq u^i$, $s^i \in t^i$, and $r^i \in u^i$. Note that $\operatorname{marg}_{S^{-i}} \lambda^i\left(t^i\right) = \mu\left(s^i\right)$ and $\operatorname{marg}_{S^{-i}} \lambda^i\left(u^i\right) = \mu\left(r^i\right)$ (the right-hand sides are independent of which $s^i \in t^i$ and $r^i \in u^i$ were chosen). If $\delta^i_1\left(t^i\right) = \delta^i_1\left(u^i\right)$ then $\mu\left(s^i\right) = \mu\left(r^i\right)$. It follows that $t^i = u^i$, as desired.

Fix $t^i \in T^i$ and also $(s^{-i}, t^{-i}) \in S^{-i} \times T^{-i}$. If, for some j, $s^j \notin t^j$, it is then immediate that t^i satisfies CI and SUFF, since

$$\lambda^i\left(t^i\right)(\bigcap_{k\neq i}\left[s^k\right]|\bigcap_{k\neq i}[t^k]) = 0 = \prod_{j\neq i}\lambda^i\left(t^i\right)(\left[s^j\right]|\bigcap_{k\neq i}[t^k]),$$
$$\lambda^i\left(t^i\right)(\left[s^j\right]|[t^j]) = 0 = \lambda^i\left(t^i\right)(\left[s^j\right]|\bigcap_{k\neq i}[t^k]).$$

So, suppose $s^j \in t^j$ for all j. First note that

$$\lambda^i\left(t^i\right)(\bigcap_{k\neq i}\left[s^k\right]\cap\bigcap_{k\neq i}[t^k]) = \mu\left(s^i\right)\left(s^{-i}\right) = \prod_{k\neq i}\mu\left(s^i\right)(\{s^k\}\times S^{-i-k}), \tag{G1}$$

where the second equality uses the fact that μ is a product measure. Write E^j for the set of all $s^j \in t^j$ and recall that $\delta^j\left(u^j\right) = \delta^j\left(t^j\right)$ only if $u^j = t^j$. Again using the fact that μ is a product measure, we have

$$\begin{aligned}\lambda^i\left(t^i\right)(\left[s^j\right]\cap\bigcap_{k\neq i}[t^k]) &= \sum_{r^k\in t^k}\mu\left(s^i\right)(\left\{s^j\right\}\times\prod_{k\neq i,j}E^k)\\ &= \mu\left(s^i\right)(\left\{s^j\right\}\times S^{-i-j})\times\prod_{k\neq i,j}\mu\left(s^i\right)\left(E^k\times S^{-i-k}\right),\end{aligned} \tag{G2}$$

where the first line is by construction and the second line follows from the fact that μ is a product measure. Similarly,

$$\lambda^i\left(s^i\right)(\bigcap_{k\neq i}[t^k]) = \mu\left(s^i\right)(\prod_{k\neq i}E^k) = \prod_{k\neq i}\mu\left(s^i\right)\left(E^k\times S^{-i-k}\right). \tag{G3}$$

Now note,

$$\begin{aligned}\prod_{j\neq i}\lambda^i\left(s^i\right)(\left[s^j\right]|\bigcap_{k\neq i}[t^k]) &= \prod_{j\neq i}\frac{\mu\left(s^i\right)(\left\{s^j\right\}\times S^{-i-j})\times\prod_{k\neq i,j}\mu\left(s^i\right)\left(E^k\times S^{-i-k}\right)}{\mu\left(s^i\right)(E^j\times S^{-i-j})\prod_{k\neq i,j}\mu\left(s^i\right)(E^k\times S^{-i-k})}\\ &= \lambda^i\left(s^i\right)(\bigcap_{k\neq i}\left[s^k\right]|\bigcap_{k\neq i}[t^k]),\end{aligned}$$

where the first line follows from G2-G3 and the second line follows from G1-G3. This establishes CI.

Finally, for each j,

$$\lambda^i\left(s^i\right)(\left[t^j\right]) = \mu\left(s^i\right)(E^j\times S^{-i-j}). \tag{G4}$$

So putting this together with G2-G3, we have

$$\lambda^i\left(s^i\right)(\left[s^j\right]|[t^j]) = \frac{\mu\left(s^i\right)\left(\left\{s^j\right\}\times S^{-i-j}\right)}{\mu\left(s^i\right)(E^j\times S^{-i-j})} = \lambda^i\left(s^i\right)(\left[s^j\right]|\bigcap_{k\neq i}[t^k]),$$

establishing SUFF. ■

Remark G1 *In the proof of Proposition G1, for each i, the random variables* $\eta_i^1, \ldots, \eta_i^{i-1}, \eta_i^{i+1}, \ldots, \eta_i^n$

are independent.

Proof. This is immediate from equations G3-G4 and the converse of Lemma C2 (which follows immediately from the proof of the forward direction). ■

Corollary G1 *Consider a game* $G = \langle S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \rangle$.

(i) Fix a bimeasurable type structure $\langle S^1, \ldots, S^n; T^1, \ldots, T^n; \lambda^1, \ldots, \lambda^n \rangle$ *where each type satisfies CI and SUFF, and has an independent assessment about the other players' hierarchies of beliefs. Suppose RCBR holds at the state* $(s^1, t^1, \ldots, s^n, t^n)$. *Then the strategy profile* $(s^1, \ldots, s^n)$ *is independent rationalizable in* G.

(ii) There is a type structure $\langle S^1, \ldots, S^n; T^1, \ldots, T^n; \lambda^1, \ldots, \lambda^n \rangle$ *such that each type satisfies CI and SUFF, and has an independent assessment about the other players' hierarchies of beliefs, and for each independent rationalizable profile* $(s^1, \ldots, s^n)$, *there is a state* $(s^1, t^1, \ldots, s^n, t^n)$ *at which RCBR holds.*

Proof. For part (i), repeat the proof of Proposition 10.1(i). Note that by Proposition 9.1, the set $\text{proj}_S \bigcap_m R_m$ is an IBRS. Part (ii) follows immediately from Proposition G1 and Remark G1. ■

Proposition G1 should be distinguished from the following: Fix a game G and associated type structure Φ. Suppose that for each player i and type t^i, the marginal on S^{-i} of the measure $\lambda^i(t^i)$ is independent. Then: (i) if there is RCBR at the state $(s^1, t^1, \ldots, s^n, t^n)$, the strategy profile $(s^1, \ldots, s^n)$ is independent rationalizable in G; and (ii) the types $t^1, \ldots, t^n$ satisfy CI. Certainly (i) is true. (Just follow the proof of Proposition 10.1(i), noting that since each $\text{marg}_{S^{-i}} \lambda^i(t^i)$ is a product measure, the set $\text{proj}_S \bigcap_m R_m$ is an IBRS.) But (ii) may be false, as the next example shows. The reason is that within a given type structure, independence need not imply conditional independence. (Of course, the fact that independence doesn't imply conditional independence is well known in probability theory.)

Example G1 *Let* $S^a = \{U, D\}$, $S^b = \{L, R\}$, *and* $S^c = \{Y\}$. *The type spaces are* $T^a = \{t^a, v^a\}$, $T^b = \{t^b, u^b\}$, *and* $T^c = \{t^c\}$, *where:*

$\lambda^a(t^a)$ *assigns probability 1 to* (L, t^b, Y, t^c);
$\lambda^a(u^a)$ *assigns probability 1 to* (R, u^b, Y, t^c);
$\lambda^b(t^b)$ *assigns probability 1 to* (U, t^a, Y, t^c);
$\lambda^b(u^b)$ *assigns probability 1 to* (D, u^a, Y, t^c);
$\lambda^c(t^c)$ *assigns probability* $\frac{1}{4}$ *to each of* $(U, t^a, L, t^b), (D, t^a, R, t^b), (D, u^a, L, u^b), (U, u^a, R, u^b)$.

Note that $\delta^a(t^a) \neq \delta^a(u^a)$ *and* $\delta^b(t^b) \neq \delta^b(u^b)$. *Figure G.1 depicts the measure* $\lambda^c(t^c)$. *Clearly, the marginal on the strategy sets of each type's measure is independent. But CI is violated. For example:*

$$\frac{1}{2} = \lambda^c(t^c)([U] \cap [L] \mid [t^a] \cap [t^b]) \neq \lambda^c(t^c)([U] \mid [t^a] \cap [t^b]) \times \lambda^c(t^c)([L] \mid [t^a] \cap [t^b]) = \frac{1}{2} \times \frac{1}{2}.$$

(Note that SUFF is satisfied. As for RCBR, we can easily add payoffs for the players–just make them all 0*'s–so that RCBR holds at every state.)*

	L	R
U	¼	0
D	0	¼

(t^a, t^b)

	L	R
U	0	¼
D	¼	0

(u^a, u^b)

Figure G.1

Appendix H Injectivity and Genericity

We start with a method of constructing measures that satisfy conditional independence and sufficiency. Fix finite sets $X^1, \ldots, X^m$, and a measure on $\prod_{i=1}^m X^i$. Suppose we can find associated finite sets $Y^1, \ldots, Y^m$ of additional variables, and for each i, an injection from X^i to Y^i. Then there is a natural way to construct a measure on $\prod_{i=1}^m \left(X^i \times Y^i\right)$ that agrees the original measure on $\prod_{i=1}^m X^i$, and which satisfies conditional independence and sufficiency defined with respect to the additional variables.

Some notation: Let $\left[x^i\right] = \left\{x^i\right\} \times X^{-i} \times Y$, and define $\left[y^i\right]$ similarly.

Proposition H1 *Let $X^1, \ldots, X^m, Y^1, \ldots, Y^m$ be finite sets and, for each $i = 1, \ldots, m$, let $f^i : X^i \to Y^i$ be an injection. Then, given a measure $\mu \in \mathcal{M}(\prod_{i=1}^m X^i)$, there is a measure $\nu \in \mathcal{M}(\prod_{i=1}^m \left(X^i \times Y^i\right))$ with:*

(i) $\operatorname{marg}_{\prod_{i=1}^m X^i} \nu = \mu$;

(ii) $\nu\left(\bigcap_{i=1}^m \left[x^i\right] \mid \bigcap_{i=1}^m \left[y^i\right]\right) = \prod_{i=1}^m \nu\left(\left[x^i\right] \mid \bigcap_{i=1}^m \left[y^i\right]\right)$ *whenever* $\nu\left(\bigcap_{i=1}^m \left[y^i\right]\right) > 0$;

(iii) for each $i = 1, \ldots, m$, $\nu(\left[x^i\right] \mid \bigcap_{j=1}^m \left[y^j\right]) = \nu\left(\left[x^i\right] \mid \left[y^i\right]\right)$ *whenever* $\nu(\bigcap_{j=1}^m \left[y^j\right]) > 0$.

Figure H.1 depicts the case $m = 2$. Since f^1 and f^2 are injective, the measure ν will assign positive probability to at most one point in each (y^1, y^2)-plane. (We'll give it the probability $\mu((f^1)^{-1}(y^1), (f^2)^{-1}(y^2))$.) Conditions (i), (ii), and (iii) are then clear.

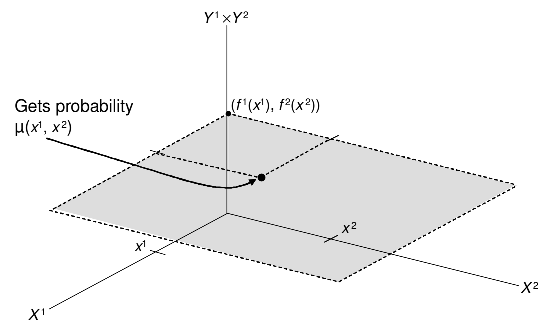

Figure H.1

Proof of Proposition H1. Define $\nu \in \mathcal{M}(\prod_{i=1}^{m}\left(X^i \times Y^i\right))$ by

$$\nu\left(x^1, y^1, \ldots, x^m, y^m\right) = \begin{cases} \mu\left(x^1, \ldots, x^m\right) & \text{if, for each } i = 1, \ldots, m,\ f^i\left(x^i\right) = y^i, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\operatorname{marg}_{\prod_{i=1}^{m} X^i} \nu = \mu$, establishing condition (i).

The proofs of (ii) and (iii) make repeated use of injectivity. For (ii), first assume $y^i = f^i\left(x^i\right)$ for all i. Then

$$\nu\left(\bigcap_{i=1}^{m}\left[x^i\right] \mid \bigcap_{i=1}^{m}\left[y^i\right]\right) = \frac{\nu\left(x^1, f^1\left(x^1\right), \ldots, x^m, f^m\left(x^m\right)\right)}{\nu\left(\bigcap_{i=1}^{m}\left[f^i\left(x^i\right)\right]\right)} = 1.$$

Also, for each i,

$$\nu\left(\left[x^i\right] \mid \bigcap_{j=1}^{m}\left[y^j\right]\right) = \frac{\nu\left(\left[x^i\right] \cap \bigcap_{j=1}^{m}\left[f^j\left(x^j\right)\right]\right)}{\nu\left(\bigcap_{j=1}^{m}\left[f^j\left(x^j\right)\right]\right)} = 1,$$

so (ii) holds. Next notice that if $y^i \neq f^i\left(x^i\right)$ for some i, then

$$\nu\left(\bigcap_{j=1}^{m}\left[x^j\right] \mid \bigcap_{j=1}^{m}\left[y^j\right]\right) = \nu\left(\left[x^i\right] \mid \bigcap_{j=1}^{m}\left[y^j\right]\right) = 0,$$

and (ii) again holds.

Turning to (iii), if $y^i = f^i\left(x^i\right)$, then

$$\nu\left(\left[x^i\right] \mid \left[y^i\right]\right) = \frac{\nu\left(\left[x^i\right] \cap \left[f^i\left(x^i\right)\right]\right)}{\nu\left(\left[f^i\left(x^i\right)\right]\right)} = 1,$$

and

$$\nu\left(\left[x^i\right] \mid \bigcap_{j=1}^{m}\left[y^j\right]\right) = \frac{\nu\left(\left[x^i\right] \cap \bigcap_{j=1}^{m}\left[y^j\right]\right)}{\nu\left(\bigcap_{j=1}^{m}\left[y^j\right]\right)} = 1,$$

so (iii) holds. If $y^i \neq f^i\left(x^i\right)$, then

$$\nu\left(\left[x^i\right] \mid \bigcap_{j=1}^{m}\left[y^j\right]\right)=\nu\left(\left[x^i\right] \mid \left[y^i\right]\right)=0.$$

and (iii) again holds. ■

We now use Proposition H1 to address the question in Section 6: Can we identify a class of games where the middle and outer sets in Figure 2.2 coincide? Here is one answer.

Fix a game G and a BRS $\prod_{i=1}^{n} Q^i$ of G. Then for every i and each $s^i \in Q^i$, there is a $\mu(s^i) \in \mathcal{M}(S^{-i})$ with $\mu(s^i)(Q^{-i})=1$, under which s^i is optimal. Say the BRS satisfies the **injectivity condition** if the measures $\mu(s^i)$ can be chosen so that $\mu\left(r^i\right) \neq \mu\left(s^i\right)$ if $r^i \neq s^i$, for $r^i, s^i \in Q^i$. That is, for every player i, each strategy in i's component of the BRS can be given a different support measure.

Proposition H2 *Fix a game $G=\left\langle S^1,\ldots,S^n;\pi^1,\ldots,\pi^n\right\rangle$ and a BRS $\prod_{i=1}^{n} Q^i$ of G that satisfies the injectivity condition. Then there is a type structure $\left\langle S^1,\ldots,S^n;T^1,\ldots,T^n;\lambda^1,\ldots,\lambda^n\right\rangle$ such that each type satisfies CI and SUFF, and for each strategy profile $\left(s^1,\ldots,s^n\right) \in \prod_{i=1}^{n} Q^i$ there is a state $\left(s^1,t^1,\ldots,s^n,t^n\right)$ at which there is RCBR.*

Proof. For each i, let T^i be a copy of the set Q^i. We now apply Proposition H1. Fix a player i. For each $j \neq i$, set $X^j=Q^j$ and $Y^j=T^j$. The identity map gives the injection f^j from X^j to Y^j.

For $t^i=s^i \in Q^i$, we construct $\lambda^i\left(t^i\right) \in \mathcal{M}\left(S^{-i} \times T^{-i}\right)$ from $\mu\left(s^i\right)$, the same way that ν is constructed from μ in Proposition H1. (For this, identify a measure on $\mathcal{M}\left(S^{-i} \times T^{-i}\right)$ with support contained in $Q^{-i} \times T^{-i}$, with a measure on $\mathcal{M}\left(Q^{-i} \times T^{-i}\right)$.)

Notice that if $t^j \neq u^j$ then $\operatorname{marg}_{S^{-j}} \lambda^j\left(t^j\right) \neq \operatorname{marg}_{S^{-j}} \lambda^j\left(u^j\right)$. From this it follows that, for any $t^j \neq u^j$, $\delta^j\left(t^j\right) \neq \delta^j\left(u^j\right)$. That is,

$$\left[t^j\right]=S^{-i} \times\left\{u^j \in T^j: \delta^j\left(u^j\right)=\delta^j\left(t^j\right)\right\} \times T^{-i-j}=S^{-i} \times\left\{t^j\right\} \times T^{-i-j}.$$

So, by Proposition H1, t^i satisfies CI and SUFF.

It remains to show that $\prod_{i=1}^{n} Q^i \subseteq \operatorname{proj}_S \bigcap_m R_m$. Fix $s^i \in T^i=Q^i$. By construction, $\operatorname{marg}_{S^{-i}} \lambda\left(s^i\right)=\mu\left(s^i\right)$ and $\lambda\left(s^i\right)\left(\left\{\left(s^{-i},s^{-i}\right): s^{-i} \in Q^{-i}\right\}\right)=1$. Certainly, if $s^i \in Q^i$ then $\left(s^i,s^i\right) \in R_1^i$. Assume inductively that, for all j, $\left(s^j,s^j\right) \in R_m^j$. Then certainly $\lambda^i(s^i)(R_m^{-i})=1$, so that $\left(s^i,s^i\right) \in R_{m+1}^i$. Thus $\left(s^i,s^i\right) \in \bigcap_m R_m^i$, establishing the result. ■

Recall that the correlated rationalizable set $\prod_{i=1}^{n} S_M^i$ is a BRS. So, Proposition H2 tells us that if the correlated rationalizable set satisfies the injectivity condition, there is a type structure such that each type satisfies CI and SUFF, and for each correlated rationalizable profile $\left(s^1,\ldots,s^n\right)$ there is a state $\left(s^1,t^1,\ldots,s^n,t^n\right)$ at which RCBR holds. We conclude that if the correlated rationalizable set satisfies the injectivity condition, then the middle and outer sets in Figure 2.2 coincide.

Next, we show this condition holds generically (in the matrix). Fix an n-player strategic game form $\left\langle S^1,\ldots,S^n\right\rangle$. A particular game can then be identified with a point $\left(\pi^1,\ldots,\pi^n\right) \in \mathbb{R}^{n \times |S|}$.

Following Battigalli-Siniscalchi [5], say the game $(\pi^1, \ldots, \pi^n)$ satisfies the **strict best-response property** if for each $s^i \in S^i_M$, there exists $\mu \in \mathcal{M}(S^{-i})$ with $\mu(S^{-i}_M) = 1$ such that s^i is the unique strategy optimal under μ. Note, if a game $(\pi^1, \ldots, \pi^n)$ satisfies the strict best-response property, then the correlated rationalizable set satisfies the injectivity condition (but not vice versa).

Proposition H3 *Let Γ be the set of games for which the strategies consistent with RCBR, CI, and SUFF are strictly contained in the correlated rationalizable strategies. The set Γ is nowhere dense in $\mathbb{R}^{n \times |S|}$.*

Proof. By Proposition H2 and the above remarks, Γ is contained in the sets of games that fail the strict best-response property. Proposition 4.4 in Battigalli-Siniscalchi [5] shows that for $n = 2$, the set of games that fail the strict best-response property is nowhere dense. Their argument readily extends to $n > 2$, giving our result. ■

Of course, genericity in the matrix is usually viewed as too strong a condition: it is well understood that many games of applied interest are non-generic (even in the tree). (See the discussions in Mertens [16, pp.582] and Marx-Swinkels [14, pp.224-225].) For this reason, we believe it is more illuminating to identify structural conditions on a game under which a particular statement–such as equality of the middle and outer sets in Figure 2.2–holds. Injectivity is one such condition. No doubt, there are other conditions of interest.

Appendix I Extrinsic Correlation contd.

Here we add a player (Nature) which doesn't have payoffs or types, and doesn't affect the payoffs of the other players. First the definition of such an extended game: A finite n-player strategic-form game **with a (payoff-irrelevant) move by Nature** is a structure $G = \langle S^0, S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \rangle$, where S^0 is the strategy set for Nature.

Definition I1 *Fix a game $G = \langle S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \rangle$, and a game with a move by Nature, viz. $\overline{G} = \langle \overline{S}^0, \overline{S}^1, \ldots, \overline{S}^n; \overline{\pi}^1, \ldots, \overline{\pi}^n \rangle$. Say $\overline{G}$ is an extension of G if, for each $i = 1, \ldots, n$, $\overline{S}^i = S^i$ and*

$$\overline{\pi}^i(s^0, s^1, \ldots, s^n) = \pi^i(s^1, \ldots, s^n)$$

for all $(s^0, s^1, \ldots, s^n) \in \prod_{j=0}^n S^j$.

For a game with a move by Nature, set $S = \prod_{i=1}^n S^i$ and, for $i = 1, \ldots, n$, $S^{-i} = \prod_{j \neq i : j = 1, \ldots, n} S^j$. Define sets S^i_m inductively by $S^i_0 = S^i$, and

$$S^i_{m+1} = \{s^i \in S^i_m : \text{there exists } \mu \in \mathcal{M}(S^0 \times S^{-i}), \text{ with } \mu(S^0 \times S^{-i}_m) = 1, \text{ such that } \pi^i(s^i, \mu) \geq \pi^i(r^i, \mu) \text{ for every } r^i \in S^i_m\}.$$

Note the similarity to the definition in Section 10. As there, for each i, we write S_M^i for the set of correlated rationalizable strategies for player i. We also define analogs to the definitions of rationality and RCBR in Section 10. The next lemma is straightforward and we omit the proof.

Lemma I1 *Fix a game G and an extension $\overline{G}$ of G. The correlated rationalizable strategies in G are the same as the correlated rationalizable strategies in $\overline{G}$.*

Now, we show that in a game with a nontrivial move by Nature, the correlated rationalizable strategies characterize CI, SUFF, and RCBR. For the definitions of CI and SUFF, redefine the random variables $\overrightarrow{s}_i^j$, $\overrightarrow{t}_i^j$, $\overrightarrow{s}_i$, and $\overrightarrow{t}_i$ from Section 9 so they are maps from the full state space (i.e. $S^0 \times S^{-i} \times T^{-i}$), and redefine δ^i, δ^{-i} so they are maps to hierarchies on $S^0 \times S^{-i}$ (i.e., take $Y_1^i = S^0 \times S^{-i}$). With this modification, Definitions 9.1 and 9.2 are appropriate to analyze the extended game.

Proposition I1 *Fix a game $G = \langle S^0, S^1, \ldots, S^n; \pi^1, \ldots, \pi^n \rangle$ with $|S^0| \geq 2$. There is a type structure for G where: (i) each type satisfies CI and SUFF; and (ii) for each correlated rationalizable strategy profile $(s^1, \ldots, s^n)$ in G there is a state $(s^0, s^1, t^1, \ldots, s^n, t^n)$ at which there is RCBR.*

This is essentially a corollary to Proposition H2: Fix the correlated rationalizable set of G. For each $s^i \in S_M^i$, there is a measure $\mu(s^i) \in \mathcal{M}(S^0 \times S^{-i})$ with $\mu(s^i)(S^0 \times S_M^{-i}) = 1$, under which s^i is optimal. Since $|S^0| \geq 2$, we can choose the measures so that $\mu(r^i) \neq \mu(s^i)$ if $r^i \neq s^i$, for $r^i, s^i \in S_M^i$.

But, to use Proposition H2, we must amend the proofs of Propositions H1 and H2 to apply to a game with a move by Nature.

Fix finite sets $X^0, X^1, \ldots, X^m, Y^1, \ldots, Y^m$. Let $[x^i] = X^0 \times \{x^i\} \times X^{-i} \times Y$, and define $[y^i]$ similarly. (We now set $X^{-i} = \prod_{j \neq i: j=1,\ldots,n} X^j$.) Proposition H1 can then be amended to say:

Proposition I2 *Let $X^0, X^1, \ldots, X^m, Y^1, \ldots, Y^m$ be finite sets and, for each $i = 1, \ldots, m$, let $f^i : X^i \to Y^i$ be an injection. Then, given a measure $\mu \in \mathcal{M}(\prod_{i=0}^m X^i)$, there is a measure $\nu \in \mathcal{M}(X^0 \times \prod_{i=1}^m (X^i \times Y^i))$ with:*

(i) $\mathrm{marg}_{\prod_{i=0}^m X^i} \nu = \mu$;

(ii) $\nu\left(\bigcap_{i=1}^m [x^i] \mid \bigcap_{i=1}^m [y^i]\right) = \prod_{i=1}^m \nu\left([x^i] \mid \bigcap_{i=1}^m [y^i]\right)$ *whenever* $\nu\left(\bigcap_{i=1}^m [y^i]\right) > 0$;

(iii) for each $i = 1, \ldots, m$, $\nu([x^i] \mid \bigcap_{j=1}^m [y^j]) = \nu([x^i] \mid [y^i])$ *whenever* $\nu(\bigcap_{j=1}^m [y^j]) > 0$.

Proof. Define $\nu \in \mathcal{M}(X^0 \times \prod_{i=1}^m (X^i \times Y^i))$ by

$$\nu(x^0, x^1, y^1, \ldots, x^m, y^m) = \begin{cases} \mu(x^0, x^1, \ldots, x^m) & \text{if, for each } i = 1, \ldots, m,\ f^i(x^i) = y^i, \\ 0 & \text{otherwise,} \end{cases}$$

and the proof follows line-by-line from the proof of Proposition H1. ■

With this proposition, the proof of Proposition H2 is readily amended. We have now shown:

Corollary I1 *Fix a game G and an extension $\overline{G} = \langle S^0, S^1, \ldots, S^n; \overline{\pi}^1, \ldots, \overline{\pi}^n \rangle$ of G with $|S^0| \geq 2$. There is a type structure for $\overline{G}$ where: (i) each type satisfies CI and SUFF; and (ii) for each correlated rationalizable strategy profile $(s^1, \ldots, s^n)$ in G, there is a state $(s^0, s^1, t^1, \ldots, s^n, t^n)$ in the type structure for $\overline{G}$ at which there is RCBR.*

Next, we comment briefly on two other sources of extrinsic correlation: (i) payoff uncertainty, and (ii) dummy players. We think both extensions are interesting. Note that both routes involve analyzing a given game G, by first changing G to a new game and then analyzing this new game. So these routes indeed involve extrinsic not intrinsic correlation.

i. Payoff uncertainty So far we have treated uncertainty over strategies. Uncertainty over payoffs is another potential source of correlation. In the text, the game was given–i.e., there was no uncertainty over the payoff functions $\pi^1, \ldots, \pi^n$. Now introduce a little uncertainty about payoff functions. Specifically, assume the payoff functions are common $(1-\varepsilon)$-belief (Monderer-Samet [18]), for some (small) $\varepsilon > 0$. (For short, say that a game G itself is common $(1-\varepsilon)$-belief.)

Go back to the game of Figure 6.1 and the associated type structure in Figure 7.2 (but without the coin toss). The idea is that now Ann's types t^a and u^a will both give probability ε to Bob's having a different payoff function from the given one, and the types will differ in what Bob's alternative payoff function is. So t^a and u^a will induce different hierarchies of beliefs (now defined over strategies and payoff functions). We do a similar construction for Bob, and this way CI and SUFF hold in the new structure. We give a general construction in an online appendix.[13]

Yet, this is not a route to understanding the correlated rationalizable strategies in the original game. In introducing payoff uncertainty, we have changed the game from the original one, in which the payoff functions were given.

Even if we allow this change to the game, there is another difficulty. If we introduce payoff uncertainty, we lose a complete characterization of correlated rationalizability. In the online appendix, we show that given $\varepsilon > 0$, we can find a game G where the conditions of RCBR, CI, and SUFF (all redefined for the case of payoff uncertainty), and common $(1-\varepsilon)$-belief of G, allow a strategy which isn't correlated rationalizable to be played in G.[14] We conclude that while payoff uncertainty can–arguably–rescue the converse direction (part (ii)) of Proposition 10.1, we then lose the forward direction (part (i)).

ii. Dummy players Here the idea is to add another player to the game. In the game of Figure 6.1, we add a fourth player ("Dummy") with a singleton strategy set. Ann's types t^a and u^a will differ in what they think Dummy thinks about the strategies chosen. Likewise with Bob, and we again get CI and SUFF. The online appendix again gives a general construction.

[13] Available on the *JET* Supplementary Materials webpage.

[14] Note that we first fix ε, and common $(1-\varepsilon)$-belief relative to this ε. Then we find a game where our conditions allow a strategy which isn't correlated rationalizable. This order is important. Epistemic conditions should be stated independent of a particular game. If the conditions are allowed to depend on the game in question, then the condition could simply be that a strategy profile we're interested in is chosen. This wouldn't be a useful epistemic analysis.

The same issue arises here. Adding a dummy player is changing the game. The basic question remains: What correlations can be understood in the original game?

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