Supplementary Online Material for "Ideologues or Pragmatists?"

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Appendix A   Proofs for Section 4.3

In Section 4.3, we discussed two reasons the Voter may "prefer" to increase the gap. The first reason was so that he can increase the size of the interval. The second reason was so that he can be made better off on the gap. This section formalizes the two arguments. In particular, we begin with the question of how the gap relates to the size of the interval. Then, we turn to the proof of Proposition 4.3, which constructs equilibria for the case of intervals of "full length." Finally, we turn to the question of whether we can improve upon Proposition 4.3, by increasing the size of the gap.

a. Increasing the Size of the Interval: Fix two intervals $[\underline{p}, \overline{p}]$ and $[\underline{q}, \overline{q}]$, where $\underline{q} > \overline{p}$. Further, take the length of $[\underline{p}, \overline{p}]$ to be $\frac{B-k^2}{2k}$. Now, let's ask: If we can find a $[\underline{p}, \overline{p}] \cup [\underline{q}, \overline{q}]$-compliant equilibrium, viz. (s_P^*, s_V^*), what is the minimum distance between $\underline{q}$ and $\overline{p}$? The answer will depend on the length of the interval $[\underline{q}, \overline{q}]$. To see this, first note that

$$s_V^*(\underline{q}) = s_V^*(\overline{q}) + \frac{2k}{B}(\overline{q} - \underline{q}).$$

(Here, we use Lemma 4.1.) So, holding $s_V^*(\overline{q})$ fixed, increasing the length of the interval increases the electoral incentives the Voter must offer the Pragmatist to induce her to choose the Voter's ideal policy when it is $\underline{q}$. However, as $s_V^*(\underline{q})$ increases, there are greater electoral incentives for choosing $\overline{p}$ over $\underline{q}$, and so we need the minimum gap between $[\underline{p}, \overline{p}]$ and $[\underline{q}, \overline{q}]$ to be larger. This last claim is formalized in the following Lemma.

Lemma A1 *Fix some interval* $[\underline{p}, \overline{p}]$ *of length* $\frac{B-k^2}{2k}$*, and some* $q > \overline{p}$*. If* (s_P^*, s_V^*) *is* $[\underline{p}, \overline{p}] \cup \{q\}$*-compliant then:*

(i) $(q - \overline{p})^2 - 2k(q - \overline{p}) + k^2 \geq s_V^*(q)B$*, and*

(ii) $(q - \overline{p}) \geq 2k$*.*

Let us comment on Lemma A1. Begin with a situation where $(q-\overline{p})^2 - 2k(q-\overline{p}) + k^2 = s_V^*(q)B$. If we must increase $s_V^*(q)$, then certainly we must increase the left-hand side. For a given k, the only way to do so is to change the gap between q and $\overline{p}$. Indeed, part (ii) says that this "change of the gap" must be an "increase the gap."

To prove Lemma A1, let us begin with a consequence of Lemma 4.1.

Lemma A2 *Fix an interval* $[\underline{p}, \overline{p}]$ *with length* $\frac{B-k^2}{2k}$*. If* (s_P^*, s_V^*) *is* $[\underline{p}, \overline{p}]$*-compliant, then, for each* $p \in [\underline{p}, \overline{p}]$*,* $s_V^*(p) = 1 + \frac{2k}{B}(\underline{p} - p)$*.*

Proof. By Lemma 4.1, for each $p \in [\underline{p}, \overline{p}]$, $s_V^*(p) = a - \frac{2k}{B}p$, for some a. (Here, a is constant across $p \in [\underline{p}, \overline{p}]$.) We will show that $a = 1 + \frac{2k}{B}\underline{p}$.

To see this, note that

$$\frac{s_V^*(\overline{p}) - s_V^*(\underline{p})}{\overline{p} - \underline{p}} = -\frac{2k}{B}.$$

Since $\overline{p} - \underline{p} = \frac{B-k^2}{2k}$, this gives that

$$s_V^*(\overline{p}) - s_V^*(\underline{p}) = -\frac{2k}{B}\frac{B-k^2}{2k} = -\frac{B-k^2}{B}.$$

Recall, $1 \geq s_V^*(\underline{p})$, so that

$$s_V^*(\overline{p}) = s_V^*(\underline{p}) - \frac{B-k^2}{B} \leq \frac{k^2}{B}.$$

Now, by Remark 4.1, $s_V^*(\overline{p}) \geq \frac{k^2}{B}$, so that, in fact,

$$s_V^*(\overline{p}) = \frac{k^2}{B}.$$

With this,

$$s_V^*(\underline{p}) = \frac{k^2}{B} + \frac{B-k^2}{B} = 1.$$

As such,

$$s_V^*(\underline{p}) = a - \frac{2k}{B}\underline{p} = 1,$$

from which the claim follows. ■

Proof of Lemma A1. Fix an equilibrium, viz. (s_P^*, s_V^*), that is $[\underline{p}, \overline{p}] \cup \{q\}$-compliant. Consider a state ω with $x_V(\omega) = \overline{p}$. By Remark 4.1 and Lemma A2,

$$-k^2 + B + 2k(\underline{p} - \overline{p}) \geq -(q - \overline{p} - k)^2 + s_V^*(q)B.$$

Using the fact established above, i.e., that the length of the interval $[\underline{p}, \overline{p}]$ is $\frac{B-k^2}{2k}$, we get that

$$0 \geq -(q - \overline{p} - k)^2 + s_V^*(q)B.$$

Part (i) follows immediately.

For Part (ii), begin by using Remark 4.1. This gives that $s_V^*(q)\,B \geq k^2$. So,

$$\begin{aligned} -(q-\overline{p}-k)^2 + s_V^*(q)\,B &\geq -(q-\overline{p}-k)^2 + k^2 \\ &= -(q-\overline{p})^2 + 2k\,(q-\overline{p})\,. \end{aligned}$$

Using the fact that

$$0 \geq -(q-\overline{p}-k)^2 + s_V^*(q)\,B,$$

it follows that

$$0 \geq -(q-\overline{p})^2 + 2k\,(q-\overline{p})\,,$$

or $(q-\overline{p}) \geq 2k$. ■

b. Proof of Proposition 4.3: It suffices to show the following two lemmata:

Lemma A3 *Fix intervals $[\underline{p}_i, \overline{p}_i]$ and $[\underline{p}_{i+1}, \overline{p}_{i+1}]$, with $\underline{p}_{i+1} > \overline{p}_i$. Suppose, further, that the length of these intervals is $\frac{B-k^2}{2k}$. If (s_P^*, s_V^*) is $[\underline{p}_i, \overline{p}_i] \cup [\underline{p}_{i+1}, \overline{p}_{i+1}]$-compliant, then $(\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{B}$.*

Lemma A4 *Fix a collection of intervals $[\underline{p}_i, \overline{p}_i]$, each of length $\frac{B-k^2}{2k}$. Suppose further that, given two such intervals $[\underline{p}_i, \overline{p}_i]$ and $[\underline{p}_{i+1}, \overline{p}_{i+1}]$, $(\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{B}$. Then, there exists some $\bigcup_i [\underline{p}_i, \overline{p}_i]$-compliant equilibrium.*

We begin with Lemma A3. We will first need the following result.

Lemma A5 *Fix intervals $[\underline{p}_i, \overline{p}_i]$ and $[\underline{p}_{i+1}, \overline{p}_{i+1}]$, with $\underline{p}_{i+1} > \overline{p}_i$. Suppose, further, that the length of these intervals is $\frac{B-k^2}{2k}$. If (s_P^*, s_V^*) is $[\underline{p}_i, \overline{p}_i] \cup [\underline{p}_{i+1}, \overline{p}_{i+1}]$-compliant, then $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$.*

Proof. Fix some state $\omega \in (x_V)^{-1}(\overline{p}_i)$. Then,

$$\begin{aligned} \mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -k^2 + B + 2k(\underline{p}_i - \overline{p}_i) \\ &\geq -(\underline{p}_{i+1} - \overline{p}_i - k)^2 + B = \mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*), \end{aligned}$$

where the first line follows from Lemma A2 and the second from Lemma A1(ii). With this,

$$2k(\underline{p}_i - \overline{p}_i) \geq -(\underline{p}_{i+1} - \overline{p}_i)^2 + 2k(\underline{p}_{i+1} - \overline{p}_i),$$

or

$$(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i),$$

as required. ∎

Lemma A6 *Fix intervals* $[\underline{p}_i, \overline{p}_i]$ *and* $[\underline{p}_{i+1}, \overline{p}_{i+1}]$*, with* $\underline{p}_{i+1} > \overline{p}_i$*. Suppose, further, that the length of* $[\underline{p}_i, \overline{p}_i]$ *is* $\frac{B-k^2}{2k}$*. Then,* $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$ *if and only if* $(\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{B}$*.*

Proof. Begin with the fact that $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$. Note that $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$ if and only if

$$(\underline{p}_{i+1} - \overline{p}_i)^2 - 2k(\underline{p}_{i+1} - \overline{p}_i) - \left(B - k^2\right) \geq 0. \tag{A1}$$

Differentiating the left-hand side of Equation A1 with respect to $(\underline{p}_{i+1} - \overline{p}_i)$ shows that the left-hand side is increasing in $(\underline{p}_{i+1} - \overline{p}_i)$ if and only if $(\underline{p}_{i+1} - \overline{p}_i)$ is greater than k. So, if $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$, then either $(\underline{p}_{i+1} - \overline{p}_i)$ is less than

$$\frac{2k - \sqrt{4k^2 + 4\left(B - k^2\right)}}{2} = k - \sqrt{B}$$

or at least

$$\frac{2k + \sqrt{4k^2 + 4\left(B - k^2\right)}}{2} = k + \sqrt{B}.$$

Recall that $\underline{p}_{i+1} > \overline{p}_i \geq \underline{p}_i$. So, $(\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i)$ implies

$$\underline{p}_{i+1} - \overline{p}_i \geq 2k\frac{\underline{p}_{i+1} - \underline{p}_i}{\underline{p}_{i+1} - \overline{p}_i} \geq 2k.$$

Since $(\underline{p}_{i+1} - \overline{p}_i) \geq 2k$, it follows that $(\underline{p}_{i+1} - \overline{p}_i)$ cannot be less than $k - \sqrt{B}$. That is, $(\underline{p}_{i+1} - \overline{p}_i)$ must be at least $k + \sqrt{B}$.

Conversely, suppose that $(\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{B}$. Note, $k + \sqrt{B} \geq 2k$. So,

$$\begin{aligned} (\underline{p}_{i+1} - \overline{p}_i)((\underline{p}_{i+1} - \overline{p}_i) - 2k) &\geq (k + \sqrt{B})(k + \sqrt{B} - 2k) \\ &= B - k^2. \end{aligned}$$

This establishes that

$$(\underline{p}_{i+1} - \overline{p}_i)^2 - 2k(\underline{p}_{i+1} - \overline{p}_i) - \left(B - k^2\right) \geq 0,$$

as required. ■

Proof of Lemma A3. Immediate from Lemmata A5-A6. ■

We need two standard results. (The proofs are standard, and so omitted.)

Remark A1 *Let $X_1, X_2, \ldots$ be disjoint events in $\mathbb{R}$. For each i, fix $f_i : X_i \to \mathbb{R}$ and define $f : \bigcup_i X_i \to \mathbb{R}$ so that $f(x) = f_i(x)$ where $x \in X_i$. Then f is measurable if and only if each f_i is measurable.*

Remark A2 *Fix measurable maps $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Let E be an event in $\mathbb{R}$ and construct $h : \mathbb{R} \to \mathbb{R}$ so that $h(x) = f(x)$ if $f(x) \in E$ and $h(x) = g(x)$ if $f(x) \in \mathbb{R}\backslash E$. Then h is measurable.*

We now turn to the proof of Lemma A4.

Proof of Lemma A4. Fix a collection of intervals as in the statement of the Lemma, and let $C = \bigcup_i [\underline{p}_i, \overline{p}_i]$. There are a countable number of such intervals. (See Proposition 5.3 in Krantz (2004)). As such, C is measurable, and so $\mathbb{R}\backslash C$ is also measurable.

Define a strategy s_V^* as follows: If $p \in \mathbb{R}\backslash C$, set $s_V^*(p) = 0$. If $p \in C$, then there exists exactly one interval i with $p \in [\underline{p}_i, \overline{p}_i] \subseteq C$. We set $s_V^*(p) = 1 + \frac{2k}{B}(\underline{p}_i - p)$. Note, the strategy s_V^* is well-defined. In particular, $s_V^*(p) \in [0,1]$. (Here, we use the fact that, if $p \in [\underline{p}_i, \overline{p}_i]$, then $p - \underline{p}_i \leq \frac{B-k^2}{2k}$.) Also, notice that s_V^* is measurable: For each i, we can construct a map $f_i : [\underline{p}_i, \overline{p}_i] \mapsto [0,1]$, that is the restriction of s_V^* to $[\underline{p}_i, \overline{p}_i]$. Each f_i is continuous, and so measurable. Now, fix a measurable set E in $[0,1]$ and note that $(s_V^*)^{-1}(E)$ is either $\bigcup_i (f_i)^{-1}(E)$ or $\bigcup_i (f_i)^{-1}(E) \cup (\mathbb{R}\backslash C)$. In either case, $(s_V^*)^{-1}(E)$ is the countable union of measurable sets, and so measurable.

Now we turn to constructing a strategy s_P^*. We will show that we can construct s_P^* so that (s_P^*, s_V^*) is a C-compliant equilibrium. We break the construction up into three steps: the case where $\omega \in (x_V)^{-1}(C)$, the case where $\omega \in (x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(\mathbb{R}\backslash C)$, and finally the case when $(x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(C)$. For each case, we show that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$ for all $p \in \mathbb{R}$. Then, we conclude the proof by showing that the constructed s_P^* is measurable. This will establish that we have a Bayesian equilibrium. The fact that the equilibrium is C-compliant will follow from the construction.

Case I: Fix some state $\omega \in (x_V)^{-1}(C)$. Set $s_P^*(\omega)(x_V(\omega)) = 1$. We show that, for each $p \in \mathbb{R}$, $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$.

Given that $\omega \in (x_V)^{-1}(C)$, there is some i, so that $x_V(\omega) \in [\underline{p}_i, \overline{p}_i]$. Note

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = -k^2 + B + 2k(\underline{p}_i - x_V(\omega)) \geq 0.$$

Fix some policy $p \in \mathbb{R}$. If $p \in \mathbb{R} \backslash C$, we certainly have that

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq 0 \geq \mathbb{E}u_P(\omega, p, s_V^*).$$

So, suppose that $p \in C$. Then, there exists some j so that $p \in [\underline{p}_j, \overline{p}_j]$. If $i = j$,

$$\begin{aligned}
\mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -k^2 + B + 2k(\underline{p}_i - x_V(\omega)) \\
&\geq -(p - x_V(\omega))^2 - k^2 + B + 2k(\underline{p}_i - x_V(\omega)) \\
&= -(p - x_V(\omega) - k)^2 + B + 2k(\underline{p}_i - p) \\
&= \mathbb{E}u_P(\omega, p, s_V^*),
\end{aligned}$$

as required. Next suppose that $j < i$. We have already seen that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, \underline{p}_i, s_V^*)$. So, to show that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$, it suffices to show that $\mathbb{E}u_P(\omega, \underline{p}_i, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$. But, this follows since

$$\begin{aligned}
\mathbb{E}u_P(\omega, \underline{p}_i, s_V^*) &= -(\underline{p}_i - x_V(\omega) - k)^2 + B \\
&\geq -(p - x_V(\omega) - k)^2 + B + 2k(\underline{p}_j - p) \\
&= \mathbb{E}u_P(\omega, p, s_V^*).
\end{aligned}$$

Finally, suppose $j > i$. Using Lemma A6, here, we have that

$$(\underline{p}_{i+1} - x_V(\omega))^2 \geq (\underline{p}_{i+1} - \overline{p}_i)^2 \geq 2k(\underline{p}_{i+1} - \underline{p}_i). \tag{A2}$$

So, using Equation A2,

$$\begin{aligned}
\mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -k^2 + B + 2k(\underline{p}_i - x_V(\omega)) \\
&\geq -(\underline{p}_{i+1} - x_V(\omega))^2 - k^2 + B + 2k(\underline{p}_{i+1} - x_V(\omega)).
\end{aligned}$$

This implies

$$\begin{aligned}
\mathbb{E}u_P(\omega, s_P^*, s_V^*) &\geq -(\underline{p}_{i+1} - x_V(\omega))^2 - k^2 + B + 2k(\underline{p}_{i+1} - x_V(\omega)) \\
&= -(\underline{p}_{i+1} - x_V(\omega) - k)^2 + B \\
&\geq -(\underline{p}_j - x_V(\omega) - k)^2 + B \\
&\geq -(p - x_V(\omega) - k)^2 + B + 2k(\underline{p}_j - p) \\
&= \mathbb{E}u_P(\omega, p, s_V^*),
\end{aligned}$$

where the third line uses the fact that $\underline{p}_j > \underline{p}_{i+1} \geq x_v(\omega) + k$.

Case II: Fix some state $\omega \in (x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(\mathbb{R}\backslash C)$. Note, there exists some i so that $x_P(\omega) \in (\overline{p}_i, \underline{p}_{i+1})$. (Here, we allow that $\overline{p}_i$ may be $-\infty$ and $\underline{p}_{i+1}$ may be ∞.) In fact, there exists some i so that $x_P(\omega) \in (\overline{p}_i, \underline{p}_{i+1})\backslash(\max\{\overline{p}_i, \underline{p}_i + k\}, \overline{p}_i + k)$. To see this last claim, suppose $\overline{p}_i < x_P(\omega) \leq \overline{p}_i + k$. Then $x_V(\omega) \leq \overline{p}_i$. Using the fact that $x_V(\omega) \in \mathbb{R}\backslash C$, this implies that $x_V(\omega) < \underline{p}_i$. It follows that $x_P(\omega) \leq \underline{p}_i + k$, as stated.

Consider the following three (disjoint) intervals: $(\overline{p}_i, \underline{p}_i + k)$, $(\overline{p}_i + k, \underline{p}_{i+1} - \sqrt{B}]$, and $(\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1})$. Note, the first of these may be empty, i.e., if $\underline{p}_i + k \leq \overline{p}_i$. The second of these however is non-empty, since $\underline{p}_{i+1} - \overline{p}_i \geq k + \sqrt{B}$. Of course, the latter is non-empty. Also note that the union of these intervals is $(\overline{p}_i, \underline{p}_{i+1})\backslash(\max\{\overline{p}_i, \underline{p}_i + k\}, \overline{p}_i + k)$.

Set $s_P^*(\omega)(\underline{p}_i) = 1$, if $x_P(\omega) \in (\overline{p}_i, \underline{p}_i + k)$. Set $s_P^*(\omega)(x_P(\omega)) = 1$, if $x_P(\omega) \in (\overline{p}_i + k, \underline{p}_{i+1} - \sqrt{B}]$. Finally, set $s_P^*(\omega)(\underline{p}_{i+1}) = 1$, if $x_P(\omega) \in (\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1})$.

We now turn to show that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$, for each $p \in \mathbb{R}$. In fact, it suffices to show that, for each $p \in [\underline{p}_i, \underline{p}_{i+1}]$, $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$. If $p < \underline{p}_i$, there exists some policy q with $s_V^*(q) \geq s_V^*(p)$ and $(q - x_P(\omega))^2 \leq (p - x_P(\omega))^2$. So, if $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, q, s_V^*)$, then certainly $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$. Likewise, if $p > \underline{p}_{i+1}$, then $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*)$ implies that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$.

First suppose that $\underline{p}_i + k > \overline{p}_i$, and specifically $\underline{p}_i + k > x_P(\omega) > \overline{p}_i$. Consider some $p \in [\underline{p}_i, \overline{p}_i]$, and note that

$$\mathbb{E}u_P(\omega, p, s_V^*) = -(p - x_P(\omega))^2 + B + 2k(\underline{p}_i - p),$$

so

$$\frac{d\mathbb{E}u_P(\omega, p, s_V^*)}{dp} = -2(p - x_P(\omega)) - 2k.$$

Note that $\underline{p}_i + k > x_P(\omega)$, and so $\mathbb{E}u_P(\omega, p, s_V^*)$ is strictly decreasing over the range $[\underline{p}_i, \overline{p}_i]$. As such,

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = \mathbb{E}u_P(\omega, \underline{p}_i, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$$

for each $p \in [\underline{p}_i, \overline{p}_i]$. Now, fix some $p \in (\overline{p}_i, \underline{p}_{i+1})$ and note that

$$\begin{aligned}
\mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -(\underline{p}_i - x_P(\omega))^2 + B \\
&\geq -k^2 + B \\
&\geq 0 \\
&\geq \mathbb{E}u_P(\omega, p, s_V^*),
\end{aligned}$$

where the second line uses the fact that $\underline{p}_i + k > x_P(\omega) > \underline{p}_i$ and the third line uses the fact that $B \geq k^2$. Finally, using the fact just established, i.e., $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq 0$, we have that

$$\begin{aligned}
\mathbb{E}u_P(\omega, s_P^*, s_V^*) &\geq 0 \\
&= -(k + \sqrt{B} - k)^2 + B \\
&\geq -(\underline{p}_{i+1} - \underline{p}_i - k)^2 + B \\
&\geq -(\underline{p}_{i+1} - x_P(\omega))^2 + B \\
&= \mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*),
\end{aligned}$$

where the third line uses the fact that $\underline{p}_{i+1} - \underline{p}_i \geq \underline{p}_{i+1} - \overline{p}_i \geq k + \sqrt{B}$ and the fourth line uses the fact (already established) that $\underline{p}_{i+1} \geq \underline{p}_i + k > x_P(\omega)$.

Next, suppose that $x_P(\omega) \in (\overline{p}_i + k, \underline{p}_{i+1} - \sqrt{B}]$. Here, $\mathbb{E}u_P(\omega, s_P^*, s_V^*) = 0$. It suffices to show that, for each $p \in [\underline{p}_i, \underline{p}_{i+1}]$, $0 \geq \mathbb{E}u_P(\omega, p, s_V^*)$. Consider first the case where $p \in [\underline{p}_i, \overline{p}_i]$. Recall, here,

$$\frac{d\mathbb{E}u_P(\omega, p, s_V^*)}{dp} = -2(p - x_P(\omega)) - 2k.$$

Now, $x_P(\omega) > \overline{p}_i + k \geq p + k$, for each $p \in [\underline{p}_i, \overline{p}_i]$. So, increasing p (over the range $[\underline{p}_i, \overline{p}_i]$) increases $\mathbb{E}u_P(\omega, p, s_V^*)$. This gives that $\mathbb{E}u_P(\omega, \overline{p}_i, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$, for each such p. Moreover,

$$\begin{aligned}
\mathbb{E}u_P(\omega, \overline{p}_i, s_V^*) &= -(\overline{p}_i - x_P(\omega))^2 + k^2 \\
&\leq -k^2 + k^2,
\end{aligned}$$

where the second line uses the fact that $x_P(\omega) > \overline{p}_i + k$. So,

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = 0 \geq \mathbb{E}u_P(\omega, \overline{p}_i, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*),$$

for each $p \in [\underline{p}_i, \overline{p}_i]$. Certainly we have that, for each $p \in (\overline{p}_i, \underline{p}_{i+1})$,

$$\mathbb{E}u_P(\omega, s_P^*, s_V^*) = 0 \geq \mathbb{E}u_P(\omega, p, s_V^*).$$

Finally, note that

$$\begin{aligned} \mathbb{E}u_P\left(\omega, s_P^*, s_V^*\right) &= -(\underline{p}_{i+1} - \underline{p}_{i+1} + \sqrt{B})^2 + B \\ &\geq -(\underline{p}_{i+1} - x_P\left(\omega\right))^2 + B \\ &= \mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*), \end{aligned}$$

where the second line uses the fact that $\underline{p}_{i+1} - \sqrt{B} \geq x_P\left(\omega\right)$.

Finally, consider the case where $x_P\left(\omega\right) \in (\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1})$. Here,

$$\begin{aligned} \mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -(\underline{p}_{i+1} - x_P\left(\omega\right))^2 + B \\ &\geq -(\underline{p}_{i+1} - \underline{p}_{i+1} + \sqrt{B})^2 + B \\ &= 0. \end{aligned}$$

So, certainly,

$$\mathbb{E}u_P\left(\omega, s_P^*, s_V^*\right) = 0 \geq \mathbb{E}u_P\left(\omega, p, s_V^*\right),$$

for each $p \in (\overline{p}_i, \underline{p}_{i+1}]$. Consider $p \in [\underline{p}_i, \overline{p}_i]$ and recall that $\mathbb{E}u_P\left(\omega, \overline{p}_i, s_V^*\right) \geq \mathbb{E}u_P\left(\omega, p, s_V^*\right)$. (Here we use the fact that $x_P\left(\omega\right) \geq \underline{p}_{i+1} - \sqrt{B} \geq \overline{p}_i + k$.) So, it suffices to show that

$$\mathbb{E}u_P\left(\omega, s_P^*, s_V^*\right) = 0 \geq \mathbb{E}u_P\left(\omega, \overline{p}_i, s_V^*\right).$$

But this is immediate, since

$$\begin{aligned} \mathbb{E}u_P\left(\omega, \overline{p}_i, s_V^*\right) &= -\left(\overline{p}_i - x_P\left(\omega\right)\right)^2 + k^2 \\ &\leq -(\overline{p}_i - \underline{p}_{i+1} + \sqrt{B})^2 + k^2 \\ &= -(\underline{p}_{i+1} - \overline{p}_i - \sqrt{B})^2 + k^2 \\ &\leq -(k + \sqrt{B} - \sqrt{B})^2 + k^2 \\ &= 0, \end{aligned}$$

where the second line uses the fact that $x_P\left(\omega\right) \geq \underline{p}_{i+1} - \sqrt{B} \geq \overline{p}$ and the fourth line uses the fact that $\underline{p}_{i+1} - \overline{p}_i \geq k + \sqrt{B}$.

Case III: Fix some state $\omega \in \left(x_V\right)^{-1}\left(\mathbb{R}\backslash C\right) \cap \left(x_P\right)^{-1}\left(C\right)$. Note, there exists some i so that $x_P\left(\omega\right) \in [\underline{p}_i, \overline{p}_i]$. In fact, since $x_V\left(\omega\right) \in \mathbb{R}\backslash C$, $x_P\left(\omega\right) < \overline{p}_i + k$. Put differently, $x_P\left(\omega\right) \in [\underline{p}_i, \overline{p}_i] \cap [\underline{p}_i, \underline{p}_i + k)$. We make use of this fact below.

Set $s_V^*\left(\omega\right)\left(\underline{p}_i\right) = 1$. We will show that, for each $p \in \mathbb{R}$, $\mathbb{E}u_P\left(\omega, s_P^*, s_V^*\right) \geq \mathbb{E}u_P\left(\omega, p, s_V^*\right)$.

In fact, it suffices to show that $\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$, for each $p \in [\underline{p}_i, \underline{p}_{i+1}]$. To see this, note that, if $p < \underline{p}_i$, then $\mathbb{E}u_P(\omega, \underline{p}_i, s_V^*) > \mathbb{E}u_P(\omega, p, s_V^*)$. (Here, we use the fact that $x_P(\omega) \geq \underline{p}_i > p$ and $s_V^*(\omega)(\underline{p}_i) \geq s_V^*(\omega)(p)$.) Likewise, if $p > \underline{p}_{i+1}$, then $\mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*) > \mathbb{E}u_P(\omega, p, s_V^*)$. (Here, we use the fact that $p > \underline{p}_{i+1} \geq x_P(\omega)$ and $s_V^*(\omega)(\underline{p}_{i+1}) \geq s_V^*(\omega)(p)$.)

Consider policies $p \in [\underline{p}_i, \overline{p}_i]$. For any such policy p, we have

$$\frac{d\mathbb{E}u_P(\omega, s_P^*, s_V^*)}{dp} = -2(p - x_V(\omega) - k) - 2k.$$

Since, for each such policy, $p > x_V(\omega)$, it follows that $\mathbb{E}u_P(\omega, s_P^*, s_V^*)$ is decreasing over the range of policies $[\underline{p}_i, \overline{p}_i]$. So certainly $\mathbb{E}u_P(\omega, \underline{p}_i, s_V^*) > \mathbb{E}u_P(\omega, p, s_V^*)$, for each $p \in (\underline{p}_i, \overline{p}_i]$. Likewise, fix some policy $p \in (\overline{p}_i, \underline{p}_{i+1})$ and note that

$$\begin{aligned}
\mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -(\underline{p}_i - x_P(\omega))^2 + B \\
&> -(\underline{p}_i - \underline{p}_i - k)^2 + B \\
&\geq 0 \\
&\geq \mathbb{E}u_P(\omega, p, s_V^*),
\end{aligned}$$

where the second line uses the fact that $\underline{p}_i + k > x_P(\omega)$, the third line uses the fact that $B \geq k^2$, and the fourth line uses the fact that $s_V^*(p) = 0$. Finally, consider the policy $\underline{p}_{i+1}$. We have that

$$\begin{aligned}
\mathbb{E}u_P(\omega, s_P^*, s_V^*) &= -(\underline{p}_i - x_P(\omega))^2 + B \\
&> -(\underline{p}_i - \underline{p}_i - k)^2 + B \\
&= -k^2 + B \\
&\geq -[(\underline{p}_{i+1} - \underline{p}_i)(\underline{p}_{i+1} - \underline{p}_i - 2k)] - k^2 + B \\
&= -(\underline{p}_{i+1} - \underline{p}_i - k)^2 + B \\
&\geq -(\underline{p}_{i+1} - x_P(\omega))^2 + B \\
&= \mathbb{E}u_P(\omega, \underline{p}_{i+1}, s_V^*),
\end{aligned}$$

where the second and sixth lines use the fact that $\underline{p}_i + k > x_P(\omega)$ and the fourth line uses the fact that $\underline{p}_{i+1} - \underline{p}_i \geq \underline{p}_{i+1} - \overline{p}_i \geq k + \sqrt{B} \geq 2k$.

Conclusion of Proof: We have constructed a measurable function s_V^*. Given this function, we were able to construct a function s_P^* so that, for each state ω and each policy p,

$\mathbb{E}u_P(\omega, s_P^*, s_V^*) \geq \mathbb{E}u_P(\omega, p, s_V^*)$. The function had the property that $s_P^*(\omega)(x_V(\omega)) = 1$, when $x_V(\omega) \in C$. So, to show that we have a C-compliant equilibrium, it suffices to show that s_P^* is measurable.

The proof will make use of Remark A1. We will construct functions, by restricting the domain of s_P^*, viz. Ω, to $(x_V)^{-1}(C)$, $(x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(\mathbb{R}\backslash C)$, and $(x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(C)$. Then, we will show that each of these maps are measurable, thereby establishing the result.

Case A: The restriction of the domain to $(\mathbf{x_V})^{-1}(\mathbf{C})$. Here, measurability follows from Theorem 14.8 in Aliprantis and Border (1999).

Case B: The restriction of the domain to $(\mathbf{x_V})^{-1}(\mathbb{R}\backslash\mathbf{C}) \cap (\mathbf{x_V})^{-1}(\mathbb{R}\backslash\mathbf{C})$. Define sets

$$\Omega_i = (x_P)^{-1}((\overline{p}_i, \underline{p}_{i+1})\backslash(\max\{\overline{p}_i, \underline{p}_i + k\}, \overline{p}_i + k)),$$

and maps $f_i : \Omega_i \to \mathbb{R}$ so that $s_P^*(\omega)(f_i(\omega)) = 1$. Now use the fact that each of the maps $\omega \mapsto \underline{p}_i$, $\omega \mapsto \underline{p}_{i+1}$, and x_P are measurable. Remark A2 gives that each of the maps f_i are measurable. Now note that $(x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(\mathbb{R}\backslash C)$ is the countable disjoint union of the sets Ω_i. So, the result follows from Remark A1 and Theorem 14.8 in Aliprantis and Border (1999).

Case C: The restriction of the domain to $(\mathbf{x_V})^{-1}(\mathbb{R}\backslash\mathbf{C}) \cap (\mathbf{x_V})^{-1}(\mathbf{C})$. Now define sets

$$\Omega_i = (x_P)^{-1}([\underline{p}_i, \overline{p}_i] \cap [\underline{p}_i, \underline{p}_i + k)),$$

and maps $f_i : \Omega_i \to \mathbb{R}$ so that $f_i(\Omega_i) = \{\underline{p}_i\}$. Now note that $(x_V)^{-1}(\mathbb{R}\backslash C) \cap (x_P)^{-1}(C)$ is the countable disjoint union of the sets Ω_i. So the result follows from Remark A1 and Theorem 14.8 in Aliprantis and Border (1999). ■

c. Proof of Lemma 4.2: The following lemma will be of use.

Lemma A7 *Fix $\underline{p}_{i+1} > \overline{p}_i + k$ and consider the function*

$$f_i(p) = -(p - \overline{p}_i - k)^2 + B - (\underline{p}_{i+1} - p)^2$$

(i) Suppose $2\sqrt{B} - k > \underline{p}_{i+1} - \overline{p}_i$. Then, $0 \geq f_i(\frac{1}{2}(\underline{p}_{i+1} + \overline{p}_i + k))$ if and only if, for each $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$, $0 \geq f_i(p)$.

(ii) Suppose $\underline{p}_{i+1} - \overline{p}_i \geq 2\sqrt{B} - k$*. Then,* $0 \geq f_i(\underline{p}_{i+1} - \sqrt{B} + k)$ *if and only if, for each* $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$*,* $0 \geq f_i(p)$*.*

Proof. Note,

$$\frac{df_i}{dp} = -2(p - \overline{p}_i - k) + 2(\underline{p}_{i+1} - p),$$

and, moreover, the second derivative of f_i with respect to p is -4. So, f_i is maximized at

$$\frac{\underline{p}_{i+1} + \overline{p}_i + k}{2}.$$

First consider the case where $2\sqrt{B} - k > \underline{p}_{i+1} - \overline{p}_i$. Here,

$$\underline{p}_{i+1} > \frac{\underline{p}_{i+1} + \overline{p}_i + k}{2} > \underline{p}_{i+1} - \sqrt{B} + k.$$

So, $0 \geq f_i(p)$, for each $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$, if and only if $0 \geq f_i(\frac{1}{2}(\underline{p}_{i+1} + \overline{p}_i + k))$.

Next consider the case where $\underline{p}_{i+1} - \overline{p}_i \geq 2\sqrt{B} - k$. Here, $f_i(\cdot)$ is decreasing over the set of policies $(\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$. So, $0 \geq f_i(p)$, for each $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$, if and only if $0 \geq f_i(\underline{p}_{i+1} - \sqrt{B} + k)$. ■

Lemma A8 *Fix intervals* $[\underline{p}_i, \overline{p}_i]$ *and* $[\underline{p}_{i+1}, \overline{p}_{i+1}]$ *each of length* $\frac{B-k^2}{2k}$*. Also, fix a* $[\underline{p}_i, \overline{p}_i] \cup [\underline{p}_{i+1}, \overline{p}_{i+1}]$*-compliant equilibrium, viz.* (s_P^*, s_V^*)*. If* $s_V^*(\omega)(x_P(\omega)) > 0$ *for some* $\omega \in (x_V)^{-1}((\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1} - k))$*, then* $(\underline{p}_{i+1} - \overline{p}_i)$ *must satisfy one of the following conditions:*

(i) $2\sqrt{B} - k \geq (\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{2B}$*, or*

(ii) $(\underline{p}_{i+1} - \overline{p}_i) \geq 2\sqrt{B}$*.*

Proof. Fix a $[\underline{p}_i, \overline{p}_i] \cup [\underline{p}_{i+1}, \overline{p}_{i+1}]$-compliant equilibrium, viz. (s_P^*, s_V^*), per the statement of the Lemma. Suppose there exists some state $\omega \in (x_V)^{-1}((\underline{p}_{i+1} - \sqrt{B}, \underline{p}_{i+1} - k))$ so that $s_V^*(\omega)(x_P(\omega)) > 0$. Write $p = x_P(\omega)$, and notice that $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$. (Here we use the fact that the Pragmatist's ideal policy is k units above the Voter's.) By Lemma A2, we must have that

$$\mathbb{E}u_P(\omega, p^*, s_V^*) = s_V^*(p) B \geq -(\underline{p}_{i+1} - p)^2 + B = \mathbb{E}u_P(\omega, \underline{q}, s_V^*),$$

and this implies that

$$s_V^*(p) \geq 1 - \frac{(\underline{p}_{i+1} - p)^2}{B}.$$

Now, consider a state, viz. $\overline{\omega}$, at which the Voter's ideal policy is $\overline{p}_i$. We need that, when the Voter's ideal policy is $\overline{p}_i$, the Pragmatist's expected payoffs from $\overline{p}_i$ are higher than her expected payoffs from each $p \in (\underline{p}_{i+1} - \sqrt{B} + k, \underline{p}_{i+1})$. Using Lemma A2, this is the requirement that, for each such p,

$$\mathbb{E}u_P\left(\overline{\omega}, \overline{p}_i, s_V^*\right) = -k^2 + k^2 \geq -\left(p - \overline{p}_i - k\right)^2 + B - \left(\underline{p}_{i+1} - p\right)^2 = \mathbb{E}u_P\left(\overline{\omega}, p, s_V^*\right). \quad \text{(A3)}$$

By Lemma A7, there are two cases to consider—where $2\sqrt{B} - k > (\underline{p}_{i+1} - \overline{p}_i)$ and where $(\underline{p}_{i+1} - \overline{p}_i) \geq 2\sqrt{B} - k$.

Case A: First consider the case where $2\sqrt{B} - k > (\underline{p}_{i+1} - \overline{p}_i)$. Using Proposition 4.3(i), $(\underline{p}_{i+1} - \overline{p}_i) > k + \sqrt{B}$. So, we can apply Lemma A7 and get that Equation A3 holds if and only if

$$0 \geq -\frac{1}{2}\left(\underline{p}_{i+1} - \overline{p}_i - k\right)^2 + B,$$

or if and only if

$$(\underline{p}_{i+1} - \overline{p}_i - k)^2 - 2B \geq 0.$$

As such, Equation A3 if and only if

$$(\underline{p}_{i+1} - \overline{p}_i)^2 - 2k(\underline{p}_{i+1} - \overline{p}_i) - \left(2B - k^2\right) \geq 0.$$

Using Proposition 4.3, $(\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{B} \geq 2k$. As such,

$$\begin{aligned}(\underline{p}_{i+1} - \overline{p}_i) &\geq \frac{2k + \sqrt{4k^2 + 8B - 4k^2}}{2} \\ &= k + \sqrt{2B}.\end{aligned}$$

From this it follows that $2\sqrt{B} - k \geq (\underline{p}_{i+1} - \overline{p}_i) \geq k + \sqrt{2B}$, as required.

Case B: Now consider the case where $(\underline{p}_{i+1} - \overline{p}_i) \geq 2\sqrt{B} - k$. Using Proposition 4.3(i), $(\underline{p}_{i+1} - \overline{p}_i) > k + \sqrt{B}$. So, we can apply Lemma A7 and get that Equation A3 holds if and only if

$$0 \geq -(\underline{p}_{i+1} - \overline{p}_i - \sqrt{B})^2 + B - (\sqrt{B} - k)^2,$$

or if and only if

$$(\underline{p}_{i+1} - \overline{p}_i)^2 - 2\sqrt{B}(\underline{p}_{i+1} - \overline{p}_i) + (\sqrt{B} - k)^2 \geq 0.$$

This gives that either $(\underline{p}_{i+1} - \overline{p}_i)$ is less than both $2\sqrt{B}$ and

$$\frac{2\sqrt{B} - \sqrt{4B - 4(\sqrt{B} - k)^2}}{2} = \sqrt{B} - \sqrt{B - (\sqrt{B} - k)^2}$$

or $(\underline{p}_{i+1} - \overline{p}_i)$ is greater than both $2\sqrt{B}$ and

$$\frac{2\sqrt{B} + \sqrt{4B - 4(\sqrt{B} - k)^2}}{2} = \sqrt{B} + \sqrt{B - (\sqrt{B} - k)^2}.$$

In the former case, we have

$$2\sqrt{B} > \sqrt{B} - \sqrt{B - (\sqrt{B} - k)^2} \geq (\underline{p}_{i+1} - \overline{p}_i) \geq 2\sqrt{B} - k.$$

This implies that

$$0 \geq -\sqrt{B - (\sqrt{B} - k)^2} \geq \sqrt{B} - k \geq 0.$$

The left-most inequality is strict when $k \neq 2\sqrt{B}$ and the right-most inequality is strict when $k \neq \sqrt{B}$. As such, this case cannot hold. In the latter case,

$$\begin{aligned}(\underline{p}_{i+1} - \overline{p}_i) &\geq 2\sqrt{B} \\ &\geq \max\{k + \sqrt{B}, 2\sqrt{B} - k, \sqrt{B} + \sqrt{B - (\sqrt{B} - k)^2}\}.\end{aligned}$$

(Here, we implicitly use Proposition 4.3(i).) This establishes the result. ■

Proof of Lemma 4.2. Suppose $\underline{p}_{i+1} - \overline{p}_i = k + \sqrt{B}$. Then $\max\{k + \sqrt{2B}, 2\sqrt{B}\} > (\underline{p}_{i+1} - \overline{p}_i)$. So the result follows from Lemma A8. ■

Appendix B Proof for Section 5

Lemma B1

(i) Fix some $k > 0$. There exists some non-empty open interval $U \subseteq \mathbb{R}_+$ so that, for each $B \in U$, $4k\sqrt{B} > B - k^2$.

(ii) Fix some $B > 0$. There exists some non-empty open interval $U \subseteq \mathbb{R}_+$ so that, for each $k \in U$, $4k\sqrt{B} > B - k^2$.

Proof. Begin with part (i). Fix some $k > 0$. Consider the non-empty open interval $(k^2, (2+\sqrt{5})^2 k^2)$. We will show that, for any B in this interval, B satisfies $4k\sqrt{B} > B - k^2$. This will establish the result.

Fix some B in this interval. Suppose, contra hypothesis, that $B - k^2 \geq 4k\sqrt{B}$. Then,

$$B - \frac{B}{(2+\sqrt{5})^2} > B - k^2 \geq 4k\sqrt{B} > \frac{4B}{(2+\sqrt{5})},$$

where the first and last inequalities use the fact that $(2+\sqrt{5})^2 k^2 > B$. It follows that

$$B((2+\sqrt{5})^2 - 1) > 4B(2+\sqrt{5}),$$

which cannot hold.

Now turn to part (ii). Fix some $B > 0$. Consider the non-empty open interval $(\sqrt{B}(\sqrt{5}-2), \sqrt{B})$. We will show that, for any k in this interval, k satisfies $4k\sqrt{B} > B - k^2$. This will establish the result.

Fix some k in this interval. Suppose, contra hypothesis, that $B - k^2 \geq 4k\sqrt{B}$. Then, using the fact that $k > \sqrt{B}(\sqrt{5}-2)$, we have

$$B \geq 4k\sqrt{B} + k^2 > 4B(\sqrt{5}-2) + B(\sqrt{5}-2)^2.$$

From this it follows that

$$1 > 4(\sqrt{5}-2) + (\sqrt{5}-2)^2 = 1,$$

which cannot hold. ■

Proof of Proposition 5.1. Begin with part (i). Fix some $k > 0$. By Lemma B1(i), there exists some non-empty open interval $U \subseteq \mathbb{R}_+$ so that, for each $B \in U$, $4k\sqrt{B} > B - k^2$. Fix some $B \in U$ and choose the prior $\mu[B]$ so that the support of $\mu[B]$ is $[-\sqrt{B}, \sqrt{B}] = (x_V)^{-1}([-\sqrt{B}, \sqrt{B}])$. By Proposition 3.1, there exists some $[-\sqrt{B}, \sqrt{B}]$-compliant equilibrium, viz. (s_P^*, s_V^*), of the game with a B-Ideologue. For this equilibrium,

$$\int_\Omega \mathbb{E}u_V(\omega, s_P^*)\, d\mu[B] = 0.$$

Now, consider the game with a (k, B)-Pragmatist. Fix some Bayesian equilibrium of this game, viz. (r_P^*, r_V^*). Note, since $B \in U$,

$$2\sqrt{B} > \frac{B - k^2}{2k}.$$

So, by Lemma A3, there is a non-empty open set of policies, viz. V, contained in $[-\sqrt{B}, \sqrt{B}]$ so that $r_P^*(\omega)(x_V(\omega)) = 0$ for each ω with $x_V(\omega) \in V$. By construction, $\mu[B](V) > 0$, and so

$$\int_\Omega \mathbb{E}u_V(\omega, r_P^*)\, d\mu[B] < 0,$$

as required.

The proof of part (ii) is analogous. ■

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