Iterated Dominance Revisited[*]

Amanda Friedenberg[†]   H. Jerome Keisler[‡]

August 12, 2011

1 Introduction

Iterated deletion of strongly dominated strategies has a long tradition in game theory, going back at least to Nash (1951, pp. 292-293). Bernheim (1984) and Pearce (1984) asserted that (up to issues of correlation) the iteratively undominated (IU) strategies correspond to the strategies consistent with common knowledge of rationality.

In many ways, this step seems intuitive and obvious. Brandenburger and Dekel (1987) and Tan and Werlang (1988) are early treatments that provide a formal statement of this claim. (See, also, Battigalli and Siniscalchi (2002) and Arieli (2010), among many others.) That is, each of these papers provide epistemic conditions for IU.

This paper argues that, in somewhat subtle ways, we still have an incomplete understanding of the epistemic conditions for IU. It then goes on to give a series of results aimed at improving our understanding. These results lead us to new epistemic conditions for IU.

Note, the modern treatment of epistemic conditions for IU rests on the idea of a **complete type structure**. Completeness is a requirement that the type structure contain all possible beliefs. As such, it is, in a sense, a requirement that the type structure is "rich." A richness condition is crucial for an epistemic characterization of IU. (We review why in Section 2.)

In Theorem 5.2, we show that, for any non-degenerate finite game, there exists a complete type structure in which no strategy is consistent with **rationality and common belief of rationality** (**RCBR**). Thus, completeness plus RCBR may not result is the set of IU strategies. The reason this can occur is simple: While a complete type structure induces all beliefs about types, it need not induce all possible hierarchies of beliefs. When a type structure induces all possible hierarchies of beliefs, the IU strategies are consistent with RCBR and so there is some state at which there is RCBR (see Proposition 6.3.) But, for a given complete type structure that does not induce all hierarchies of beliefs, the same conclusion need not follow.

This then raises the question: Which hierarchies of beliefs are induced by a complete type structure? We show that a complete type structure need not induce all possible third-order beliefs—or even all possible third-order beliefs with finite support (Proposition 7.6). But, complete type structures are—what we refer to as—**finitely terminal for all atomic type structures** (Proposition 7.14). An implication of this fact is that complete type structures induce all m^{th}-order beliefs that are induced by type structures where each type has finite support. We use this fact to inspire new epistemic conditions for IU (Theorem 8.1).

One message of this paper is that topological assumptions on the set of types implicitly impose substantive assumptions on players' reasoning. Typically, the literature assumes that the type sets are Polish.[1] When possible, this paper will refrain from making that assumption. As a result, in Section 7, we will encounter certain exceptional cases, involving measurable cardinals, that are not covered by the usual axioms of set theory.

The paper proceeds as follows. Section 2 reviews the literature and previews the results. Sections 3-4 provide the framework and a formal statement of the known epistemic conditions. Section 5 shows that there is a complete type structure for which there is no state at which there is RCBR. Sections 6-7 discuss the implications, in terms of which hierarchies of beliefs are induced by a complete type structure. Section 8 returns to the question of epistemic conditions for IU, to provide new such conditions. Section 9 discusses some conceptual and technical aspects of the paper. Section 10 concludes.

---

[1]There is even such an implicit assumption in the so-called "topology-free" approach to type structures, see e.g., Heifetz and Samet (1998, 1999). When the underlying set of uncertainty is Polish, the literature typically constructs a "large type structure" whose type sets turn out to be Polish.

2 Review of the Literature and the Approach

The purpose of this section is to review the literature and provide an informal preview of the framework and main results.

2.1 The Framework

Refer to Figure 2.1, in which Ann chooses rows and Bob chooses columns. Note, U (resp. M, resp. D) is Ann's best choice, if Bob plays L (resp. C, resp. R). Similarly, L and C are Bob's best choices, if Ann plays D.

	L	C	R
U	4, 4	1, 1	0, 0
M	1, 1	5, 5	0, 0
D	0, 1	0, 1	6, 0

Figure 2.1

What are the implications of the requirement that each player is rational, each player believes the other player is rational, etc.? To say if a strategy is rational for Ann, we need to say what she believes about the strategy Bob employs: For instance, whether U is rational for Ann depends on the likelihood that she assigns to Bob playing L versus C versus R. Likewise, what Ann thinks about Bob's rationality versus irrationality depends on what she thinks Bob thinks about her own play of the game. And so on. So, to talk about the implications of rationality and common belief of rationality, we need to enrich the description of the game—to describe Ann's and Bob's hierarchies of beliefs about the play of the game.

To specify these hierarchies of beliefs, we will make use of Harsanyi's (1967) **type structure** model. There are three ingredients: First, for each player, there is an **underlying set of uncertainty**. Here, Ann's underlying set of uncertainty, viz. X_a, is Bob's strategy set $S_b = \{L, C, R\}$ and Bob's underlying set of uncertainty, viz. X_b, is Ann's strategy set $S_a = \{U, M, D\}$. Second, for each player, there is a set of **(belief) types**. For instance, take Ann's type set to be $T_a = \{t_a, u_a, v_a\}$ and Bob's type set to be $T_b = \{t_b, u_b\}$. Third, for each player, there is a **belief map**. A belief map for a player maps each type of that player to a belief about the player's underlying set of uncertainty and the type of the other player (i.e., here, to a belief about the strategy and type of the other player). Figure 2.2 describes the belief maps β_a and β_b. For instance, under the belief map β_a (resp. β_b), type t_a of Ann (resp. t_b of Bob) assigns probability one to Bob playing L (resp. Ann playing U) and being of type t_b (resp. t_a). And so on.

$\beta_a(t_a)$	t_b	u_b
L	1	
C		
R		

$\beta_a(u_a)$	t_b	u_b
L		
C		1
R		

$\beta_a(v_a)$	t_b	u_b
L		
C		
R	.5	.5

$\beta_b(t_b)$	t_a	u_a	v_a
U	1		
M			
D			

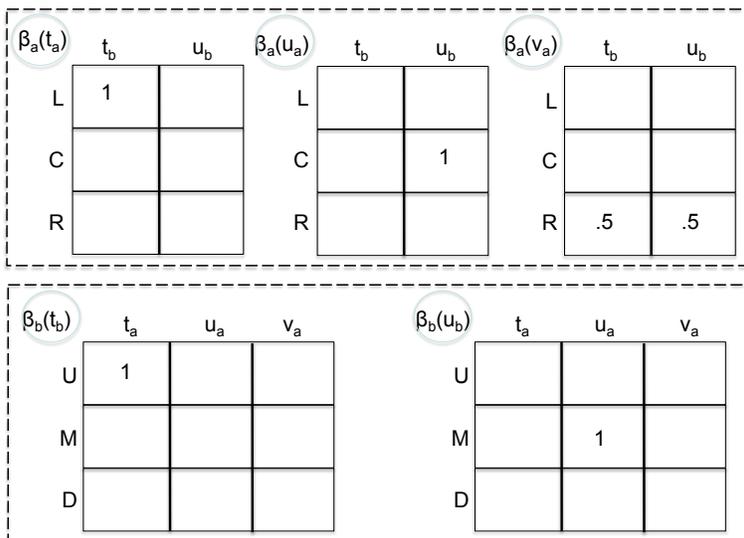| $\beta_b(u_b)$	t_a	u_a	v_a
U			
M		1	
D			

Figure 2.2

Taken together, Figures 2.1-2.2 describe the rules of the game, payoff functions, and the players' beliefs. They make up an **epistemic game**. Note, an epistemic game has two parts: First is a game G. Second, a type structure that is **compatible with the game G**, i.e., where the underlying set of uncertainty for Ann (resp. Bob) is the strategy set of Bob (resp. Ann) in the game G. Note, players' beliefs are part of the description of the epistemic game and so part of the description of the strategic situation.

Within the epistemic game, we can study the implications of strategic reasoning. In particular, a **state** specifies a strategy-type pair for each player. Intuitively, strategic reasoning should impose a restriction on the states that we focus on. Let us consider the implications of one such restriction: rationality and common belief of rationality.

Start with the requirement of rationality. Above, we argued that a strategy may be rational given some beliefs (about the play of the game) but irrational given other beliefs. Types specify beliefs (via belief maps). So, rationality can be seen as a property of a strategy-type pair. For instance, the strategy-type pair (s_a, t_a) is **rational** if s_a is a best response given the belief associated with t_a, viz. $\beta_a(t_a)$, i.e., if $s_a = $ U. Continuing along these lines, the set of rational strategy-type pairs for Ann is

$$\mathrm{R}_a^1 = \{(\mathrm{U}, t_a), (\mathrm{M}, u_a), (\mathrm{D}, v_a)\}$$

4

and the set of rational strategy-type pairs for Bob is

$$R_b^1 = \{(L, t_b), (C, u_b)\}.$$

Now turn to the requirement of belief in rationality. Since types are associated with beliefs (via the belief map), this is a requirement on types of a player. A type **believes** an event E if it assigns probability one to E under the belief map. For instance, $\beta_a(t_a)$ assigns probability one to (L, t_b), which is a rational strategy-type pair, i.e., is in R_b^1. So, t_a believes the event "Bob is rational." And, likewise, $\beta_a(u_a)$ assigns probability one to the event R_b^1. But, $\beta_a(v_a)$ does not. So, the set of strategy-type pairs of Ann that are "rational and 'believe Bob is rational'" is

$$R_a^2 = \{(U, t_a), (M, u_a)\}.$$

Likewise, the set of strategy-type pairs for Bob that are "rational and 'believe Ann is rational'" is

$$R_b^2 = \{(L, t_b), (C, u_b)\}.$$

Proceeding inductively, we can see that the set of strategy-type pairs for Ann consistent with **rationality and m^{th}-order belief of rationality (RmBR)** is $R_a^{m+1} = R_a^2$. And, likewise, $R_b^{m+1} = R_b^2$. Thus, here, the set of states at which there is **rationality and common belief of rationality (RCBR)** is $\bigcap_m R_a^m \times \bigcap_m R_b^m = R_a^2 \times R_b^2$.

For the epistemic game given by Figures 2.1-2.2, the prediction associated with RCBR is that a strategy profile in $\{U, M\} \times \{L, C\}$ will be played. Note, this is also the set of iteratively undominated (IU) strategies.

2.2 A Connection Between RCBR and IU

Return to Figures 2.1-2.2. There, the prediction associated with RCBR was exactly the set of IU strategies. This is an instance of a more general result:

Result 2.1 (Brandenburger and Dekel, 1987) *Fix a game. There is a compatible type structure so that the set of strategies consistent with RCBR is the IU strategy set.*

Result 2.1 does not hold for every epistemic game: for a given epistemic game, the set of strategies consistent with RCBR need not be the IU strategy set. To see this, instead append to the game the type structure in Figure 2.3, where each player has exactly one
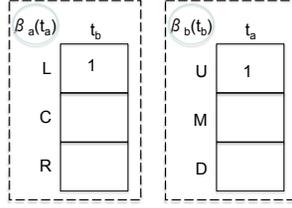
Figure 2.3

type. In the associated epistemic game, we have

$$\mathrm{R}_a^m \times \mathrm{R}_b^m = \{(\mathrm{U}, t_a)\} \times \{(\mathrm{L}, t_b)\},$$

for each m. The prediction of RCBR is then $\{\mathrm{U}\} \times \{\mathrm{L}\}$. So, we don't get all of the IU strategies. But we do get a subset. This is true more generally.

Result 2.2 (Brandenburger and Dekel, 1987) *Fix an epistemic game. The set of strategies consistent with RCBR is contained in the IU strategy set.*

2.3   An Epistemic Justification for IU

Taken together, Results 2.1-2.2 give a connection between RCBR and IU: Given an epistemic game, every strategy consistent with RCBR is an IU strategy. And, each IU strategy is consistent with RCBR in some epistemic game. This says that the analyst can justify IU play as resulting from RCBR, if the analyst looks at RCBR across all type structures.

This might appear to be the end of the matter. But, under the epistemic game theory approach, beliefs are part of the description of the strategic situation. Thus, when we write down a type structure, we specify what beliefs players do versus do not consider possible. From the perspective of the players, other type structures are simply irrelevant: They may have types that the players do not themselves consider possible. Or they may not include types the players do consider possible. So, while RCBR may justify IU from the perspective of an analyst who looks across all type structures, arguably, it does not justify IU from the perspective of the players themselves.

This raises the question: Can the players themselves see all the IU strategies as the result of a certain thought process? To provide a positive answer, we will need to restrict the players reasoning at two levels:

- By restricting the (class of) type structure(s) we analyze.

- By restricting the states within the restricted (class of) type structure(s).

Thus, a positive answer requires restricting both the set of epistemic games to be analyzed (i.e., to rule out the epistemic game in Figures 2.1 and 2.3) and the states within the restricted set of epistemic games (i.e., so that we don't consider all possible states). If we find a restriction of this kind that gives all the IU strategies, then we can justify IU from the perspective of the players themselves. In this case, we will say that we have found an **epistemic justification for IU**.

The restriction that we seek on states is RCBR. What is the desired restriction on type structures? To get at an answer, return to the epistemic game given by Figures 2.1 and 2.3. Why did we not get all of the IU strategies? Note, there was no type of Ann that assigns strictly positive probability to Bob playing C. So there is no type of Ann for which M is a best response—M must be inconsistent with rationality. Even if there were such a type, there is no type of Bob for which C is a best response. So, Ann could not assign strictly positive probability to Bob playing C, if Ann believes Bob is rational. And so on. This suggests that, to justify IU (from the players' perspective), we need a requirement that "players consider enough beliefs possible." Put differently, we need a requirement that the type structure is "rich."

What is this "richness" condition? For a given game and IU set thereof, we can tailor the type structure to be "rich enough," and, thereby, get the IU strategies as an output of RCBR. This is a consequence of Result 2.1. Yet, this does not give a satisfactory epistemic justification for IU. To see this, note that to get the type structure (as constructed by Brandenburger and Dekel, 1987), we begin with a particular game and use properties of the IU set to construct the type structure in question. In particular, the construction will have different beliefs for different games and we cannot describe the beliefs without first specifying the game in question. That is, the construction fails a certain "game independence" property.[2] This may seem like a technical point, i.e., that the construction is game dependent. But, from the perspective of epistemic game theory, it is an important conceptual point: a good epistemic justification for a solution concept should be game independent. If we allowed, quite generally, for a game dependent epistemic justification, then we could simply look at a game, decide (based on intuition) which behavior is "desirable" for that game, and tailor an epistemic condition for that game so that (by definition) we get the desired behavior. There would be no discipline to the epistemic programme.

The goal is to identify a richness condition that can be described independent of the game. For now, we note that Tan and Werlang (1988) identified one such richness condition. They restricted attention to the canonical construction of the so-called **universal type**

---

[2]Note, we use the phrase "game independence" in a limited manner. See Section 9.

structure, in Mertens and Zamir (1985). Section 9 clarifies the sense in which this can be described independent to the game. For now note that this is a particular construction of a type structure that induces all hierarchies of beliefs about the play of the game. The result is:

Result 2.3 (Tan and Werlang, 1988) *Fix a finite game and the compatible universal type structure.*

  (i) The set of strategies consistent with RmBR is the set of (m+1)-undominated strategies.

  (ii) The set of strategies consistent with RCBR is the set of IU strategies.

Tan and Werlang stated the result without proof. In the course of attempting to prove the result, the literature identified properties of the canonical construction (of a universal type structure) that suffice for a proof. Thus, the literature was able to modify the richness condition—expanding it beyond the single universal type structure of Mertens and Zamir (1985).

The key property (due to Brandenburger, 2003) is completeness. A type structure is **complete** if the belief maps (β_a and β_b, in our example) are onto. Thus, for every belief a player can hold (about the strategies and types of the other players), there is a type of the player which induces that belief. That is, the type structure has all possible beliefs about types. Now:

Result 2.4 *[Folk Result; Proposition 4.9] Fix a finite game and a compatible complete type structure.*

  (i) The set of strategies consistent with RmBR is the set of (m+1)-undominated strategies.

  (ii) If, in addition, the type sets are compact and the belief maps are continuous, then the set of strategies consistent with RCBR is the set of IU strategies.[3]

2.4 Technical Assumptions May Be Substantive Assumptions

Revisit Result 2.4(ii). It says that we can justify the entire IU set as an output of players' reasoning: IU is an output of RCBR in a type structure that is complete with compact type sets and continuous belief maps. Note, then, compactness, continuity, completeness, and RCBR give an epistemic condition for IU, and this epistemic condition encompasses the epistemic condition of RCBR given by the single universal type structure.

---

[3]This "folk result" is a special case of Proposition 6 in Battigalli and Siniscalchi (2002).

How should these conditions be interpreted? RCBR is a condition on players' reasoning. Likewise, we can interpret the completeness condition as an output of players' reasoning—it is the condition that all possible beliefs are present (and so all beliefs are "considered" by the players). But, compactness and continuity are technical conditions. As technical conditions, they are not easily interpretable.

Why do we find compactness and continuity in the statement of Result 2.4(ii)? If the answer is "for technical convenience," then these requirements can be ignored. But, if the answer is "because they are necessary for the result," then they are integral components of the epistemic condition for IU. In the latter case, arguably, we have not yet understood conditions on players' reasoning that give IU—precisely, because compactness and continuity restrict players' reasoning in a way that is not transparent.

In Section 5 we show that these seemingly technical requirements cannot be dispensed with.

Result 2.5 *[Theorem 5.2] For each non-trivial finite game, there exists a compatible complete type structure so that:*

(i) The set of strategies consistent with RmBR is the set of $(m+1)$-undominated strategies.

(ii) There is no state at which there is RCBR.

At first glance, the constructed type structure appears rich, in so far as it contains all possible beliefs about types, i.e., it is complete. But, this result tells us that it is not rich enough to give the IU strategy set—or even some state at which there is RCBR.

To understand why we can construct such a complete type structure, refer back to Figure 2.1. Note, iterated dominance gives:

	Set of Strategies that Survive
Round 1	$\{U, M, D\} \times \{L, C\}$
Round 2	$\{U, M\} \times \{L, C\}$
Round 3	$\{U, M\} \times \{L, C\}$
Round 4	. . .

Now, refer to Result 2.4(i): It says that a strategy s_a survives m rounds of deletion if and only if (in a complete type structure) the set of types t_a so that (s_a, t_a) satisfies "rationality and $(m-1)^{th}$-order belief of rationality" is nonempty. Write $\mathbb{T}[s_a, m]$ for this set. Note, in our example then, for any complete type structure, we have:

9

$$\mathbb{T}[U,1] \neq \emptyset \quad \mathbb{T}[M,1] \neq \emptyset \quad \mathbb{T}[D,1] \neq \emptyset$$
$$\mathbb{T}[U,2] \neq \emptyset \quad \mathbb{T}[M,2] \neq \emptyset \quad \mathbb{T}[D,2] = \emptyset$$
$$\mathbb{T}[U,3] \neq \emptyset \quad \mathbb{T}[M,3] \neq \emptyset \quad \mathbb{T}[D,3] = \emptyset$$
$$\cdots \qquad\qquad \cdots \qquad\qquad \cdots$$

The question we then have is: If a strategy s_a survives IU, is it the case that there is a type t_a so that t_a believes "Bob is rational," "Bob is rational and 'Bob believes I am rational,'" etc.? This is equivalent to, if a strategy s_a survives IU, is it the case that there is a type t_a so that $t_a \in \mathbb{T}[s_a,1], \mathbb{T}[s_a,2], \ldots$? That is, if s_a survives IU, is $\bigcap_m \mathbb{T}[s_a,m]$ nonempty?
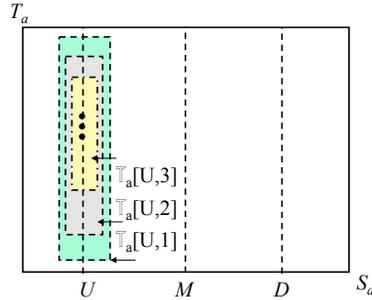

Figure 2.4

Refer to Figure 2.4. The strategy U survives IU and so we are looking for a type in $\bigcap_m \mathbb{T}[U,m]$. The sets $\mathbb{T}[U,m]$ are shrinking, i.e., $\mathbb{T}[U,m+1] \subseteq \mathbb{T}[U,m]$. In fact, because the type structure it complete, the sets $\mathbb{T}[U,m]$ must be strictly shrinking, i.e., $\mathbb{T}[U,m+1] \subsetneq \mathbb{T}[U,m]$.[4] In principle, then, we might have that the intersection is empty.

Result 2.5(ii) tells us that, when the type sets are compact and the belief maps are continuous, we must have a non-empty intersection—the reason this is the case is that, then, the sets $\mathbb{T}[U,m]$ are closed subsets of a compact set. But, the result shows that we can construct some complete type structure so that the intersection of these sets is empty. In particular, we construct a complete type structure that has Polish (but not compact) type sets and continuous belief maps, but for which the intersection is empty. (This is Theorem 5.2.) As a by-product, we also get an example of a complete type structure that has compact types sets and discontinuous belief maps, but for which the intersection is empty. (This is Corollary 5.9.)

2.5   Completeness and Hierarchies of Beliefs

The following known result gives sufficient conditions for a complete type structure to induce (i) all finite order beliefs, and (ii) all hierarchies of beliefs.

---

[4]This is not particular to the example—it must hold in any non-trivial game. The proof is omitted.

Result 2.6 (Friedenberg, 2010; Theorem 3.1) *Let $\mathcal{T}$ be a complete type structure.*

(i) *If T_a and T_b are Polish (or even analytic) spaces, then $\mathcal{T}$ induces all finite-order beliefs (i.e., is finitely terminal).*

(ii) *If $\mathcal{T}$ is compact and continuous, then $\mathcal{T}$ induces all hierarchies of beliefs (i.e., is terminal).*

Return to Result 2.5. To put that result in perspective, let us ask: Which hierarchies of beliefs are induced by the complete type structure $\mathcal{T}$ constructed there? By Result 2.6(i), $\mathcal{T}$ induces all finite-order beliefs. But, $\mathcal{T}$ does not induce all hierarchies of beliefs. If it did, Result 2.5(ii) could not hold—the strategies consistent with RCBR must be the IU strategies. So, while $\mathcal{T}$ induces all beliefs about types, it does not induce all hierarchies of beliefs.

This raises the question: Which hierarchies of beliefs are induced by every complete type structure? We show that a complete type structure induces all finite-order beliefs that are induced by types in *atomic type structures*. (See Proposition 7.14). A consequence of this result is that a complete type structure induces all finite-order beliefs that are induced by types in *simple type structures*. (The notions of atomic type structures and simple type structures will be defined in Section 7.) While a complete type structure induces all finite-order beliefs that are induced by types in simple type structures, this does not imply that it induces all finite-order beliefs with finite support. In fact, we show that, under a set-theoretic hypothesis that is known to be consistent, there is a complete type structure that does not induce all finite-order beliefs are given by a measure with finite support.

2.6 Epistemic Conditions for IU Revisited

Return to Result 2.4(i): For an epistemic game with a complete type structure, the set of strategies consistent with RmBR is the set of $(m+1)$-undominated strategies. If the type set is analytic, then by Result 2.6(i), the type structure induces all finite order beliefs. But, Result 2.4(i) does not impose this technical requirement on the type sets. And, we have seen that, absent such a requirement, the complete type structure need not induce all finite-order beliefs. Thus, the completeness condition used in Result 2.4(i) cannot be interpreted as the requirement that the type structure induces all finite-order beliefs.

This raises the question: How should we interpret the role of the completeness condition in 2.4(i)? The discussion above suggests an answer—it is a requirement that the type structure induces all finite-order beliefs induced by types in atomic type structures. Indeed,

if a type structure induces all finite-order beliefs induced by types in atomic structures, then RmBR is characterized by the $(m + 1)$-undominated strategies. This implies that, if a type structure induces all finite-order beliefs induced by types in simple structures, then RmBR is characterized by the $(m + 1)$-undominated strategies.

Result 2.7 *[Theorem 8.1] Fix a finite game and a compatible type structure.*

(i) *Suppose the type structure induces all finite-order beliefs that are induced by types in simple type structures. Then, the set of strategies consistent with RmBR is the set of $(m + 1)$-undominated strategies.*

(ii) *Suppose the type structure induces all hierarchies of beliefs that are induced by types in simple type structures. Then the set of strategies consistent with RCBR is the set of IU strategies.*

By Result 2.4(ii), completeness, compactness, and continuity give an epistemic condition for IU. By Result 2.6(ii), a type structure that is complete, compact, and continuous induces all hierarchies of beliefs. Result 2.7(ii) above gives another epistemic condition for IU. It replaces completeness, compactness, and continuity by a new condition that makes explicit reference to players' hierarchies of beliefs—namely, that the type structure induces all hierarchies of beliefs that are induced by types in simple type structures.

3 Type Structures

Throughout we take the following conventions: When we write Ω, we refer to a metrizable space. Given a metrizable space Ω, we endow Ω with the Borel sigma-algebra. The set of Borel probability measures on Ω is $\mathcal{P}(\Omega)$. We endow $\mathcal{P}(\Omega)$ with the topology of weak convergence. We endow the product of metrizable spaces with the product topology.

There are two players, viz. a and b.[5] Write c for an arbitrary player from a, b and write d for the other player. Each player c has a basic set of uncertainty (or set of parameters), viz. X_c. We always take the set of parameters X_c to be metrizable.

Definition 3.1 *An (X_a, X_b)-**based type structure**, viz. $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$, consists of metrizable **parameter sets** X_a, X_b, metrizable **type spaces** T_a, T_b and measurable **belief maps***

$$\beta_a : T_a \to \mathcal{P}(X_a \times T_b) \quad and \quad \beta_b : T_b \to \mathcal{P}(X_b \times T_a).$$

---

[5]The restriction to two players is immaterial, up to issues of correlation.

*We refer to a quadruple (s_a, t_a, s_b, t_b) as a **state**.*

We will be interested in type structures that satisfy certain properties.

Definition 3.2 *Fix an (X_a, X_b)-based type structure, viz. $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$. Say the type structure is **compact** (resp. **Polish**) if T_a and T_b are compact (resp. Polish). Say the type structure is **continuous** if β_a and β_b are continuous.*

Next is the requirement that a type structure contain all possible beliefs:

Definition 3.3 (Brandenburger, 2003) *An (X_a, X_b)-based type structure, viz. $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$, is **complete** if β_a and β_b are onto.*

4 Epistemic Games and RCBR

Fix a finite two-player strategic-form game $G = (S_a, S_b, \pi_a, \pi_b)$, where S_a, S_b are the finite strategy sets a $\pi_a : S_a \times S_b \to \mathbb{R}, \pi_b : S_b \times S_a \to \mathbb{R}$ are payoff functions.

One case of particular interest is where each player faces uncertainty about the strategy the other player employs. Here, then, player a's space of basic uncertainty X_a is S_b. Thus, we append to $G = (S_a, S_b, \pi_a, \pi_b)$ an (S_b, S_a)-based type structure $\mathcal{T} = (S_b, S_a; T_a, T_b; \beta_a, \beta_b)$.

Definition 4.1 *An **epistemic game** is a pair $(G, \mathcal{T})$, where $G = (S_a, S_b, \pi_a, \pi_b)$ is a finite game and $\mathcal{T}$ is an (S_b, S_a)-based type structure.*

Within an epistemic game, viz. $(G, \mathcal{T})$, we can define rationality and common belief of rationality. To do so, begin by extending π_c to $\mathcal{P}(S_c) \times \mathcal{P}(S_d)$ in the usual way, i.e. $\pi_c(\sigma_c, \sigma_d) = \sum_{(s_c, s_d) \in S_c \times S_d} \sigma_c(s_c)\sigma_d(s_d)\pi_c(s_c, s_d)$.

Definition 4.2 *Say s_c is **optimal under** $\sigma_c \in \mathcal{P}(S_d)$ if $\pi_c(s_c, \sigma_d) \geq \pi_c(r_c, \sigma_d)$ for all $r_c \in S_c$.*

Definition 4.3 *A strategy-type pair (s_c, t_c) is **rational** if s_c is optimal under $\text{marg}_{S_d}\beta_c(t_c)$.*

Definition 4.4 *Say $\mu \in \mathcal{P}(\Omega)$ **believes E** if E is an event and $\mu(E) = 1$.*

Definition 4.5 *Say a type t_c **believes E_d** if $\beta_c(t_c)$ believes E_d.*

Write

$$\mathrm{B}_c(E_d) = S_c \times \{t_c : t_c \text{ believes } E_d\},$$

for the set of strategy-type pairs that believe E_d.

Set $\mathrm{R}_c^0 = S_c \times T_c$ and take R_c^1 for the set of rational strategy-type pairs of player c. For each $m \geq 1$, set $\mathrm{R}_c^{m+1} = \mathrm{R}_c^m \cap \mathrm{B}_c(\mathrm{R}_d^m)$.

Definition 4.6 *The sets of states at which there is **rationality and $\mathbf{m^{th}}$-order belief of rationality (RmBR)** is $\mathrm{R}_a^{m+1} \times \mathrm{R}_b^{m+1}$. The sets of states at which there is **rationality and common belief of rationality (RCBR)** is $\bigcap_{m=1}^{\infty}[\mathrm{R}_a^m \times \mathrm{R}_b^m]$.*

Now define the set of strategies that survive iterative elimination of strongly dominated strategies.

Definition 4.7 *Fix $Y_a \times Y_b \subseteq S_a \times S_b$. A strategy $s_c \in Y_c$ is **strongly dominated with respect to** $Y_c \times Y_d$ if there exists $\sigma_c \in \mathcal{P}(S_c)$ with $\sigma_c(Y_c) = 1$ and $\pi_c(\sigma_c, s_d) > \pi_c(s_c, s_d)$ for every $s_d \in Y_d$. Otherwise, say s_c is **undominated with respect to** $Y_c \times Y_d$.*

Note, then, if $Y_b = \emptyset$, then the convention is that each $s_c \in Y_c$ is strongly dominated with respect to $Y_c \times Y_d$.

Definition 4.8 *Set $S_c^0 = S_c$ and define inductively*

$$S_c^{m+1} = \{s_c \in S_c^m : s_c \text{ is undominated with respect to } S_c^m \times S_d^m\}.$$

*A strategy $s_c \in S_c^m$ is called **m-undominated**. A strategy $s_c \in \bigcap_{m=1}^{\infty} S_c^m$ is called **iteratively undominated (IU)**.*

The following epistemic justification for the m-undominated and the IU undominated strategies is well-known:

Proposition 4.9 (Folk-Result) *Fix an epistemic game $(G, \mathcal{T})$, where $\mathcal{T}$ is complete.*

(i) For each $m \geq 1$, $\mathrm{proj}_{S_a} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m$.

(ii) If $\mathcal{T}$ is compact and continuous, then

$$\mathrm{proj}_{S_a} \bigcap_{m=1}^{\infty} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \bigcap_{m=1}^{\infty} \mathrm{R}_b^m = \bigcap_{m=1}^{\infty} S_a^m \times \bigcap_{m=1}^{\infty} S_b^m.$$

14

Part (i) says that, in a complete type structure, the set of strategies consistent with RmBR is the set of strategies that survive $(m + 1)$ rounds of eliminating dominated strategies.[6] Part (ii) says that if, in addition, the complete type structure is compact and continuous, then the set of strategies consistent with RCBR is the set of IU strategies.

5   A Negative Result

Following Marx and Swinkels (1997), say a strategy s_c is **very weakly dominant** if, for each $r_c \in S_c$ and $s_d \in S_d$, $\pi_c(s_c, s_d) \geq \pi_c(r_c, s_d)$.

Definition 5.1 *Say a game $G = (S_a, S_b, \pi_a, \pi_b)$ is **non-trivial** if no player has a very weakly dominant strategy.*

Note, if a game is non-trivial, then, for each strategy s_c, there is some $\sigma_d \in \mathcal{P}(S_d)$ so that s_c is not optimal under σ_d. The game in Figure 2.1 is non-trivial.

Theorem 5.2 *Fix a finite non-trivial game G. There exists a Polish, continuous, and complete (S_b, S_a)-based type structure $\mathcal{T}$ so that*

(i) For each $m \geq 1$, $\mathrm{proj}_{S_a} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m$.

(ii) $\bigcap_{m=1}^{\infty} \mathrm{R}_a^m = \emptyset$ and $\bigcap_{m=1}^{\infty} \mathrm{R}_b^m = \emptyset$.

Part (i) is immediate from Proposition 4.9(i). We now turn to the proof of Part (ii).

Recall, the idea from Section 2. The set $\mathbb{T}[s_c, m]$ was the set of types t_c so that $(s_c, t_c) \in \mathrm{R}_c^m$. In a complete type structure, the sets $\mathbb{T}[s_c, m]$ are shrinking. The goal then is to construct a complete type structure so that the intersection of these sets is empty.

To construct this type structure, it will be useful to express the m-undominated strategies as an output of an **$m^{\mathbf{th}}$-order best response maps**: For each $m \geq 0$, let

$$\mathrm{BR}_c^{m+1} : \{\sigma_d \in \mathcal{P}(S_d) : \sigma_d(S_d^m) = 1\} \to 2^{S_c} \setminus \{\emptyset\}.$$

map each probability measure $\sigma_d \in \mathcal{P}(S_d)$ that believes S_d^m into the set of strategies that are optimal under that measure. We refer to an element of the range of BR_c^m as an **$m^{\mathbf{th}}$-order best response set** or, simply, a **best response set** if $m = 1$. Write $\mathbb{S}_c^m$ for the range of the mth-order best response map, i.e., the collection of all m^{th}-order best response sets.

---

[6]This is typically proved in set-ups where T_a and T_b are Polish. However, an inspection of the proofs shows that the full-force of the Polish requirement is not used.

Back to Figure 2.1. There, $\mathbb{S}_a^1$ is the set of all non-empty subsets of S_a and, for each $m \geq 2$, $\mathbb{S}_a^m = \{\{U\}, \{M\}, \{U, M\}\}$.

Clearly, for each m, the sets S_c^m and $\mathbb{S}_c^m$ are non-empty. Note the following properties of $\mathbb{S}_c^m$:

Properties 5.3

(i) If $Q_c \in \mathbb{S}_c^m$, then $Q_c \subseteq S_c^m$

(ii) If $s_c \in S_c^m$, then there exists some $Q_c \in \mathbb{S}_c^m$ so that $s_c \in Q_c$.

Conditions (i)-(ii) say that the sets $\mathbb{S}_c^m$ characterize the set of strategies that survive m-rounds of iterated dominance.

In constructing the type structure for Theorem 5.2, we will "match up" subsets of types with m^{th}-order best response sets. Specifically, for each $Q_c \in \mathbb{S}_c^m$, we will have a subset of types $\mathrm{T}[Q_c, m]$. We will later show that we can choose the belief maps so that $\mathrm{T}[Q_c, m]$ is the set of all types t_c so that (i) the set of strategies optimal under $\mathrm{marg}_{S_d} \beta_c(t_c)$ is Q_c and (ii) t_c believes R_d^{m-1}.

The Baire space is $\mathbb{N}^{\mathbb{N}}$, where we take as an open basis the family of "cones"

$$\{\{(n_0, n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}} : (n_0, \ldots, n_k) = (o_0, \ldots, o_k)\} \text{ for each } k \in \mathbb{N} \text{ and } o_0, \ldots, o_k \in \mathbb{N}.$$

Take T_a and T_b to be copies of the Baire space. So, each element of T_a and T_b is then some $(n_0, n_1, n_2, \ldots)$, where each n_k is a natural number. The spaces T_a and T_b are Polish, but not compact.

It will be useful to index the sets in the range of the 1st-order best response map, i.e., setting $\mathbb{S}_c^1 = \{Q_{c,0}, \ldots, Q_{c,K}\}$. Then, for each integer k with $K > k \geq 0$ set

$$\mathrm{T}[Q_{c,k}, 1] = \{(n_0, n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}} : n_0 = k\}$$

and set

$$\mathrm{T}[Q_{c,K}, 1] = \{(n_0, n_1, n_2, \ldots) \in \mathbb{N}^{\mathbb{N}} : n_0 \geq K\}.$$

Figure 5.1 illustrates the construction of $\mathrm{T}[\{U\}, 1]$ and $\mathrm{T}[\{D\}, 1]$, for the game in Figure 2.1. In the illustration, each type $t^a \in \mathbb{N}^{\mathbb{N}}$ corresponds to a path that contains one point in each row. The set $\mathbb{S}_a^1$ is the set of non-empty subsets of S_a. List

$$\mathbb{S}_a^1 = \{\{U\}, \{M\}, \{U, M\}, \{D\}, \{U, D\}, \{M, D\}, \{U, M, D\}\}$$

and note that, in the illustration, the heading of the kth column is the kth set $Q_{a,k}$ in this list. For example, $\{D\} = Q_{a,3}$ is the heading of column 3. So, $\mathrm{T}\left[\{D\}, 1\right]$ is the set of all points in $n \in \mathbb{N}^{\mathbb{N}}$ such that $n_0 = 3$, and corresponds to the set of all paths that select one red dot in each row. Thus, the zeroth row of a type can be seen as "tracking an associated best response set."
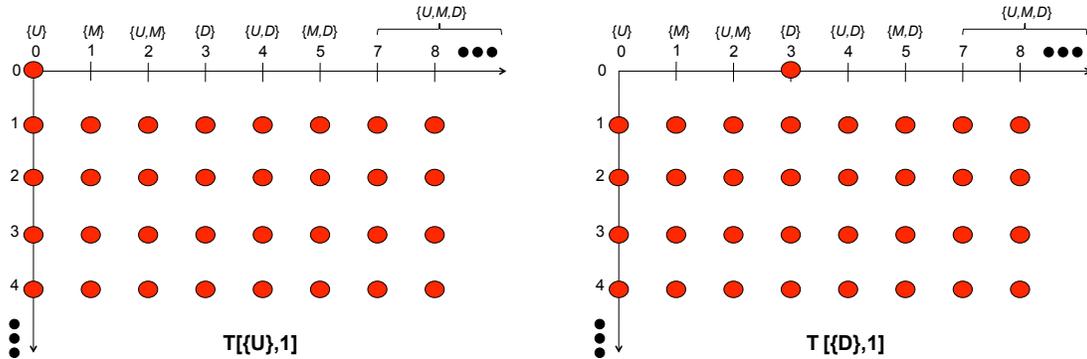

Figure 5.1

Next, for each $m \geq 1$, set

$$\mathrm{T}\left[Q_{c,k}, m+1\right] = \{(n_0, n_1, n_2, \ldots) \in \mathrm{T}\left[Q_{c,k}, m\right] : n_1 \geq m+1\}$$

if $Q_{c,k} \in \mathbb{S}_c^m$, and set

$$\mathrm{T}\left[Q_{c,k}, m+1\right] = \emptyset$$

otherwise. So, referring to our example, $\mathrm{T}\left[\{D\}, m\right] = \emptyset$, for all $m \geq 2$. Figure 5.2 illustrates the sets $\mathrm{T}\left[\{U\}, 2\right]$ and $\mathrm{T}\left[\{U\}, 3\right]$. Note, then, the first row can be seen as "tracking whether a best response set is an mth-order best response set," for $m \geq 2$.
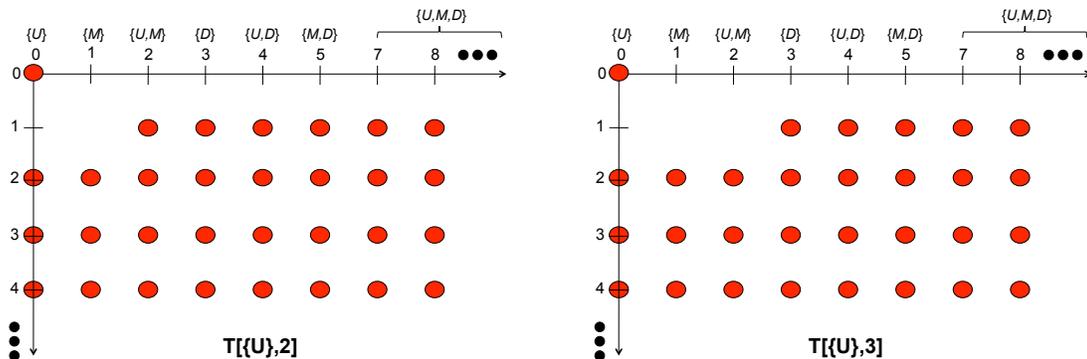

Figure 5.2

17

By **a Baire space**, we mean a topological space which is homeomorphic to the Baire space. With this, we have the following properties of the sets $T[Q_c, m]$:

Properties 5.4 *For each $m \in \mathbb{N}$ with $m > 0$ and each $Q_c \in \mathbb{S}_m^c$, we have:*

(i) $T[Q_c, m+1] \subseteq T[Q_c, m]$.

(ii) $\bigcap_m T[Q_c, m] = \emptyset$.

(iii) $T[Q_c, m+1]$ *is clopen in* T_c.

(iv) *If* $T[Q_c, m+1] \neq \emptyset$, $T[Q_c, m+1]$ *is a Baire space.*

(v) *If* $T[Q_c, m] \neq \emptyset$, $T[Q_c, m] \backslash T[Q_c, m+1]$ *is a Baire space.*

Let $P_c^0 = S_c \times T_c$ and, for each $m \geq 1$, define

$$P_c^m = \bigcup \{Q_c \times T[Q_c, m] : Q_c \in \mathbb{S}_c^m\}$$

The plan is to have $T[Q_c, 1]$ map onto the set of all probability measures $\psi_d \in \mathcal{P}(S_d \times T_d)$ so that $Q_c = \mathrm{BR}_c^1(\mathrm{marg}_{S_d} \psi_d)$, and to map $T[Q_c, m+1]$ onto the set of all probability measures $\psi_d \in \mathcal{P}(S_d \times T_d)$ so that $Q_c = \mathrm{BR}_c^1(\mathrm{marg}_{S_d} \psi_d)$ and P_d^m is believed under ψ_d. Then, the desire that $\bigcap_m \mathbb{T}[s_c, m] = \emptyset$ will follow from the fact that $\bigcap_m T[Q_c, m] = \emptyset$ for each Q_c that contains s_c.

With this in mind, we will extend the idea of mth-order best response maps in the space of strategies cross types: Specifically, for each $m \geq 0$ set

$$\mathbb{BR}_c^{m+1} : \{\psi_d \in \mathcal{P}(S_d \times T_d) : \psi_d(P_d^m) = 1\} \to 2^{S_c} \setminus \{\emptyset\}$$

so that $\mathbb{BR}_c^{m+1}(\psi_d)$ is the set of all strategies that are optimal under $\mathrm{marg}_{S_d} \psi_d$. Then, for each $m \geq 1$, $(\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is the set of probability measures ψ_d so that Q_c is the set of strategies optimal under $\mathrm{marg}_{S_d} \psi_d$ and ψ_d believes P_d^m.

We will want to construct maps $\beta_a : T_a \to \mathcal{P}(S_b \times T_b)$ and $\beta_b : T_b \to \mathcal{P}(S_a \times T_a)$. We next show that, if these maps "match" each of the sets $T[Q_c, m]$ to the sets $(\mathbb{BR}_c^m)^{-1}(Q_c)$, then the RmBR states are exactly $P_a^{m+1} \times P_b^{m+1}$.

Lemma 5.5 *Suppose β_a and β_b are such that, for each $m \geq 1$ and each $Q_c \in \mathbb{S}_c^1$,*

$$(\beta_c)^{-1}((\mathbb{BR}_c^m)^{-1}(Q_c)) = T[Q_c, m]. \tag{1}$$

Then, for each $m \geq 1$, $\mathrm{R}_c^m = P_c^m$.

Proof. The proof is by induction on m.

$\mathbf{m = 1}$: Suppose $(s_c, t_c) \in P_c^1$. Then, there exists some $Q_c \in \mathbb{S}_c^1$ so that $(s_c, t_c) \in Q_c \times$ T$[Q_c, 1]$. By Equation (1), $Q_c = \mathbb{BR}_c^1(\beta_c(t_c))$. Since $s_c \in Q_c$, it follows that s_c is optimal under marg $_{S_d}\beta_c(t_c)$, i.e., $(s_c, t_c) \in \mathrm{R}_c^1$.

Conversely, suppose $(s_c, t_c) \in \mathrm{R}_c^1$. Then, $s_c \in \mathrm{BR}_1^c(\mathrm{marg}\,_{S_d}\beta_c(t_c))$. Write $Q_c = \mathrm{BR}_1^c(\mathrm{marg}\,_{S_d}\beta_c(t_c))$ and note that $s_c \in Q_c \in \mathbb{S}_c^1$. Moreover, $t_c \in (\beta_c)^{-1}((\mathbb{BR}_c^1)^{-1}(Q_c))$ and so, by Equation (1), $t_c \in \mathrm{T}[Q_c, 1]$. That is, $(s_c, t_c) \in Q_c \times \mathrm{T}[Q_c, 1] \subseteq P_c^1$, as required.

$\mathbf{m \geq 2}$: Assume the result was shown for m. We will show that the same holds for $m + 1$.

Suppose $(s_c, t_c) \in P_c^{m+1}$. Then, there exists some $Q_c \in \mathbb{S}_c^{m+1}$ so that $(s_c, t_c) \in Q_c \times$ T$[Q_c, m + 1]$. By Equation (1), $Q_c = \mathbb{BR}_c^{m+1}(\beta_c(t_c))$. Since $s_c \in Q_c$, it follows that s_c is optimal under marg $_{S_d}\beta_c(t_c)$. Moreover, since $\beta_c(t_c)$ is in the domain of $\mathbb{BR}_c^{m+1}$, it follows from the induction hypothesis that $\beta_c(t_c)$ believes $P_d^m = \mathrm{R}_d^m$. Moreover, by the induction hypothesis and Property 5.4(iii), each $\mathrm{R}_d^1, \ldots, \mathrm{R}_d^m$ is Borel. Thus, t_c believes each $\mathrm{R}_d^1, \ldots, \mathrm{R}_d^m$ and so $(s_c, t_c) \in \mathrm{R}_c^{m+1}$.

Conversely, suppose $(s_c, t_c) \in \mathrm{R}_c^{m+1}$. Then, using the induction hypothesis, $\beta_c(t_c)$ believes $\mathrm{R}_d^m = P_d^m$. It follows that $s_c \in \mathrm{BR}_c^{m+1}(\mathrm{marg}\,_{S_d}\beta_c(t_c))$. Write $Q_c = \mathrm{BR}_{m+1}^c(\mathrm{marg}\,_{S_d}\beta_c(t_c))$ and note that $s_c \in Q_c \in \mathbb{S}_c^{m+1}$. Moreover, $t_c \in (\beta_c)^{-1}((\mathbb{BR}_c^{m+1})^{-1}(Q_c))$ and so, by Equation (1), $t_c \in \mathrm{T}[Q_c, m + 1]$. That is, $(s_c, t_c) \in Q_c \times \mathrm{T}[Q_c, m + 1] \subseteq P_c^{m+1}$, as required. ∎

Lemma 5.6 *For each $m \geq 1$, proj $_{S_a}P_a^m \times$ proj $_{S_b}P_b^m = S_a^m \times S_b^m$.*

Proof. If $s_c \in$ proj $_{S_c}P_c^m$, then there exists $Q_c \in \mathbb{S}_c^m$ so that $s_c \in Q_c$. By Property 5.3(i), $s_c \in S_c^m$. Conversely, fix $s_c \in S_c^m$. Then, by Property 5.3(ii), there exists $Q_c \in \mathbb{S}_c^m$ so that $s_c \in Q_c$. It follows that $s_c \in$ proj $_{S_c}P_c^m$. ∎

Corollary 5.7 *Suppose β_a and β_b are such that, for each $m \geq 1$ and each $Q_c \in \mathbb{S}_c^1$,*

$$(\beta_c)^{-1}((\mathbb{BR}_c^m)^{-1}(Q_c)) = \mathrm{T}[Q_c, m].$$

Then, for each $m \geq 1$, proj $_{S_a}\mathrm{R}_a^m \times$ proj $_{S_b}\mathrm{R}_b^m = S_a^m \times S_b^m$.

Proof of Theorem 5.2. Take T_a and T_b to be copies of the Baire space. We will show that we can find onto continuous maps β_a and β_b so that for each $m \geq 1$ and each $Q_c \in \mathbb{S}_c^1$,

$$(\beta_c)^{-1}((\mathbb{BR}_c^m)^{-1}(Q_c)) = \mathrm{T}[Q_c, m]. \tag{2}$$

If this holds, then part (i) follows from Corollary 5.7 and part (ii) follows from Lemma 5.5 and Property 5.4(ii).

Consider the collection of sets of types

$$\mathcal{T}_c = \{\mathrm{T}\,[Q_c, m]\backslash\mathrm{T}\,[Q_c, m+1] : Q_c \in \mathbb{S}_c^1 \text{ and } m \geq 1\}$$

and the collection of sets of probabilities

$$\mathcal{P}_c = \{(\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}(Q_c) : Q_c \in \mathbb{S}_c^1 \text{ and } m \geq 1\}.$$

Note, $\mathcal{T}_c$ partitions T_c into pairwise disjoint sets and $\mathcal{P}_c$ partitions $\mathcal{P}(S_d \times T_d)$ into pairwise disjoint sets. (See Lemmata A.1-A.2.) We will first show that we can find a continuous mapping from each element of $\mathcal{T}_c$ to its "matching set" in $\mathcal{P}_c$. Then, we will use the fact that these collections form a partition to construct an onto map β_c satisfying the desired requirements.

Step 1: Fix some $Q_c \in \mathbb{S}_c^1$ and some $m \geq 1$. Note, $\mathrm{T}\,[Q_c, m]\backslash\mathrm{T}\,[Q_c, m+1]$ is nonempty if and only if $(\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is nonempty. (See Lemma A.6, which makes use of the fact that the game is nontrivial.) Moreover, using Properties 5.4(iii)-(v), if $\mathrm{T}\,[Q_c, m]\backslash\mathrm{T}\,[Q_c, m+1]$ is nonempty, then it is a Baire space clopen in T_c. Similarly, $(\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is a Polish space. (See Lemma A.4.) So, if $\mathrm{T}\,[Q_c, m]\backslash\mathrm{T}\,[Q_c, m+1]$ (and so $(\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}(Q_c)$) is nonempty, there is a continuous mapping, viz. $\beta_c^{[Q_c, m]}$, from $\mathrm{T}\,[Q_c, m]\backslash\mathrm{T}\,[Q_c, m+1]$ onto $(\mathbb{BR}_c^m)^{-1}(Q_c)\backslash(\mathbb{BR}_c^{m+1})^{-1}(Q_c)$. (See Kechris, 1995, Theorem 7.9)

Step 2: Take β_c to be the union of the maps $\beta_c^{[Q_c, m]}$. Note, this is well-defined since $\mathcal{T}_c$ partitions T_c. Using the fact that $\mathcal{P}_c$ partitions $\mathcal{P}(S_d \times T_d)$ and each of the maps $\beta_c^{[Q_c, m]}$ is onto, it follows that the map β_c is onto. Again using the fact that $\mathcal{P}_c$ partitions $\mathcal{P}(S_d \times T_d)$, it follows that Equation (2) is satisfied. Finally, continuity follows from Lemma A.7. ∎

Theorem 5.2 says that, for each non-trivial finite game, there exists a Polish, continuous, and complete type structure, so that the RCBR set is empty. We will now show that, for each non-trivial finite game, there exists a compact and complete type structure, so that the RCBR set is empty. This will be a corollary of the following result:

Lemma 5.8 *Fix a finite non-trivial game G and two uncountable Polish spaces T_a, T_b.*

There exists a complete (S_b, S_a)-based type structure

$$\mathcal{T} = (S_b, S_a; T_a, T_b; \beta_a, \beta_b)$$

so that

(i) For each $m \geq 1$, $\operatorname{proj}_{S_a} R_a^m \times \operatorname{proj}_{S_b} R_b^m = S_a^m \times S_b^m$.

(ii) $\bigcap_{m=1}^{\infty} R_a^m = \emptyset$ and $\bigcap_{m=1}^{\infty} R_b^m = \emptyset$.

Recall, every uncountable Polish space has the cardinality of the continuum, $\mathfrak{c}$ (See Kechris, 1995, Corollary 6.5.) Let Ω and Φ be metrizable. If $f : \Omega \to \Phi$ is a measurable map, define $\underline{f} : \mathcal{P}(\Omega) \to \mathcal{P}(\Phi)$ so that $\underline{f}(\mu)$ is the image measure of f under μ. The function $\underline{f}$ is measurable. (See Lemma B.1.) If Ω, Φ are Polish and f is a Borel bijection, then $\underline{f}$ is also a Borel bijection. (cite). Moreover, in this case, $\underline{f}$ also maps Borel sets to Borel sets. (See Kechris, 1995, Corollary 15.2.)

Proof of Lemma 5.8. By Theorem 5.2, there is a complete type structure

$$\mathcal{U} = (S_b, S_a, U_a, U_b, \gamma_a, \gamma_b),$$

so that U_a, U_b are uncountable Polish spaces, γ_a, γ_b are continuous, and conditions (i)-(ii) are satisfied. Both T_c and U_c have the same cardinality, namely $\mathfrak{c}$. So, by the Borel isomorphism theorem (see Kechris, 1995, Theorem 15.6), there is a Borel bijection α_c from T_c to U_c. Write $\operatorname{id}_d : S_d \to S_d$ for the identity map. Then, $\underline{\operatorname{id}_d \times \alpha_d} : \mathcal{P}(S_d \times T_d) \to \mathcal{P}(S_d \times U_d)$ is a Borel bijection. Take $\beta_c = (\underline{\operatorname{id}_d \times \alpha_d})^{-1} \circ \gamma_c \circ \alpha_c$. Note, β_c is onto and Borel. (The map β_c may not be continuous.) It follows that $\mathcal{T}$ satisfies conditions (i) and (ii). ∎

Note that Lemma 5.8 fails when T_c is countable (or even of cardinality $< \mathfrak{c}$), because in that case there are no complete (S_b, S_a)-based type structures with type sets T_a, T_b. We do not know what happens when T_c is metrizable of cardinality $\geq \mathfrak{c}$ but not Polish. A consequence of Lemma 5.8 is:

Corollary 5.9 *Fix a finite non-trivial game G. There exists a compact and complete (S_b, S_a)-based type structure $\mathcal{T} = (S_b, S_a; T_a, T_b; \beta_a, \beta_b)$ so that*

(i) For each $m \geq 1$, $\operatorname{proj}_{S_a} R_a^m \times \operatorname{proj}_{S_b} R_b^m = S_a^m \times S_b^m$.

(ii) $\bigcap_{m=1}^{\infty} R_a^m = \emptyset$ and $\bigcap_{m=1}^{\infty} R_b^m = \emptyset$.

To see Corollary 5.9, take $T_a = T_b$ to be copies of the Cantor space $\{0,1\}^{\mathbb{N}}$. This is an uncountable compact metrizable space. Now apply Lemma 5.8.

Return to Proposition 4.9(ii): Given an epistemic game $(G, \mathcal{T})$, where $\mathcal{T}$ is complete, compact, and continuous, the strategies consistent with RCBR are exactly the IU strategies. But, per Theorem 5.2, this conclusion need not follow, if $\mathcal{T}$ is complete and continuous, but not compact. Per Corollary 5.9, this conclusion also need not follow, if $\mathcal{T}$ is complete and compact, but not continuous. Thus, what seem to be technical conditions (compactness and continuity) are actually substantive assumptions about players' reasoning.

This then raises the question: What do compactness and continuity buy us, as a restriction on players reasoning? By Result 2.6 (ii), a complete type structure that is compact and continuous induces all hierarchies of beliefs. This leads to the conjecture that the substantive assumption we are making (i.e., about reasoning) in Proposition 4.9(i) is that the type structure induces all hierarchies of beliefs.

We will next see that this conjecture is incorrect, assuming that the Lebesgue measure cannot be extended to all subsets of $[0, 1]$.

6 Type Structures and Hierarchies of Beliefs

An (X_a, X_b)-based type structure, viz. $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$, induces hierarchies of beliefs about X_a and X_b. To see this, set $Z_c^1 = X_c$ and inductively define $Z_c^{m+1} = Z_c^m \times \mathcal{P}(Z_d^m)$. Set $\rho_c^m : X_c \times T_d \to Z_c^m$ so that

$$\rho_c^1(x_c, t_d) = x_c$$
$$\rho_c^{m+1}(x_c, t_d) = (\rho_c^m(x_c, t_d), \delta_d^m(t_d)),$$

where $\delta_c^m : T_c \to \mathcal{P}(Z_c^m)$ is defined by $\delta_c^m = \underline{\rho}_c^m \circ \beta_c$. Note, the maps ρ_c^m and δ_c^m are measurable and so well-defined. (See Lemma B.2.) Type $\mathbf{t_c}$'s $\mathbf{m^{th}}$-order belief is $\delta_c^m(t_c)$.

Set $\delta_c : T_c \to \prod_{m=1}^{\infty} \mathcal{P}(Z_c^m)$ so that, for each t_c, $\delta_c(t_c) = (\delta_c^1(t_c), \delta_c^2(t_c), \ldots)$. Type $\mathbf{t_c}$'s **hierarchy of beliefs** is $\delta_c(t_c)$. Note, the hierarchy of belief is **coherent**, i.e., for each m, $\delta_c^m(t_c) = \text{marg }_{Z_c^m} \delta_c^{m+1}(t_c)$.

In what follows, we will often compare pairs of (X_a, X_b)-based type structures

$$\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b), \quad \mathcal{T}^* = (X_a, X_b; T_a^*, T_b^*; \beta_a^*, \beta_b^*).$$

Definition 6.1 *Given an (X_a, X_b)-based type structures $\mathcal{T}$ and $\mathcal{T}^*$, say that $\mathcal{T}^*$ is **finitely***

terminal for $\mathcal{T}$ *if, for each m, and each type $t_c \in T_c$, there is a type $t_c^* \in T_c^*$ with*

$$(\delta_c^{*,1}(t_c^*), \dots, \delta_c^{*,m}(t_c^*)) = (\delta_c^1(t_c), \dots, \delta_c^m(t_c)).$$

*Say $\mathcal{T}^*$ is **finitely terminal** if it is finitely terminal for every (X_a, X_b)-based type structure $\mathcal{T}$.*

Note, here, t_c^* can depend both on the t_c and m.

Definition 6.2 *An (X_a, X_b)-based type structure $\mathcal{T}^*$ is **terminal for** $\mathcal{T}$ if for each type $t_c \in T_c$, there is a type $t_c^* \in T_c^*$ with*

$$\delta_c^*(t_c^*) = \delta_c(t_c).$$

*Say $\mathcal{T}^*$ is **terminal** if it is terminal for every (X_a, X_b)-based type structure $\mathcal{T}$.*

Definition 6.1 says that the type structure $\mathcal{T}^*$ is finitely terminal if, for each type t_c that occurs in some type structure and each m, there is a type $t_c^{*,m}$ in $\mathcal{T}^*$ whose hierarchy agrees with t_c up to level m. Definition 6.2 says that the type structure is $\mathcal{T}^*$ terminal if, for each type t_c that occurs in some type structure, there is a type t_c^* with the same hierarchy of beliefs as t_c. Thus, terminality captures the idea that a type structure induces all hierarchies of beliefs.[7] (See, Result 2.1 in Friedenberg, 2010 for an alternate argument for this interpretation.)

Terminal type structures exist: When X_a, X_b are Polish, we can apply the bottom-up construction in Mertens and Zamir (1985) or the top-down construction in Brandenburger and Dekel (1993) to get a terminal type structure.[8] When X_a, X_b are arbitrary metric spaces, the embedding construction in Heifetz and Samet (1998) gives a type structure that is terminal.

Turning back to the IU application, we have the following alternate epistemic condition for IU:

Proposition 6.3 *Fix an epistemic game $(G, \mathcal{T})$.*

---

[7] Here we use the phrase "terminal" in the spirit of Böge and Eisele's (1979) original usage. Some authors reserve the phrase "terminal" for a type structure that satisfies a stronger embedding property. (See, e.g., Meier, 2006.) No confusion should result.

[8] Mertens and Zamir (1985) state their result when X_a, X_b are compact, but it is well-known that their analysis applies even when X_a, X_b are Polish. Proposition 3.4 in Battigalli and Siniscalchi (1999) shows that the construction in Brandenburger and Dekel (1993) is terminal.

(i) *If $\mathcal{T}$ is finitely terminal, then* $\mathrm{proj}_{S_a} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m$, *for each $m \geq 1$.*

(ii) *If $\mathcal{T}$ is terminal, then* $\mathrm{proj}_{S_a} \bigcap_{m=1}^{\infty} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \bigcap_{m=1}^{\infty} \mathrm{R}_b^m = \bigcap_{m=1}^{\infty} S_a^m \times \bigcap_{m=1}^{\infty} S_b^m$.

We omit the proof, as it will be a special case of Theorem 8.1 to come. Part (i) says that, if the type structure induces all finite-order beliefs, then, for each m, RmBR is characterized by the $(m+1)$-undominated strategies. Part (ii) says that, if the type structure induces all hierarchies of beliefs, then RCBR is characterized by the IU strategies. Note, Proposition 4.9(ii) is a special case of Proposition 6.3(ii), since a complete, compact, and continuous type structure is terminal.

As a consequence of Theorem 5.2 and Proposition 6.3, we have the following corollary:

Definition 6.4 *Say X_c is **non-degenerate** if it has at least two elements.*

Corollary 6.5 *Suppose X_a and X_b are finite and non-degenerate. Then:*

(i) *There exists a complete, Polish, and continuous (X_a, X_b)-based type structure that is not terminal.*

(ii) *There exists a complete and compact (X_a, X_b)-based type structure that is not terminal.*

If X_a and X_b are finite and non-degenerate, then we can construct a non-trivial game where we take $X_a = S_b$ and $X_b = S_a$. Then, we can apply Theorem 5.2, Corollary 5.9, and Proposition 6.3 to get the desired result.

Comparing the above corollary with Proposition 4.9(i), we might be led to believe that a complete type structure is finitely terminal. The next section shows that this is not the case—at least under the assumption that Lebesgue measure cannot be extended to all subsets of $[0, 1]$.

7 Completeness and Finite Terminality

This section first shows a negative result about complete type structures—that they need not induce all 3rd-order beliefs with finite support. The implication is that a complete type structure need not be finitely terminal. Then, it goes on to show a positive result, describing the finite-order beliefs that a complete structure does induce. For both results, we will want to avoid certain exceptional cases involving large cardinals.

Call $\mu \in \mathcal{P}(\Omega)$ **simple** if the support of μ is finite. Call $\mu \in \mathcal{P}(\Omega)$ **atomic** if every set of positive measure contains an atom. Note that every simple measure is atomic. If Ω is Polish, μ is atomic if and only if the (countable) set of **point masses** (**of** μ), viz.

$$\mathrm{PM}_\mu = \{\omega \in \Omega : \mu(\{\omega\}) > 0\},$$

has measure one. It is clear that, on a Polish space, every probability measure with countable support is atomic. But there are also atomic measures μ with full support, i.e., when PM_μ is dense.

Definition 7.1 *Fix an (X_a, X_b)-based type structure $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$ and a type $t_c \in T_c$. Call t_c **simple** (resp. **atomic**) if $\delta_c^m(t_c)$ is simple (resp. atomic) for each m.*

Definition 7.2 *Fix an (X_a, X_b)-based type structure $\mathcal{T}^*$. Say $\mathcal{T}^*$ **induces all simple (resp. atomic) m^{th} order beliefs** if, for every (X_a, X_b)-based type structure $\mathcal{T}$ and every simple (resp. atomic) type $t_c \in T_c$, there is a type $t_c^* \in T_c^*$ such that $\delta_c^{*,m}(t_c^*) = \delta_c^m(t_c)$.*

Note that every finitely terminal type structure induces all atomic m^{th} order beliefs for all m.

Definition 7.3 *Call an (X_a, X_b)-based type structure $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$ **c-simple** (resp. **c-atomic**) if each $t_c \in T_c$ is simple (resp. atomic). Say $\mathcal{T}$ is **simple** (resp. **atomic**) if it is a- and b-simple (resp. a- and b-atomic).*

The negative result will be that a complete type structure need not induce all simple 3rd-order beliefs. To get the result, we will avoid a particular case involving large cardinals—we will assume that the continuum is not atomlessly measurable. (See Fremlin, 2008.)

Definition 7.4 *Say the continuum is **atomlessly measurable** if the Lebesgue measure on $[0,1]$ can be extended to a probability measure on $([0,1], 2^{[0,1]})$.*[9]

Remark 7.5 *The continuum is not atomlessly measurable if and only if every set X of cardinality $\mathfrak{c}$, every probability measure on $(X, 2^X)$ has a countable set of measure one.*

Proposition 7.6 *Assume the continuum is not atomlessly measurable. Suppose X_a, X_b are non-degenerate and are discrete topological spaces of cardinality at most $\mathfrak{c}$. Then there is a type structure $\mathcal{T}^*$ that is complete and atomic, but which does not induce all simple 3rd-order beliefs (and hence is not finitely terminal).*

---

[9]Strictly speaking, the literature speaks of atomlessly measurable cardinals, and this definition means that the cardinality $\mathfrak{c}$ of the continuum is atomlessly measurable.

Thus, in the case that the continuum is not atomlessly measurable, there are complete type structures that are not finitely terminal. In the case that the continuum is atomlessly measurable, we do not know whether or not there are complete type structures that are not finitely terminal. We now turn to the proof of Proposition 7.6.

Definition 7.7 *We say that $\mathcal{T}$ **misses** $\mathcal{T}^*$ **at level** m if there are no types $t_c \in T_c$ and $t_c^* \in T_c^*$ with $\delta_c^m(t_c) = \delta_c^{*,m}(t_c^*)$. Say that $\mathcal{T}$ **misses** $\mathcal{T}^*$ if $\mathcal{T}$ misses $\mathcal{T}^*$ at some level m.*

So, if $\mathcal{T}$ misses $\mathcal{T}^*$ at level m, then, for each player, the set of mth-order beliefs of types in $\mathcal{T}$ is disjoint from the set of mth-order beliefs of types in $\mathcal{T}^*$.

Remark 7.8 *If $\mathcal{T}$ misses $\mathcal{T}^*$ at level m, then*

  (i) $\mathcal{T}^*$ misses $\mathcal{T}$ at level m, and

  (ii) $\mathcal{T}^*$ is not finitely terminal for $\mathcal{T}$.

Lemma 7.9 *Fix some (X_a, X_b), where X_a is non-degenerate. Then, there is an a-simple $\mathcal{T}$ that misses every atomic type structure at level 3.*

Proof. Fix $x_a \neq y_a$ in X_a. Construct $\mathcal{T}$ as follows: Take $T_a = [0, 1]$ and take $T_b = \{t_b\}$. Let $\beta_a(t_a)(\{(x_a, t_b)\}) = t_a$ and $\beta_a(t_a)(\{(y_a, t_b)\}) = 1 - t_a$. Fix $x_b \in X_b$ and let $\beta_b(t_b)$ be the measure μ on $X_b \times [0, 1]$ such that $\mu(\{x_b\} \times [u, v]) = (v - u)$ for each $0 \leq u \leq v \leq 1$.

Note, the set $A = \{x_a, y_a\} \times T_b$ is finite and

$$\delta_a^m(t_a)(\rho_a^m(A)) \geq \beta_a(t_a)(A) = 1,$$

for all $t_a \in T_a$. So, $\mathcal{T}$ is a-simple.

Now, let $\mathcal{T}^*$ be atomic. We want to show that $\mathcal{T}$ misses $\mathcal{T}^*$ at level 3. We start with a preliminary result.

Result: There is no $\mathbf{t_b^* \in T_b^*}$ with $\delta_b^{*,2}(\mathbf{t_b^*}) = \delta_b^2(\mathbf{t_b})$. Suppose, contra hypothesis, we have found such a type t_b^*. Note, for each $t_a \in T_a$, $\delta_a^1(t_a)$ is the measure with support $\{x_a, y_a\}$ that gives $\{x_a\}$ measure t_a. Moreover, $\rho_b^2(x_b, t_a) = (x_b, \delta_a^1(t_a))$. Therefore

$$(\rho_b^2)^{-1}(\{(x_b, \delta_a^1(t_a))\}) = \{(x_b, t_a)\}.$$

26

It follows that, for any $z \in Z_b^2 = X_b \times \mathcal{P}(X_a)$, $(\rho_b^2)^{-1}(\{z\})$ has cardinality at most one and, so, has measure zero. With this,

$$\delta_b^2(t_b)(\{z\}) = \beta_b(t_b)((\rho_b^2)^{-1}(\{z\})) = 0.$$

Therefore, $\delta_b^2(t_b)$ is not atomic, contradicting the assumption that $\delta_b^{*,2}(t_b^*) = \delta_b^2(t_b)$ and $\delta_b^{*,2}(t_b^*)$ is atomic.

Now we return to show that $\mathcal{T}$ misses $\mathcal{T}^*$ at level 3. We suppose $\mathcal{T}$ does not miss $\mathcal{T}^*$ at level 3 and get a contradiction.

Case 1: There is some $\mathbf{t_b^*} \in \mathbf{T_b^*}$ with $\delta_{\mathbf{b}}^{*,\mathbf{3}}(\mathbf{t_b^*}) = \delta_{\mathbf{b}}^{\mathbf{3}}(\mathbf{t_b})$. By coherency, $\delta_b^{*,2}(t_b^*) = \delta_b^2(t_b)$, contradicting the result.

Case 2: There is some $\mathbf{t_a} \in \mathbf{T_a}$ and $\mathbf{t_a^*} \in \mathbf{T_a^*}$ with $\delta_{\mathbf{a}}^{*,\mathbf{3}}(\mathbf{t_a^*}) = \delta_{\mathbf{a}}^{\mathbf{3}}(\mathbf{t_a})$. Note that

$$\rho_a^3(x_a, t_b) = (x_a, \delta_1^b(t_b), \delta_2^b(t_b)) \quad \text{and} \quad \rho_a^3(y_a, t_b) = (y_a, \delta_1^b(t_b), \delta_2^b(t_b)).$$

So,

$$\rho_a^3(A) = \{x_a, y_a\} \times \{(\delta_1^b(t_b), \delta_2^b(t_b))\}.$$

Write $B = \rho_a^3(A)$ and note that

$$
\begin{aligned}
\beta_a^*(t_a^*)((\rho_a^{*,3})^{-1}(B)) &= \delta_a^{*,3}(t_a^*)(B) \\
&= \delta_a^3(t_a)(B) \\
&= \beta_a(t_a)((\rho_a^3)^{-1}(B)) \\
&\geq \beta_a(t_a)(A) = 1,
\end{aligned}
$$

where the first and third lines are by definition, the second line is by the assumption of Case 2, and the last line uses the fact that $(\rho_a^3)^{-1}(B) = (\rho_a^3)^{-1}(\rho_a^3(A)) \supseteq A$. This implies that $(\rho_a^{*,3})^{-1}(B)$ is non-empty. So, there exists a pair $(q_a, t_b^*) \in X_a \times T_b^*$ with

$$\rho_a^{*,3}(q_a, t_b^*) = (q_a, \delta_b^{*,1}(t_b^*), \delta_b^{*,2}(t_b^*)) \in B = \rho_a^3(\{x_a, y_a\} \times \{(\delta_1^b(t_b), \delta_2^b(t_b))\}),$$

and so $\delta_b^{*,2}(t_b^*) = \delta_b^2(t_b)$. But, this contradicts the result. ∎

A type structure $\mathcal{T}$ is **$\mathfrak{c}$-discrete** if each T_c is the discrete topological space on a set of cardinality $\mathfrak{c}$.

Lemma 7.10 *Assume that the continuum is not atomlessly measurable. Suppose X_a and X_b are discrete topological spaces of cardinality at most $\mathfrak{c}$. Then each $\mathfrak{c}$-discrete (X_a, X_b)-based type structure is atomic.*

Proof. Let $\mathcal{T}$ be $\mathfrak{c}$-discrete. Then $X_c \times T_d$ is discrete and has cardinality $\mathfrak{c}$. Since the continuum is not atomlessly measurable, by Remark 7.5, every probability measure in $\mathcal{P}(X_c \times T_d)$ has a countable set of measure one. Therefore for each $t_c \in T_c$, $\beta_c(t_c)$ has a countable set of measure one. Then by Lemma C.1, $\mathcal{T}$ is atomic. ∎

Lemma 7.11 *Assume that the continuum is not atomlessly measurable. Fix (X_a, X_b), where X_a and X_b are discrete topological spaces of cardinality at most $\mathfrak{c}$ and X_a is non-trivial. Then there exists a complete type structure that is $\mathfrak{c}$-discrete.*

Proof. Let $T_a = T_b = [0, 1]$ and endow T_a and T_b with the discrete topology. Since the continuum is not atomlessly measurable, each probability measure in $\mathcal{P}(X_c \times T_d)$ has a countable set of measure one, and thus is determined by a countable sequence of elements of $X_c \times T_d \times [0, 1]$, corresponding to a sequence of points and measures of points. Therefore, $\mathcal{P}(X_c \times T_d)$ has cardinality $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. Hence there are bijective (continuous) functions β_c from T_c onto $\mathcal{P}(X_c \times T_d)$. For any such functions β_a and β_b, $(X_a, X_b; T_a, T_b; \beta_a, \beta_b)$ is a complete type structure that is $\mathfrak{c}$-discrete. ∎

Proof of Proposition 7.6. By Lemma 7.11, there exists a complete type structure that is $\mathfrak{c}$-discrete. Lemma 7.10 gives that this type structure is atomic. So, by Lemma 7.9, there is an a-simple type structure $\mathcal{T}$ that misses this type structure at level 3. ∎

We saw that a complete type structure need not induce all finite-order simple beliefs—and so need not induce all finite-order atomic beliefs. Next, we will see that a complete type structure is finitely terminal for all atomic type structures (and, so, is finitely terminal for all simple type structures). The result will require that we assume the cardinality of the parameter set is not too large.

Definition 7.12 *A cardinal κ is **measurable** if there is a set X and a probability measure ν on $(X, 2^X)$ such that*

(i) $|X| = \kappa$,

(ii) $\nu(E) \in \{0, 1\}$ *for each subset E of X, and*

(iii) $\nu(F) = 0$ *for each finite subset F of X.*

28

We will assume that the cardinalities of the parameter sets X_a and X_b are not measurable. This, for instance, is satisfied when X_a and X_b are Polish. (See Fremlin, 2008.)

Definition 7.13 *Fix an (X_a, X_b)-based type structure $\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b)$. Say $\mathcal{T}$ is **countable** (resp. **finite**) if T_a and T_b are countable (resp. finite) and endowed with the discrete topology. Say $\mathcal{T}$ is **countably atomic** (resp.**finitely simple**) if it is countable and atomic (resp. finite and simple).*

If a type structure is countably atomic then it is countable. But the converse need not hold. (For instance, we may have $X_a = X_b = [0, 1]$, $T_a = \{t_a\}$, and $T_b = \{t_b\}$, but $\beta_a(t_a)$ non-atomic. If so, $\delta_a^1(t_a)$ is not atomic.)

Proposition 7.14 *Fix an (X_a, X_b)-based complete type structure $\mathcal{T}$.*

(i) *The type structure $\mathcal{T}$ is finitely terminal for all countable type structures.*

(ii) *If $|X_a|$ and $|X_b|$ are not measurable, then the type structure $\mathcal{T}$ is finitely terminal for all atomic type structures.*

What is the role of the requirement that $|X_a|$ and $|X_b|$ are not measurable? Recall, if X_c is a Polish space, then μ is atomic if and only if the set of point masses of μ, viz. PM_μ, gets probability one under μ. We show in Appendix C that this consequence follows even when X_c is not Polish, so long as the cardinality of X_c is not measurable. It is this property that is used to prove Proposition 7.14(ii).

We do not know whether Proposition 7.14(ii) can be proved without the hypothesis that $|X_a|$ and $|X_b|$ are not measurable.

8 Epistemic Conditions for IU Revisited

Now we return to the question of an epistemic justication for IU. In light of Proposition 4.9(i) and Proposition 7.14, a natural conjecture is that if a type structure is finitely terminal for all atomic type structures, then the set of RmBR strategies is the set of strategies that survive $(m + 1)$-rounds of iterated dominance. This is the case. In fact, we weaken this atomic requirement.

Theorem 8.1 *Fix an epistemic game $(G, \mathcal{T})$.*

(i) If $\mathcal{T}$ is finitely terminal for all finitely simple type structures, then

$$\mathrm{proj}_{S_a} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m.$$

(ii) If $\mathcal{T}$ is terminal for all finitely simple type structures, then

$$\mathrm{proj}_{S_a} \bigcap_{m=1}^{\infty} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \bigcap_{m=1}^{\infty} \mathrm{R}_b^m = \bigcap_{m=1}^{\infty} S_a^m \times \bigcap_{m=1}^{\infty} S_b^m.$$

Part (ii) is an epistemic condition for IU. It replaces the completeness, compactness, and continuity conditions in Proposition 4.9 with a condition that makes explicit reference to players' hierarchies of beliefs.

To understand why the result holds, fix a finite game $G = (S_a, S_a; \pi_a, \pi_b)$. By the argument in Brandenburger and Dekel (1987), there exists a finite (S_b, S_a)-based type structure $\mathcal{T} = (S_b, S_b; T_a, T_b; \beta_a, \beta_b)$ so that, RCBR is characterized by IU within that game.[10] More precisely:

Lemma 8.2 *Fix a finite game $G = (S_a, S_b; \pi_a, \pi_b)$. There exists an (S_b, S_a)-based type structure $\mathcal{T} = (S_b, S_a; T_a, T_b, \beta_a, \beta_b)$ with $|T_a| \leq |S_a|$ and $|T_b| \leq |S_b|$ so that*

(i) For each m, $\mathrm{proj}_{S_a} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m$.

(ii) $\mathrm{proj}_{S_a} \bigcap_{m=1}^{\infty} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \bigcap_{m=1}^{\infty} \mathrm{R}_b^m = \bigcap_{m=1}^{\infty} S_a^m \times \bigcap_{m=1}^{\infty} S_b^m$.

We will want to show that the RmBR set is preserved when going from $\mathcal{T}$ to any type structure $\mathcal{T}^*$ that is finitely terminal for $\mathcal{T}$. The key lemma is:

Lemma 8.3 *Fix epistemic games $(G, \mathcal{T})$ and $(G, \mathcal{T}^*)$, where T_a, T_b are finite. If $\delta_c^m(t_c) = \delta_c^{*,m}(t_c^*)$, then $(s_c, t_c) \in R_c^m$ only if $(s_c, t_c^*) \in R_c^{*,m}$.*

Fix an epistemic game $(G, \mathcal{T})$ where $\mathcal{T}$ has a finite number of types. Lemma 8.3 says if (s_c, t_c) is consistent with RmBR in the epistemic game $(G, \mathcal{T})$, then for any epistemic game $(G, \mathcal{T}^*)$ and any type t_c^* that induces the same $(m + 1)^{th}$-order belief at t_c, (s_c, t_c^*) is also consistent with RmBR in the epistemic game $(G, \mathcal{T}^*)$.

As a corollary of Lemmata 8.2-8.3, we get one direction of the equalities in Theorem 8.1:

---

[10] The result of Brandenburger and Dekel (1987) pertains to "common knowledge of rationality" and not to "rationality and m^{th}-order belief of rationality" or "rationality and common belief of rationality," as stated here. A proof of this statement (i.e., Lemma 8.2) can be found by following the proof of Theorem 10.1(ii) in Brandenburger and Friedenberg (2008).

Corollary 8.4 *Fix an epistemic game* $(G, \mathcal{T})$.

(i) *If* $\mathcal{T}$ *is finitely terminal for all finitely simple type structures, then*

$$S_a^m \times S_b^m \subseteq \operatorname{proj}_{S_a} \mathrm{R}_a^m \times \operatorname{proj}_{S_b} \mathrm{R}_b^m.$$

(ii) *If* $\mathcal{T}$ *is terminal for all finitely simple type structures, then*

$$\bigcap_{m=1}^{\infty} S_a^m \times \bigcap_{m=1}^{\infty} S_b^m \subseteq \operatorname{proj}_{S_a} \bigcap_{m=1}^{\infty} \mathrm{R}_a^m \times \operatorname{proj}_{S_b} \bigcap_{m=1}^{\infty} \mathrm{R}_b^m.$$

Proof. By Lemma 8.2, there is a type structure $\mathcal{T}^* = (S_b, S_a; T_a^*, T_b^*, \beta_a^*, \beta_b^*)$ with finite type sets so that

- For each m, $\operatorname{proj}_{S_a} \mathrm{R}_a^{*,m} \times \operatorname{proj}_{S_b} \mathrm{R}_b^{*,m} = S_a^m \times S_b^m$.

- $\operatorname{proj}_{S_a} \bigcap_{m=1}^{\infty} \mathrm{R}_a^{*,m} \times \operatorname{proj}_{S_b} \bigcap_{m=1}^{\infty} \mathrm{R}_b^{*,m} = \bigcap_{m=1}^{\infty} S_a^m \times \bigcap_{m=1}^{\infty} S_b^m$.

For part (i): Fix $s_c \in S_c^m$. There exists some $t_c^* \in T_c^*$ so that $(s_c, t_c^*) \in \mathrm{R}_c^{*,m}$. Since T_a^* and T_b^* are finite, $\mathcal{T}^*$ is simple and so $\mathcal{T}$ is finitely terminal for $\mathcal{T}^*$. Therefore there exists $t_c \in T_c$ with $\delta_c^m(t_c) = \delta_c^{*,m}(t_c^*)$. By Lemma 8.3, $(s_c, t_c) \in \mathrm{R}_c^m$.

For part (ii): Fix $s_c \in \bigcap_m S_c^m$. There exists some $t_c^* \in T_c^*$ so that $(s_c, t_c^*) \in \bigcap_m \mathrm{R}_c^{*,m}$. Since T_a^* and T_b^* are finite, $\mathcal{T}^*$ is simple and so $\mathcal{T}$ is terminal for $\mathcal{T}^*$. Therefore there exists $t_c \in T_c$ with $\delta_c(t_c) = \delta_c^*(t_c^*)$. By Lemma 8.3, $(s_c, t_c) \in \bigcap_m \mathrm{R}_c^m$. ∎

The other direction of Theorem 8.1 always holds:

Lemma 8.5 *Fix an epistemic game* $(G, \mathcal{T})$.

(i) *For each* m, $\operatorname{proj}_{S_a} \mathrm{R}_a^m \times \operatorname{proj}_{S_b} \mathrm{R}_b^m \subseteq S_a^m \times S_b^m$.

(ii) $\operatorname{proj}_{S_a} \bigcap_{m=1}^{\infty} \mathrm{R}_a^m \times \operatorname{proj}_{S_b} \bigcap_{m=1}^{\infty} \mathrm{R}_b^m \subseteq \bigcap_{m=1}^{\infty} S_a^m \times \bigcap_{m=1}^{\infty} S_b^m$.

Proof of Theorem 8.1. Immediate from Corollary 8.4 and Lemma 8.5. ∎

Proof of Proposition 6.3. Immediate from Theorem 8.1. ∎

9   Discussion

Epistemic Conditions for IU   The focus of this paper is on providing an epistemic justification for IU, with an aim toward justifying IU from the perspective of the players' themselves. That is, for each game G, we seek a "line of reasoning" for the players that yields, as a prediction, the IU set of G.

An epistemic justification for IU can have two ingredients. One is a restriction on the set of type structures compatible with a game. This is a substantive assumption about which beliefs players do vs. do not consider possible. We refer to this as a **contextual assumption**, since such assumptions typically reflect an unmodeled "context" within which the game is played.[11] The second is a restriction on players' reasoning, within an epistemic game $(G, \mathcal{T})$. Such a restriction is known as an **epistemic condition** and typically reflects "strategically sophisticated reasoning" on the part of players.

With the above in mind, we will say we have found an **epistemic justification for IU** if we can find some pair $(\mathfrak{T}, O)$ so that

1. $\mathfrak{T}$ is a class of type structures where, for each finite game G, $\mathfrak{T}$ contains (at least) one type structure compatible with G,

2. O is an operation that maps each epistemic game $(G, \mathcal{T})$ with $\mathcal{T} \in \mathfrak{T}$ to a set $O(G, \mathcal{T})$ of states in $(G, \mathcal{T})$, and

3. the projection of each $O(G, \mathcal{T})$ onto the strategy sets in G is the IU set of G.

This says that we can find contextual assumptions (given by $\mathfrak{T}$) and epistemic conditions (given by the operation O), so that the implied prediction (given by the projection of $O(G, \mathcal{T})$ onto $S_a \times S_b$) is the IU set of G.

Referring to Proposition 4.9(ii), one example of an epistemic justification for IU is given by taking $\mathfrak{T}$ to be the class of all complete, compact, and continuous type structures and taking O to map epistemic games to the set of states consistent with RCBR (in those games). Proposition 4.9(i) might be thought to suggest a second epistemic justification for IU, i.e., by taking $\mathfrak{T}$ to be the class of all complete type structures and looking at the intersection over m of all sets $\text{proj}_{S_a} R_a^m \times \text{proj}_{S_b} R_b^m$. But, it appears that there is no operation O that would satisfy Requirements 2-3, so this does not give an epistemic justification for IU.

---

[11]The phrase "contextual assumption" is due to Pierpaolo Battigalli and Marciano Siniscalchi.

Game-Independent Epistemic Justification   The core of the exercise in this paper is the desire to obtain more than simply *any* epistemic justification for IU. Refer back to Section 2.3, where we explained why we do not want to be able to tailor an epistemic justification to the solution concept. Thus, we seek what we referred to as a "game independent" epistemic justification for IU. This should be seen as an epistemic justification that corresponds to a "line of reasoning" (for the players) that is the same for all games.

Of course, any epistemic justification depends on the game in some manner.[12] We use the phrase "game independent epistemic justification" in a particular way: We should be able to describe the contextual assumption and the epistemic condition in words, without first making reference to the game in question.

In Section 2.3 we ruled out a game-dependent "richness" condition where the type structure associated with G is just "rich enough to deliver IU in G." An example of such a condition was given by the type structure constructed in the proof of Proposition 2.1. In particular, that construction makes reference to both the solution concept of IU and the epistemic condition of RCBR. As such, the contextual assumption cannot be described without reference to the game in question.

Another example of a game-dependent analysis is suggested by Proposition 4.9(i). For each finite game G, and for each associated epistemic game $(G, \mathcal{T})$ with $\mathcal{T}$ complete, the set of strategies consistent with R(M-1)BR is the set of strategies that survive IU, where $M = |S_a| \times |S_b| < \infty$. This delivers an epistemic justification for IU, but one in which the epistemic condition restricts the set of states in a game-dependent manner (since M depends on the game in question). Thus, the epistemic condition cannot be described without reference to the game in question.

Justifying IU from the Perspective of the Players   In Section 2.3, we said that the analyst can justify IU play as resulting from RCBR, if the analyst looks at RCBR across all type structures. The bulk of the literature focuses on justifying IU from the perspective of the players themselves. Why not stop at justifying IU from the perspective of the analyst looking across all type structures? Is it really important to justify IU from the perspective of the players themselves?

One answer is that understanding IU is a benchmark to help us understand more interesting epistemic analyses. Two that stand out are: epistemic analyses of extensive-form

---

[12]For instance, the contextual assumption that a type structure contains all possible hierarchies of beliefs depends on the game, in so far as the hierarchies are about the strategies played in the game. Likewise, the epistemic condition of "rationality" also depends on the game.

games and epistemic analyses of weak dominance. Each of these are important for applications.[13] The natural analogues to IU are extensive-form rationalizability (Pearce, 1984) and iterated weak dominance (i.e., maximal simultaneous deletion). Both concepts give sharp predictions in games of applied interest. But, we cannot justify these concepts by looking across all type structures—at least, in the analyses that are known to date. (The reason is that, in those cases, there is a natural non-monotonicity that arises. See Battigalli and Siniscalchi, 2002 and Brandenburger, Friedenberg and Keisler, 2008.) As such, in those cases, it becomes important to justify the concepts from the perspective of the players themselves. A useful natural benchmark case is IU.

It is worth noting that continuity plays an important role in the literature on weak dominance. In particular, compare the output of Theorem 10.1 in Brandenburger, Friedenberg and Keisler (2008) versus the output of Theorems 4.2-4.4 in Keisler and Lee (2010). There is a striking difference: In the former case, there is no prediction whereas in the latter case the output is the set of strategies that survive iterated weak dominance (i.e., maximal simultaneous deletion of weakly dominated strategies). Yet, the input is remarkably similar, differing only based on a complete and continuous type structure versus a complete and discontinuous type structure. This raises the question: How do these type structures differ in terms of players' reasoning (i.e., hierarchies of beliefs)? This is an open question. The hope is that the ideas here are a step toward answering this question.

Topologies and Substantive Assumptions about Reasoning This paper highlights the fact that topological assumptions on the type structure may implicitly impose important substantive assumptions on players' reasoning. This message is reminiscent of—but distinct from—the goal of the so-called "topology-free approach to type structures." (See, e.g., Heifetz and Samet, 1998, 1999 and subsequent work.) Let us review.

That literature begins with an underlying set of uncertainty, e.g., X_a and X_b, that may not be a topological space. That is, it is endowed with a sigma-algebra that may not be the Borel sigma-algebra. It then explicitly imposes a sigma-algebra on the set of beliefs on this set, e.g., on $\mathcal{P}(X_a)$ and $\mathcal{P}(X_b)$. For instance, if X_c is a Polish space, the sigma-algebra imposed on $\mathcal{P}(X_c)$ coincides with the Borel sigma algebra on $\mathcal{P}(X_c)$ when endowed with the weak topology. The sigma-algebra implicitly imposes restrictions on players' reasoning and, in turn, has implications for constructing type structures. For instance, when X_c is not Polish, there may be a hierarchy of beliefs that cannot be induced by a type in any type

---

[13]Weak dominance is an important criterion for analyzing strategic-form games that naturally have ties, e.g., auctions, voting games, Bertrand-like games, etc.

structure. (See, e.g., Heifetz and Samet, 1999.)

In this paper, we begin with an underlying set of uncertainty that is topological. We show that adding certain topological assumptions to the type structure may implicitly impose substantive assumptions about players' reasoning. (So, for instance, even if we begin with sets of uncertainty that are compact metrizable, adding topological assumptions to the type structure may impose additional restrictions on players' reasoning.) Thus, the message here is more closely related to Brandenburger and Keisler (2006), Friedenberg (2010), Friedenberg and Meier (2010), and Kets (2010).

Measurable Cardinals   Propositions 7.6 and 7.14 have assumptions about large cardinals that are not covered by the usual Zermelo-Fraenkel axioms of set theory (ZFC).

It is known that if ZFC is consistent, then it remains consistent if one adds the hypothesis that the continuum is not atomlessly measurable. This is the hypotheses used in Proposition 7.6. The message of Proposition 7.6 is that one cannot prove that every complete atomic type structure induces all simple beliefs. In fact, there are counterexamples in every model of ZFC in which the continuum is not atomlessly measurable. More precisely, Proposition 7.6 shows that if ZFC is consistent, then one cannot prove from the axioms of ZFC that every complete atomic type structure induces all simple beliefs. Hence one cannot prove from the axioms of ZFC that every complete atomic type structure is finitely terminal.

Similarly, if ZFC is consistent then it remains consistent if one adds the hypothesis that there are no measurable cardinals. If measurable cardinals exist, then they are very large, much larger than the continuum or even the first inaccessible cardinal. Every Polish space has cardinality at most $\mathfrak{c}$, and hence is not measurable. The message of Proposition 7.14 (ii) is that unless $X_a \cup X_b$ has enormously large cardinality, every complete (X_a, X_b)-based type structure is finitely terminal for all atomic type structures. In particular, if X_a and X_b are Polish, then every complete (X_a, X_b)-based type structure is finitely terminal for all atomic type structures. And one cannot prove that there is a complete type structure that isn't finitely terminal for all atomic type structures.

RCBR   Lemma 5.8 says that, given a game G and two uncountable Polish spaces T_a, T_b, we can construct a complete (S_b, S_a)-based type structure with type sets T_a, T_b so that there is no state consistent with RCBR. Note, too, we can also construct a complete (S_b, S_a)-based type structure with the same type sets T_a, T_b, so that there is a state consistent with RCBR:

Result 9.1 *Fix a game G and two uncountable Polish spaces T_a, T_b. There is a complete*

(S_b, S_a)-*based type structure*

$$\mathcal{T} = (S_b, S_a; T_a, T_b; \beta_a, \beta_b)$$

so that

(i) For each $m \geq 1$, $\mathrm{proj}_{S_a} \mathrm{R}_a^m \times \mathrm{proj}_{S_b} \mathrm{R}_b^m = S_a^m \times S_b^m$.

(ii) $\bigcap_{m=1}^{\infty} \mathrm{R}_a^m \neq \emptyset$ and $\bigcap_{m=1}^{\infty} \mathrm{R}_b^m \neq \emptyset$.

To see this, note that there is a complete, compact, and continuous (S_b, S_a)-based type struc-
ture viz., $\mathcal{U} = (S_b, S_a, U_a, U_b, \gamma_a, \gamma_b)$. Within the epistemic game $(G, \mathcal{U})$, the set of strategies
consistent with RCBR is the IU strategy set, and hence is non-empty. (See Proposition 4.9.)
So, now we can repeat the proof of Lemma 5.8 to get the desired conclusion.

Varieties of Terminality   In our analysis, we encountered two variants on terminality. A
type structure is terminal (for all type structures) if it induces all hierarchies of beliefs, and
is terminal for finitely simple type structures if it induces all hierarchies of beliefs associated
with finitely simple type structures. Figure 9.1 reviews the relationship between these two
variants on terminality and the corresponding variants on finite terminality.

We also encountered several variants on finite terminality that lie between the extremes
of finite terminality (for all type structures) and finite terminality for finitely simple type
structures. Figure 9.2 reviews the relationship between them. Here, we can provide addi-
tional relationships, showing an equivalence between finite terminality for all simple (resp.
atomic) type structures and finite terminality for all finitely simple (resp. countably atomic)
type structures. These equivalences are given by Proposition 9.2.

Proposition 9.2

*(i) An (X_a, X_b)-based type structure is finitely terminal for all simple type structures if
and only if it is finitely terminal for all finitely simple type structures.*

*(ii) Suppose $|X_c|$ is not measurable. An (X_a, X_b)-based type structure is finitely terminal
for all atomic type structures if and only if it is finitely terminal for all countably
atomic type structures.*

Part (ii) assumes $|X_c|$ is not measurable. It is used in the proof of Proposition 7.14(ii).
Part (i) was not used in our analysis, but it is useful right. For instance, it can be used to
show that we can construct a simple type structure that is finitely terminal for all simple
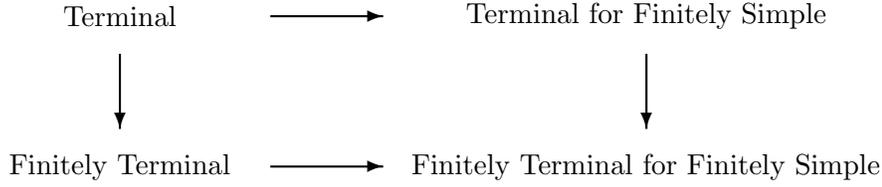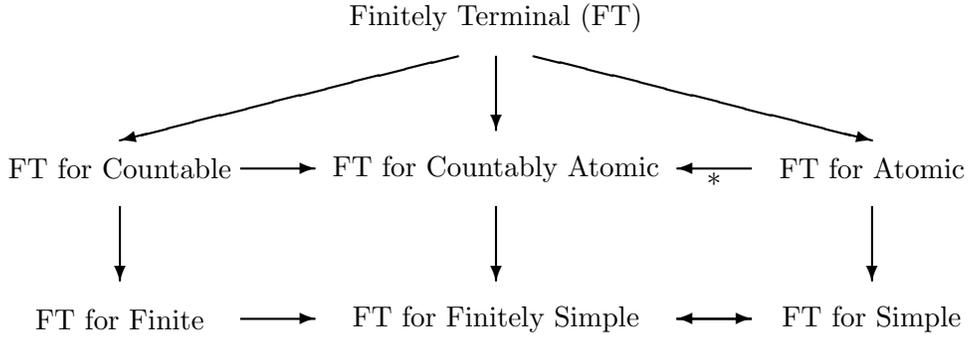
Terminal $\longrightarrow$ Terminal for Finitely Simple

Finitely Terminal $\longrightarrow$ Finitely Terminal for Finitely Simple

Figure 9.1: Variants on Terminality

Finitely Terminal (FT)

FT for Countable $\longrightarrow$ FT for Countably Atomic $\xleftarrow{\ *\ }$ FT for Atomic

FT for Finite $\longrightarrow$ FT for Finitely Simple $\longleftrightarrow$ FT for Simple

*: Equivalent if $|T^c|$ is not measurable

Figure 9.2: Variants on Finite Terminality

type structures. Appendix E provides the formal construction. It is analogous to the embedding construction in Heifetz and Samet (1998). (See, also, Yildiz, 2009.)

10   Conclusion

We began with the classical result: Every complete type structure has the property that the set of RmBR strategies is the set of strategies that survive $(m+1)$ rounds of iterated dominance. (See Result 2.4(i).) Theorem 8.1(i) says that every type structure that is finitely terminal for all finitely simple type structures also has that property. And Proposition 7.14(i) says that every complete type structure is finitely terminal for all countable type structures, and hence for all finitely simple type structures. Thus, Theorem 8.1(i) and Proposition 7.14(i) together are a sharpening of Result 2.4(i). That is, they provide a better understanding of the epistemic conditions for the $(m+1)$-undomianted strategies.

It is known that every complete type structure with Polish type spaces is finitely termi-

nal. (See Result 2.6.) Theorem 8.1(ii) says that every type structure that is terminal for all finitely simple type structures has the property that the set of RCBR strategies is the set of IU strategies. Theorem 5.2 shows that there are complete type structures with Polish type spaces that do not have that property. Hence these complete type structures are finitely terminal, but not terminal for all finitely simple type structures.

Proposition 7.14(ii) tells us that every complete type structure whose type sets are not "extremely large" is finitely terminal for all atomic type structures. But Proposition 7.6 shows that if the continuum is not atomlessly measurable, then there are complete type structures with type spaces of cardinality $\mathfrak{c}$ that do not induce all finite order simple beliefs. These complete type structures are finitely terminal for all atomic type structures but not finitely terminal.

Appendix A   Proofs for Section 5

Lemma A.1 *The collection*

$$\{\mathrm{T}\,[Q_c, m] \backslash \mathrm{T}\,[Q_c, m+1] : Q_c \in \mathbb{S}_c^1 \text{ and } m \geq 1\}$$

partitions T_c.

Proof. Clearly, elements of the collection are pairwise disjoint. It suffices to show that each t_c is contained in $\mathrm{T}\,[Q_c, m] \backslash \mathrm{T}\,[Q_c, m+1]$, for some $Q_c \in \mathbb{S}_c^1$ and some $m \geq 1$. Certainly, each $t_c \in \mathrm{T}\,[Q_c, 1]$, for some Q_c. Moreover, using the fact that $\bigcap_m \mathrm{T}\,[Q_c, m] = \emptyset$ (Property 5.4(ii)), it follows that there is some m^* so that $t_c \notin \mathrm{T}\,[Q_c, m^*]$ and, so, there exists some m so that $t_c \in \mathrm{T}\,[Q_c, m] \backslash \mathrm{T}\,[Q_c, m+1]$. ∎

Lemma A.2 *The collection of sets of probabilities*

$$\{(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c) : Q_c \in \mathbb{S}_c^1 \text{ and } m \geq 1\}.$$

partitions $\mathcal{P}(S_d \times T_d)$.

Proof. Clearly, elements of the collection are pairwise disjoint. It suffices to show that each ψ_d is contained in $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ for some $Q_c \in \mathbb{S}_c^1$ and some $m \geq 1$. To show this, note that each $\psi_d \in (\mathbb{BR}_c^1)^{-1}(Q_c)$ for some Q_c. Moreover, using Property 5.4(ii), it can be seen that $\bigcap_m P_d^m = \emptyset$. So, no ψ_d can believe each P_d^m, i.e., there must be some m^* so that $\psi_d \notin (\mathbb{BR}_c^{m^*})^{-1}(Q_c)$. So, for each ψ_d, there exists some m with $\psi_d \in (\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$. ∎

Lemma A.3 *Fix some $Q_c \in \mathbb{S}_c^1$ and some $m \geq 1$. The set $(\mathbb{BR}_c^m)^{-1}(Q_c)$ is closed.*

Proof. Note,

$$(\mathbb{BR}_c^m)^{-1}(Q_c) = \{\psi_d : \mathbb{BR}_c^1(\psi_d) = Q_c\} \cap \{\psi_d : \psi_d(P_c^{m-1}) = 1\}.$$

A standard application of the Portmanteau Theorem (Kechris, 1995, Theorem 17.20(i)-(ii)) gives that the set $\{\psi_d : \mathbb{BR}_c^1(\psi_d) = Q_c\}$ is closed. It follows from Property 5.4(iii) that P_c^{m-1} is clopen. So, the Portmanteau Theorem (Kechris, 1995, Theorem 17.20(i)-(v)) gives that the set $\{\psi_d : \psi_d(P_c^{m-1}) = 1\}$ is closed. This establishes the result. ∎

Lemma A.4 *Fix some $Q_c \in \mathbb{S}_c^1$ and some $m \geq 1$. The set $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is a Polish space.*

Proof. It suffices to show that $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is a G_δ set, i.e., a countable intersection of open sets. If so, then the claim follows from Kechris (1995, Theorem 3.11).

Note,

$$(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c) = (\mathbb{BR}_c^m)^{-1}(Q_c) \cap [\mathcal{P}(S_d \times T_d) \backslash \{\psi_d : \psi_d(P_c^m) = 1\}].$$

By Lemma A.3 and Proposition 3.7 in Kechris (1995), $(\mathbb{BR}_c^m)^{-1}(Q_c)$ is a G_δ set. The Portmanteau Theorem (Kechris, 1995, Theorem 17.20(i)-(v)) gives that the set $\{\psi_d : \psi_d(P_c^m) = 1\}$ is closed, i.e., $\mathcal{P}(S_d \times T_d) \backslash \{\psi_d : \psi_d(P_c^m) = 1\}$ is open. So, $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is a countable intersection of open sets, as required. ∎

Lemma A.5 *For each $m \geq 0$ and each $s_c \in S_c^m$, there exists a type t_c so that $(s_c, t_c) \in P_c^m \backslash P_c^{m+1}$.*

Proof. Throughout, fix $s_c \in S_c^m$. We break the proof into two cases.

$m = 0$: Since the game is non-trivial, there exists some σ_d so that s_c is not optimal under σ_d, i.e., $s_c \notin \mathrm{BR}_c^1(\sigma_d)$. For any $t_c \in \mathrm{T}[\mathrm{BR}_c^1(\sigma_d), 1]$, $(s_c, t_c) \in P_c^0 \backslash P_c^1$.

$m \geq 1$: Using Property 5.3, there exists some $Q_c \in \mathbb{S}_c^m$ so that $s_c \in Q_c$. It follows from the construction that there is some $t_c \in \mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m + 1]$. So, $(s_c, t_c) \in P_c^m \backslash P_c^{m+1}$. ∎

Lemma A.6 *For each $Q_c \in \mathbb{S}_c^1$ and m, $\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m + 1]$ is nonempty if and only if $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is nonempty.*

Proof. Note, by construction, $\mathrm{T}[Q_c, m] \backslash \mathrm{T}[Q_c, m+1]$ is nonempty if and only if $\mathrm{T}[Q_c, m]$ is nonempty or, equivalently, if and only if $Q_c \in \mathbb{S}_c^m$. So, it suffices to show that $Q_c \in \mathbb{S}_c^m$ if and only if $(\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$ is nonempty.

First, suppose that $Q_c \in \mathbb{S}_c^m$. Then, there exists some $\sigma_d \in \mathcal{P}(S_d)$ so that $\mathrm{BR}_c^1(\sigma_d) = Q_c$ and σ_d believes S_d^{m-1}. We will show that we can find some ψ_d with $\mathbb{BR}_c^m(\psi_d) = Q_c$ but $\psi_d(P_d^m) \neq 1$. This will establish the result.

Note, by Lemma A.5, there exists a mapping $g : S_d \to T_d$ so that, for each $s_d \in S_d^m$, $g(s_d) \in P_d^{m-1} \backslash P_d^m$. Construct ψ_d so that $\psi_d(\{s_d, g(s_d)\}) = \sigma_d(\{s_d\})$, for all s_d. Then $\mathrm{marg}_{S_d} \psi_d = \sigma_d$, $\psi_d(P_d^{m-1}) = 1$, and $\psi_d(P_d^m) \neq 1$, as required.

Now suppose that $\psi_d \in (\mathbb{BR}_c^m)^{-1}(Q_c) \backslash (\mathbb{BR}_c^{m+1})^{-1}(Q_c)$. Then, $\mathrm{BR}_c^1(\mathrm{marg}_{S_d} \psi_d) = Q_c$ and $\psi_d(Q_d^{m-1}) = 1$. Using Lemma 5.6, $\mathrm{marg}_{S_d} \psi_d(S_d^{m-1}) = 1$. This establishes that $Q_c \in \mathbb{S}_c^m$. ∎

Lemma A.7 *Fix Polish spaces Ω and Φ. Let $\{\Omega^1, \Omega^2, \ldots\}$ be a partition of Ω and $\{\Phi^1, \Phi^2, \ldots\}$ be a partition of Φ so that each Ω^n and Φ^n is clopen. Suppose, for each n, there is a continuous map $f^n : \Omega^n \to \Phi^n$. Let $f : \Omega \to \Phi$ be such that $f(\omega) = f^n(\omega)$ if $\omega \in \Omega^n$. Then, f is continuous.*

Proof. Since Ω is Polish, it suffices to show that, if ω_k converges to ω in Ω, then $f(\omega_k)$ converges to $f(\omega)$ in Φ. To show this, note that there exists some n so that so that $\omega \in \Omega^n$ and $f(\omega) \in \Phi^n$. Since Ω^n is clopen, $\omega_k \in \Omega^n$, for k large. So, by construction, $f(\omega_k) \in \Phi^n$ for k sufficiently large. Using the fact that f is continuous on Ω^n, it follows that $f(\omega_k)$ converges to $f(\omega)$, as required. ∎

Appendix B   Proofs for Section 6

Lemma B.1 *Let Ω and Φ be metrizable and let $f : \Omega \to \Phi$ be measurable. Then $\underline{f}$ is measurable.*

Proof. Note, an open sub-basis for $\mathcal{P}(\Phi)$ is given by the family of sets of the form

$$U(\overline{\nu}, G, \varepsilon) = \{\nu \in \mathcal{P}(\Phi) : \nu(G) > \overline{\nu}(G) - \varepsilon\},$$

for $\overline{\nu} \in \mathcal{P}(\Omega)$, G open in Φ, and $\varepsilon > 0$. (See Billingsley, 1968, page 236.) It suffices to show that, for each set $U = U(\overline{\nu}, G, \varepsilon)$ in this open sub-basis, $\underline{f}^{-1}(U)$ is Borel in $\mathcal{P}(\Omega)$.

Fix $U = U(\overline{\nu}, G, \varepsilon)$. Let $r = \overline{\nu}(G) - \varepsilon$ and note that $f^{-1}(G)$ is Borel in Ω. With this, $\mu \in \underline{f}^{-1}(U)$ if and only if $\underline{f}(\mu) \in U$ if and only if $\underline{f}(\mu)(G) > r$ if and only if $\mu(f^{-1}(G)) > r$. So, by Lemma 15.16 in Aliprantis and Border (2007), $\underline{f}^{-1}(U)$ is Borel in $\mathcal{P}(\Omega)$. ∎

Lemma B.2 *For each m, the maps ρ_c^m and δ_c^m are well-defined and measurable.*

Proof. The proof is by induction on m.

Clearly, ρ_c^1 is measurable and, so, by Lemma B.1 and the fact that β_c is measurable, δ_c^1 is well-defined and measurable.

Assume the result holds for m. Then, by the induction hypothesis, ρ_c^{m+1} is well-defined and measurable. Again applying Lemma B.1 and the fact that β_c is measurable, δ_c^{m+1} is measurable. ∎

Appendix C   Proofs for Section 7

Lemma C.1 *Let $\mathcal{T}$ be a type structure such that, for each c and each t_c, $\beta_c(t_c)$ has a countable set (resp. finite set) of measure one. Then $\mathcal{T}$ is atomic (resp. simple).*

Proof. Let $t_c \in T_c$. By hypothesis, there is a countable (resp. finite) set $E \subseteq X_c \times T_d$ with $\beta_c(t_c)(E) = 1$. Then, the image $\rho_c^m(E)$ is countable (resp. finite) and $E \subseteq (\rho_c^m)^{-1}(\rho_c^m(E))$. So,

$$\delta_c^m(t_c)(\rho_c^m(E)) = \beta_c(t_c)((\rho_c^m)^{-1}(\rho_c^m(E))) \geq \beta_c(t_c)(E) = 1.$$

So, $\delta_c^m(t_c)$ is atomic (resp. simple). ∎

Now, we turn to prove Proposition 7.14. Part (i) will follow from the following Lemma.

Lemma C.2 *Any (X_a, X_b)-based complete type structure is finitely terminal for all countably atomic type structures.*

Proof. Let $\mathcal{T}$ be countable and $\mathcal{T}^*$ be complete. We show by induction on m that, for each $t_c \in T_c$, there exists $t_c^* \in T_c^*$, such that $\delta_c^{*,m}(t_c^*) = \delta_c^m(t_c)$. The result will then follow by coherency.

$\mathbf{m = 1}$: Fix a type $t_c \in T_c$. By completeness, there exists a type $t_c^* \in T_c^*$ so that $\text{marg}_{X_c} \beta_c^*(t_c^*) = \text{marg}_{X_c} \beta_c(t_c)$. It follows that for each event E in $Z_c^1 = X_c$,

$$\delta_c^{*,1}(t_c^*)(E) = \text{marg}_{X_c} \beta_c^*(t_c^*)(E) = \text{marg}_{X_c} \beta_c(t_c)(E) = \delta_c^1(t_c)(E),$$

as required.

m ≥ 2 : Assume the result holds for m and we will show that it also holds for $m + 1$.

Fix a type t_c. Since T_d is countable, there is a set $E_d \subseteq T_d$ so that $\beta_c(t_c)(X_c \times E_d) = 1$ and $\beta_c(t_c)(X_c \times \{t_d\}) > 0$ for each $t_d \in E_d$. By the induction hypothesis, there is a mapping $\tau_d^m : T_d \to T_d^*$ so that $\delta_d^{*,m}(\tau_d^m(t_d)) = \delta_d^m(t_d)$. Construct a probability measure $\psi \in \mathcal{P}(X_d \times T_d^*)$ that satisfies the following: For each event $F_c \times \{t_d\}$ in $X_c \times E_d$, $\psi(F_c \times \{\tau_d^m(t_d)\}) = \beta_c(t_c)(F_c \times [t_d])$, where we write $[t_d] := (\tau_d^m)^{-1}(\{\tau_d^m(t_d)\})$.

By completeness, there exists a type $t_c^* \in T_c^*$ with $\beta_c^*(t_c^*) = \psi$. We will show that $\delta_c^{*,m+1}(t_c^*) = \delta_c^{m+1}(t_c)$.

Fix an event G in Z_c^{m+1}. Note,

$$\delta_c^{m+1}(t_c)(G) = \beta_c(t_c)((\rho_c^{m+1})^{-1}(G)).$$

Note, there are sets $F_c^k \times [t_d^k]$ so that $(\rho_c^{m+1})^{-1}(G) = \bigcup_{k=1}^K (F_c^k \times [t_d^k])$. (Note, we choose these sets so that each t_d^k and t_d^j are distinct.) So,

$$\delta_c^{m+1}(t_c)(G) \quad = \quad \beta_c(t_c)(\bigcup_{k=1}^K (F_c^k \times [t_d^k])) \tag{3}$$

We also have

$$\begin{aligned} \delta_c^{*,m+1}(t_c^*)(G) \quad &= \quad \psi((\rho_c^{*,m+1})^{-1}(G)) \\ &= \quad \psi((\rho_c^{*,m+1})^{-1}(G) \cap (X_c \times \tau_d^m(T_d))) \end{aligned}$$

So, to show that $\delta_c^{m+1}(t_c)(G) = \delta_c^{*,m+1}(t_c^*)(G)$, it suffices to show that

$$(\rho_c^{*,m+1})^{-1}(G) \cap (X_c \times \tau_d^m(T_d)) = \bigcup_{k=1}^K (F_c^k \times \{\tau(t_d^k)\}). \tag{4}$$

If so, then

$$\begin{aligned} \delta_c^{*,m+1}(t_c^*)(G) \quad &= \quad \psi(\bigcup_{k=1}^K (F_c^k \times \{\tau(t_d^k)\})) \\ &= \quad \beta_c(t_c)(\bigcup_{k=1}^K (F_c^k \times [t_d^k])) \tag{5} \end{aligned}$$

Putting Equation (3) and Equation (5) together gives the desired result.

To show Equation (4): Fix some $(x_c, t_d^*) \in (\rho_c^{*,m+1})^{-1}(G) \cap (X_c \times \tau_d^m(T_d))$. Then,

$\rho_c^{*,m+1}(x_c, t_d^*) \in G$ and there exists $t_d \in T_d$ with $\tau_d^m(t_d) = t_d^*$. It follows that $\rho_c^{m+1}(x_c, t_d) = \rho_c^{*,m+1}(x_c, t_d^*)$ and so $(x_c, t_d) \in G$. So, there exists some k so that $(x_c, t_d) \in F_c^k \times \{t_d^k\}$. It follows that $(x_c, t_d^*) \in F_c^k \times \{\tau_d^m(t_d^k)\}$, as desired.

Conversely, fix some (x_c, t_d^*) in some set $F_c^k \times \{\tau_d^m(t_d^k)\}$. Then, $(x_c, t_d^k) \in F_c^k \times \{t_d^k\}$ and so $\rho_c^{m+1}(x_c, t_d^k) \in G$. It follows that $\rho_c^{m+1}(x_c, t_d^k) \in G$ and so $(x_c, t_d^*) \in (\rho_c^{*,m+1})^{-1}(G) \cap (X_c \times \tau_d^m(T_d))$. ∎

Part (ii) of Proposition 7.14 will now follow quickly from Part (ii) and Proposition 9.2.[14]

Proof of Proposition 7.14. Part (i) is by Lemma C.2. Part (ii) is by part (i) and Proposition 9.2. ∎

Appendix D Proofs for Section 8

Lemma D.1 *Fix an epistemic game $(G, \mathcal{T})$. A strategy-type pair (s_c, t_c) is rational if and only if s_c is optimal under $\delta_c^1 \in \mathcal{P}(S_d)$.*

Proof. It suffices to show that $\mathrm{marg}_{S_d} \beta_c(t_c) = \delta_c^1(t_c)$. But this follows since, for each event E_d in S_d

$$\delta_c^1(t_c)(E_d) = \beta_c(t_c)((\rho_c^1)^{-1}(E_d)) = \beta_c(t_c)(E_d \times T_d),$$

as required. ∎

Proof of Lemma 8.3. To show the lemma, we will prove the following points together, by induction on m.

(1) If $\delta_c^m(t_c) = \delta_c^{*,m}(t_c^*)$, then $(s_c, t_c) \in R_c^m$ only if $(s_c, t_c^*) \in R_c^{*,m}$.

(2) $(\rho_d^{*,m+1})^{-1}(\rho_d^{m+1}(\mathrm{R}_c^m)) \subseteq \mathrm{R}_c^{*,m}$.

$\mathbf{m = 1}:$ Begin with part 1. Suppose $\delta_c^1(t_c) = \delta_c^{*,1}(t_c^*)$. Then, s_c is optimal under $\delta_c^1(t_c)$ if and only if s_c is optimal under $\delta_c^{*,1}(t_c^*)$. So, applying Lemma D.1, $(s_c, t_c) \in \mathrm{R}_c^1$ if and only if $(s_c, t_c^*) \in \mathrm{R}_c^{*,1}$.

For part 2, fix $(s_c, t_c^*) \in (\rho_d^{*,2})^{-1}(\rho_d^2(\mathrm{R}_c^1))$. Then, there exists some t_c such that $(s_c, t_c) \in \mathrm{R}_c^1$ and $\rho_d^{*,2}(s_c, t_c^*) = \rho_d^2(s_c, t_c)$. So, $\delta_c^{*,1}(t_c^*) = \delta_c^1(t_c)$. By Lemma D.1, $(s_c, t_c^*) \in \mathrm{R}_c^{*,1}$.

$\mathbf{m \geq 2}:$ Assume the result holds for m and we will show that it also holds for $m + 1$.

To prove part 1, suppose $\delta_c^{m+1}(t_c) = \delta_c^{*,m+1}(t_c^*)$ and $(s_c, t_c) \in \mathrm{R}_c^{m+1}$. By coherency, $\delta_c^m(t_c) = \delta_c^{*,m}(t_c^*)$. So, by part 1 of the induction hypothesis, $(s_c, t_c) \in \mathrm{R}_c^{*,m}$. As such, it suffices to show that t_c^* believes $\mathrm{R}_d^{*,m}$.

Since $(s_c, t_c) \in \mathrm{R}_c^{m+1}$, t_c believes R_d^m, i.e., $\beta_d(t_d)(\mathrm{R}_d^m) = 1$. Note, $\rho_c^{m+1}(\mathrm{R}_d^m)$ is finite and so Borel. (Here, we use the fact that R_d^m is finite, since both the game and the type sets are finite.) So,

$$\delta_c^{m+1}(t_c)(\rho_c^{m+1}(\mathrm{R}_d^m)) = \beta_c(t_c)((\rho_c^{m+1})^{-1}(\rho_c^{m+1}(\mathrm{R}_d^m))) \geq \beta_c(t_c)(\mathrm{R}_d^m) = 1.$$

Using the fact that $\delta_c^{*,m+1}(t_c^*) = \delta_c^{m+1}(t_c)$,

$$\delta_c^{*,m+1}(t_c^*)(\rho_c^{m+1}(\mathrm{R}_d^m)) = 1.$$

So,

$$\beta_c^*(t_c^*)((\rho_c^{*,m+1})^{-1}(\rho_c^{m+1}(\mathrm{R}_d^m))) = 1.$$

By part 2 of the induction hypothesis, $(\rho_c^{*,m+1})^{-1}(\rho_c^{m+1}(\mathrm{R}_d^m)) \subseteq \mathrm{R}_d^{*,m}$, so that $\beta_c^*(t_c^*)(\mathrm{R}_d^{*,m}) = 1$, as desired.

To prove part 2, fix $(s_c, t_c^*) \in (\rho_c^{*,m+2})^{-1}(\rho_d^{m+2}(\mathrm{R}_c^{m+1}))$. Then, there exists some $t_c \in T_c$ so that $(s_c, t_c) \in \mathrm{R}_c^{m+1}$ and $\rho_d^{*,m+2}(s_c, t_c^*) = \rho_d^{m+2}(s_c, t_c)$. So, $\delta_c^{*,m+1}(t_c^*) = \delta_c^{m+1}(t_c)$. By part 1 for $m + 1$, $(s_c, t_c^*) \in \mathrm{R}_c^{*,m+1}$. $\blacksquare$

The following lemma is standard and so the proof is omitted:

Lemma D.2 *The strategy $s_c \in Y_c$ is undominated given $Y_a \times Y_b$ if and only if there exists some $\sigma_d \in \mathcal{P}(S_d)$ with*

(i) $\pi_c(s_c, \sigma_d) \geq \pi_c(r_c, \sigma_d)$, for all $r_c \in Y_c$, and

(ii) $\sigma_d(Y_d) = 1$.

Proof of Lemma 8.5. Part (ii) is immediate from part (i). We show part (i) by induction on m.

$\mathbf{m = 1}:$ If $(s_c, t_c) \in \mathrm{R}_c^1$ then s_c is optimal under $\mathrm{marg}_{S_d} \beta_c(t_c)$. So, by Lemma D.2, s_c is undominated.

$\mathbf{m \geq 2}$:   Assume the result for m. If $(s_c, t_c) \in \mathrm{R}_c^{m+1}$ then s_c is optimal under $\mathrm{marg}_{S_d}\beta_c(t_c)$ and $\beta_c(\mathrm{R}_d^m) = 1$. It follows that $\mathrm{marg}_{S_d}\beta_c(t_c)(\mathrm{proj}_{S_d}\mathrm{R}_d^m) = 1$. So, by the induction hypothesis, $\mathrm{marg}_{S_d}\beta_c(t_c)(S_d^m) = 1$. Now the claim follows from Lemma D.2. ∎

Appendix E   Proofs for Section 9

To prove Proposition 9.2, we will need to add some definitions.

Definition E.1 *Fix two (X_a, X_b)-based type structures, viz.*

$$\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b) \ and \ \mathcal{T}^* = (X_a, X_b; T_a^*, T_b^*; \beta_a^*, \beta_b^*).$$

*Say (τ_a, τ_b) is a **type morphism** from $\mathcal{T}$ to $\mathcal{T}^*$ if each $\tau_c : T_c \to T_c^*$ is a measurable map with*

$$(\mathrm{id}_c \times \tau_d) \circ \beta_c = \beta_c^* \times \tau_c,$$

where $\mathrm{id}_c \times \tau_d : X_c \times T_d \to X_c \times T_d^*$ satisfies $(\mathrm{id}_c \times \tau_d)(x_c, t_d) = (x_c, \tau_d(t_d))$.

Lemma E.2 *Fix two (X_a, X_b)-based type structures, viz.*

$$\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b) \ and \ \mathcal{T}^* = (X_a, X_b; T_a^*, T_b^*; \beta_a^*, \beta_b^*)$$

so that (τ_a, τ_b) is a type morphism from $\mathcal{T}$ to $\mathcal{T}^*$. Then, for each $t_c \in T_c$, $\delta_c(t_c) = \delta_c^*(\tau(t_c))$.

Lemma E.2 is standard (see, e.g., Heifetz and Samet, 1998, Proposition 5.1) and so the proof is omitted.[15]

Definition E.3 *A family of (X_a, X_b)-based type structures $\{\mathcal{T}^i : i \in I\}$ is said to be **pairwise disjoint** if T_c^i is disjoint from T_c^j, for each $i, j \in I$ with $i \neq j$ and each $c \in \{a, b\}$.*

Definition E.4 *Given a pairwise disjoint family of (X_a, X_b)-based type structures $\{\mathcal{T}^i : i \in I\}$, the **disjoint union** $\mathcal{T}^* = \bigsqcup_{i \in I} \mathcal{T}^i$ is the sextuple*

$$\mathcal{T}^* = (X_a, X_b; T_a^*, T_b^*; \beta_a^*, \beta_b^*)$$

such that the following hold.

---

[15] Heifetz and Samet (1998) define hierarchies of beliefs somewhat differently than here. That said, their proof can be replicated in this formalism.

(i) $T_c^* = \bigcup_{i \in I} T_c^i$.

(ii) A set $U \subseteq T_c^*$ is open in T_c^* if and only if $U \cap T_c^i$ is open in T_c^i for each $i \in I$.

(iii) $\beta_c^* : T_c^* \to \mathcal{P}(X_c \times T_d^*)$ is the mapping such that for each $i \in I$, $t_c^i \in T_c^i$, and Borel set $E \subseteq X_c \times T_d^i$, $\beta_c^*(t_c^i)(E) = \beta_c^i(t_c^i)(E)$.

We make several easy observations in the following remark.

Remark E.5 *Let $\{\mathcal{T}^i : i \in I\}$ be a family of type structures and let $\mathcal{T}^* = \bigsqcup_{i \in I} \mathcal{T}^i$ be the disjoint union.*

   (i) *The topological spaces T_a^* and T_b^* are metrizable.*

  (ii) *The disjoint union $\mathcal{T}^*$ is a type structure if and only if the functions β_a^* and β_b^* are measurable.*

 (iii) *If the index set I is finite or countable, then $\mathcal{T}^*$ is a type structure.*

 (iv) *If $\mathcal{T}^i$ is discrete for each $i \in I$, then $\mathcal{T}^*$ is a discrete type structure.*

The next set of observations pertains to the case where the disjoint union is a type structure.

Remark E.6 *Let $\{\mathcal{T}^i : i \in I\}$ be a family of type structures and let the disjoint union $\mathcal{T}^* = \bigsqcup_{i \in I} \mathcal{T}^i$ be a type structure.*

   (i) *Write $\mathrm{id}_c : T_c^i \to T_c^*$ for the identity maps. Then $(\mathrm{id}_a, \mathrm{id}_b)$ is a type morphsim from $\mathcal{T}^i$ to $\mathcal{T}^*$. For each m and each $t_c \in T_c^i$, $\delta_c^{i,m}(t_c) = \delta_c^{*,m}(t_c)$.*

  (ii) *For each m and each $t_c \in T_c^i$, $\delta_c^{i,m}(t_c) = \delta_c^{*,m}(t_c)$.*

 (iii) *If the index set I is finite and the family of type structures is each finitely simple, then $\mathcal{T}^*$ is a finitely simple type structure.*

 (iv) *If the index set I is countable and the family of type structures is each countably atomic, then $\mathcal{T}^*$ is a countably atomic type structure.*

Note, part (ii) follows from (i) and Lemma E.2. Parts (iii)-(iv) follow from part (ii).

We now introduce two related Lemmata:

Lemma E.7 *Fix a simple type structure*

$$\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b).$$

For each type $t_c \in T_c$ and each m, there exists some finitely simple type structure

$$\overline{\mathcal{T}} = (X_a, X_b; \overline{T}_a, \overline{T}_b; \overline{\beta}_a, \overline{\beta}_b)$$

and some type $\overline{t}_c \in \overline{T}_c$ with $\overline{\delta}_c^m(\overline{t}_c) = \delta_c^m(t_c)$.

Lemma E.8 *Suppose $|X_a|$ and $|X_b|$ are not measurable. Fix an atomic type structure*

$$\mathcal{T} = (X_a, X_b; T_a, T_b; \beta_a, \beta_b).$$

For each type $t_c \in T_c$ and each m, there exists some countably atomic type structure

$$\overline{\mathcal{T}} = (X_a, X_b; \overline{T}_a, \overline{T}_b; \overline{\beta}_a, \overline{\beta}_b)$$

and some type $\overline{t}_c \in \overline{T}_c$ with $\overline{\delta}_c^m(\overline{t}_c) = \delta_c^m(t_c)$.

We will show the second of these results and then comment on the proof of the first. To show the second, we will need the following preliminary Lemma. (Recall, given a measure $\mu \in \mathcal{P}(\Omega)$, PM_μ is the set of point masses of μ, i.e., the set of ω with $\mu(\{\omega\}) > 0$.)

Lemma E.9 *Suppose $|X_a|$ and $|X_b|$ are non-measurable. Then, $\mu \in \mathcal{P}(Z_c^m)$ is atomic if and only if $\mu(\mathrm{PM}_\mu) = 1$.*

We return to the proof of Lemma E.9 below. First the proof of Lemma E.8.
Proof of Lemma E.8. The proof is by induction on m.

$\mathbf{m = 1}$: Fix some type $t_c \in T_c$. Construct $\overline{\mathcal{T}}$ as follows. Take $\overline{T}_c = \{\overline{t}_c\}$. Choose $\overline{T}_d \neq \emptyset$ to be finite. Take $\overline{\beta}_c(\overline{t}_c)$ so that, for each event E_c in X_c,

$$\overline{\beta}_c(\overline{t}_c)(E_c \times \overline{T}_d) = \beta_c(t_c)(E_c \times T_d).$$

Choose $\overline{\beta}_d$ arbitrarily but such that each $\overline{\beta}_d(\overline{t}_d)$ is simple (and so atomic).

Note, for each event E_c in X_c

$$\overline{\delta}_c^1(\overline{t}_c)(E_c) = \overline{\beta}_c(\overline{t}_c)(E_c \times \overline{T}_d) = \beta_c(t_c)(E_c \times T_d) = \delta_c^1(t_c)(E_c).$$

So $\bar{\delta}_c^1(\bar{t}_c) = \delta_c^1(t_c)$. By construction, the sets $\overline{T}_a$ and $\overline{T}_b$ are finite. To show that the constructed type structure is atomic, note that by Lemma C.1, it suffices to show that, for each $\bar{t}_c$ and each $\bar{t}_d$, $\overline{\beta}_c(\bar{t}_c)$ and $\overline{\beta}_d(\bar{t}_d)$ are atomic. By construction, each $\overline{\beta}_d(\bar{t}_d)$ is simple and so atomic. To see that each $\overline{\beta}_c(\bar{t}_c)$ is atomic, note that, since $\bar{\delta}_c^1(\bar{t}_c) = \delta_c^1(t_c)$ is atomic and $|X_c|$ is not measurable, by Lemma E.9 there exists a countable set E_c with

$$\overline{\beta}_c(\bar{t}_c)(E_c \times \overline{T}_d) = \bar{\delta}_c^1(\bar{t}_c)(E_c) = 1,$$

i.e., so $E_c \times \overline{T}_d$ is a countable set that gets probability one under $\overline{\beta}_c(\bar{t}_c)$, as required.

$\mathbf{m \geq 2}$: Assume the result holds for each player, each type in $\mathcal{T}$, and m. We will show that the same holds for $m + 1$.

Fix a type $t_c \in T_c$. Since $\mathcal{T}$ is atomic and $|X_a|, |X_a|$ are not measurable, there is a countable set of distinct points

$$E = \{z^1, z^2, \ldots\} \subseteq Z_c^{m+1}$$

with $\delta_c^{m+1}(t_c)(E) = 1$ and $\delta_c^{m+1}(t_c)(\{z^k\}) = \alpha^k > 0$ for each $z^k \in E$. (See Lemma E.9.) Note, E is the set of point masses of $\delta_c^m(t_c)$ and so depends on both t_c and m.

For each $k = 1, \ldots, K$, we have

$$\alpha^k = \beta_c(t_c)((\rho_c^{m+1})^{-1}(\{z^k\}) > 0.$$

So, for each $k = 1, \ldots$ there exists a point $(x_c^k, t_d^k) \in X_c \times T_d$ with $\rho_c^{m+1}(x_c^k, t_d^k) = z^k$. By the induction hypothesis, for each $k = 1, \ldots,$ there is a countably atomic type structure

$$\overline{\mathcal{T}}^k = (X_a, X_b; \overline{T}_a^k, \overline{T}_b^k; \overline{\beta}_a^k, \overline{\beta}_b^k)$$

and a type $\bar{t}_d^k \in \overline{T}_d^k$ with $\bar{\delta}_d^{k,m}(\bar{t}_d^k) = \delta_d^m(t_d)$. By renaming points, we can take the type structures $(\overline{\mathcal{T}}^1, \overline{\mathcal{T}}^2, \ldots)$ to be pairwise disjoint.

Construct $\overline{\mathcal{T}}$ as follows: Let $\mathcal{T}^*$ be the disjoint union $\mathcal{T}^* = \bigsqcup_k \overline{\mathcal{T}}^k$. Note, by Remark E.5(iii) and Remark E.6(iii), $\mathcal{T}^*$ is a countably atomic type structure. By construction, the points $\bar{t}_d^1, \bar{t}_d^2, \ldots$ are distinct points in T_d^*.

Construct a new type structure, viz. $\overline{\mathcal{T}}$, as follows: Take a new point $\bar{t}_c \notin T_c^*$ and set $\overline{T}_c = T_c^* \cup \{\bar{t}_c\}$. Set $\overline{T}_d = T_d^*$. Endow $\overline{T}_c$ and $\overline{T}_d$ with the discrete topology. Choose the maps $\overline{\beta}_c$ and $\overline{\beta}_d$ so that the identity maps form a type morphism from $\overline{\mathcal{T}}$ to $\mathcal{T}^*$. For $\bar{t}_c$, let

$\overline{\beta}_c(\overline{t}_c)$ be an atomic probability measure on $X_c \times \overline{T}_d$ such that

$$\overline{\beta}_c(\overline{t}_c)(\{(x_c^k, \overline{t}_d^k)\}) = \alpha^k$$

for each $k = 1, \ldots$. Since $\mathcal{T}^*$ is countably atomic and $\overline{\beta}_c(\overline{t}_c)$ is atomic, it follows that $\overline{\mathcal{T}}$ is countably atomic.

It remains to show that $\overline{\delta}_c^{m+1}(\overline{t}_c) = \delta_c^{m+1}(t_c)$: Note, since the identity maps are a type morphism from $\overline{\mathcal{T}}$ to $\mathcal{T}^*$, it follows that

$$\overline{\delta}_d^m(\overline{t}_d^k) = \overline{\delta}_d^{k,m}(\overline{t}_d^k) = \delta_d^m(t_d^k).$$

So,

$$\overline{\rho}_c^{m+1}(x_c^k, \overline{t}_d^k) = \rho_c^{m+1}(x_c^k, t_t^k) = z^k.$$

It follows that

$$\overline{\delta}_c^{m+1}(\overline{t}_c)(\{z^k\}) = \overline{\beta}_c(\overline{t}_c)((\overline{\rho}_c^{m+1})^{-1}(\{z^k\})) \geq \overline{\beta}_c(\overline{t}_c)(\{(x_c^k, \overline{t}_d^k)\}) = \alpha^k.$$

Using the fact that $\sum_k \alpha^k = 1$, it follows that $\overline{\delta}_c^{m+1}(\overline{t}_c)(\{z^k\}) = \alpha^k$ for each $k = 1, \ldots$. So, $\overline{\delta}_c^{m+1}(\overline{t}_c) = \delta_c^{m+1}(t_c)$, as desired. ∎

For the proof of Lemma E.7: Repeat the proof of Lemma E.8. Now take $\mathcal{T}$ to be simple. The proof is the same except that the word "countable" is everywhere replaced by "finite," "countably" is everywhere replaced by "finitely," "atomic" is everywhere replaced by "simple," and the index k ranges over some $1, \ldots, K$. Note, in this case, we need not require that $|X_c|$ is not measurable: If $\delta_c^m(t_c)$ is simple, there is necessarily a finite set of points in Z_c^m that gets probability one under $\delta_c^m(t_c)$, irrespective of whether or not $|Z_c^m|$ is measurable.

Proof of Proposition 9.2. Immediate from Lemma E.7. ∎

Now, we return to prove Lemma E.9. First some facts about non-measurable cardinals:

- If κ and λ are non-measurable cardinals, then so is $\kappa \times \lambda$.

- If $|X|$ is non-measurable, then $2^{|X|}$ is non-measurable.

- $\mathfrak{c}$ is non-measurable.

Lemma E.10 *Fix a space X and a sigma-algebra on X. Write $\mathcal{P}(X)$ for the set of probability measures on X. If $|X|$ is non-measurable then $|\mathcal{P}(X)|$ is not-measurable.*

49

Proof. Note, $\mathcal{P}(X)$ is a set of functions from the set of measurable subsets of X into the reals. Therefore $|\mathcal{P}(X)|$ is at most $\mathfrak{c}^{2^{|X|}}$, which is equal to $2^{\aleph_0 \times 2^{|X|}}$. Hence if $|X|$ is non-measurable then $|\mathcal{P}(X)|$ is non-measurable. $\blacksquare$

Lemma E.11 *Suppose $|X_a|$ and $|X_b|$ are non-measurable. Then, for each c and each m, $|Z_m^c|$ is non-measurable*

Proof. The proof is by induction on m. The case of $m = 1$ is immediate from the fact that $|X_a|$ and $|X_b|$ are non-measurable. Assume the result holds for m. Then, by the induction hypothesis, $|Z_c^m|$ and $|Z_d^m|$ are non-measurable. By Lemma E.10, $|\mathcal{P}(Z_d^m)|$ is non-measurable. So, $|Z_c^{m+1}| = |Z_c^m \times \mathcal{P}(Z_d^m)|$ is non-measurable. $\blacksquare$

Fix some $\mu \in \mathcal{P}(\Omega)$, where Ω is metrizable. Recall, $\mathrm{PM}_\mu = \{\omega \in \Omega : \mu(\{\omega\}) > 0\}$ is the set of point masses of μ. We will show that whenever the cardinality of Ω is not measurable, μ is atomic if and only if $\mu(\mathrm{PM}_\mu) = 1$.

Lemma E.12 *The following are equivalent:*

(i) Ω does not have a discrete subset of measurable cardinality.

(ii) For every $\mu \in \mathcal{P}(\Omega)$, every atom of μ contains a point mass of μ.

(iii) $\mu \in \mathcal{P}(\Omega)$ is atomic if and only if $\mu(\mathrm{PM}_\mu) = 1$.

Lemma E.13 *[Billingsley, 1968, Theorem 2, page 235] The following are equivalent:*

(i) Ω does not have a discrete subset of measurable cardinality.

(ii) Every $\mu \in \mathcal{P}(\Omega)$ has a separable support.

Proof of Lemma E.12. It is easily seen that (ii) and (iii) are equivalent. Thus, it suffices to show (ii) implies (i) and (i) implies (iii).

(ii) implies (i): Assume that (i) fails, so there is a discrete subset E of Ω of measurable cardinality. Then there is a Borel probability measure μ on Ω such that $\mu(F) = 0$ for every finite $F \subseteq E$, $\mu(E) = 1$, and E is an atom of μ. Then E is an atom that does not contain a point mass, so (ii) fails.

(i) implies (iii): Suppose (i) holds. Let μ be an atomic Borel probability measure on Ω. By Lemma E.13, μ has a separable support. This means that there is a closed separable set C such that $\mu(C) = 1$. Let D be a countable dense subset of C, and for each $n > 0$ let D_n be the union of all $\frac{1}{n}$-balls centered at elements of D. Then $C \subseteq D_n$ for each $n > 0$, and $\bigcap_n D_n = D$. Therefore $\mu(D) = 1$. Since $\mu(D) = \sum_{x \in D} \mu(\{x\})$, it follows that D is a union of point masses for μ, so $\mu(\mathrm{PM}_\mu) = 1$. ∎

Corollary E.14 *If the cardinality of Ω is not measurable, then $\mu \in \mathcal{P}(\Omega)$ is atomic if and only if $\mu(\mathrm{PM}_\mu) = 1$.*

Proof of Lemma E.9. Immediate from Lemma E.11 and Corollary E.14. ∎

Finally, we turn to a result eluded to in the discussion.

Proposition E.15 *Fix (X_a, X_b).*

   (i) *There is a simple (X_a, X_b)-based type structure that is terminal for all finitely simple type structures.*

   (ii) *There is a countably atomic (X_a, X_b)-based type structure that is terminal for all countably atomic type structures.*

As a corollary of this result plus Proposition 9.2:

Corollary E.16 *Fix (X_a, X_b).*

   (i) *There is a simple (X_a, X_b)-based type structure that is finitely terminal for all simple type structures.*

   (ii) *If $|X_a|$ and $|X_b|$ are not measurable, there is a countably atomic (X_a, X_b)-based type structure that is finitely terminal for all atomic type structures.*

The idea of the proof of Proposition E.15 is clear: Construct a type structure that is the disjoint union of all finitely simple (resp. countably atomic) type structures. But this does not work because the class of all finitely simple (resp. countably atomic) type structures is a proper class. Instead, we construct a type structure that is the disjoint union of a set of finitely simple (resp. countably atomic) type structures that contains a copy of every finitely simple (resp. countably atomic) type structure.

Proof of Proposition E.15.   Let $\{\mathcal{T}^i : i \in I\}$ be the set of all finitely simple (resp. countably atomic) type structures whose type sets are finite subsets (resp. any subset) of $\mathbb{N}$. This set of type structures is not pairwise disjoint, but we can replace it by a pairwise disjoint set in the following way: For each $i \in I$, let

$$\hat{\mathcal{T}}^i = (X_a, X_b; \hat{T}_a^i, \hat{T}_b^i; \hat{\beta}_a^i, \hat{\beta}_b^i),$$

where $\hat{T}_c^i = T_c^i \times \{i\}$ and $\hat{\beta}_c^i(E \times \{i\}) = \beta_c^i(E)$ for each event E in $X_c \times T_d^i$. Then, $\{\hat{\mathcal{T}}^i : i \in I\}$ is a pairwise disjoint set of finitely simple (resp. countably atomic) type structures that contains a copy of every finitely simple (resp. countably atomic) type structure.

Let $\mathcal{T}^* = \bigsqcup_{i \in I} \hat{\mathcal{T}}^i$. Since each $\mathcal{T}^i$ is discrete, it follows from Remark E.5(iv) that $\mathcal{T}^*$ is a discrete type structure. By Remark E.6(i) and Lemma E.2, $\delta_c^*(t_c^i) = \hat{\delta}_c^i(t_c^i)$, for each $t_c \in \hat{T}_c^i$. It follows that $\mathcal{T}^*$ is terminal for all finitely simple (resp. countably atomic) type structures. Moreover, since each $\hat{\mathcal{T}}^i$ is simple (resp. countably atomic), $\mathcal{T}^*$ is simple (resp. countably atomic). ■

References

Aliprantis, C.D. and K.C. Border. 2007. *Infinite dimensional analysis: a hitchhiker's guide.* Springer Verlag.

Arieli, I. 2010. "Rationalizability in Continuous Games." *Journal of Mathematical Economics* 46(5):912–924.

Battigalli, P. and M. Siniscalchi. 1999. "Hierarchies of Conditional Beliefs and Interactive Epistemology in Dynamic Games." *Journal of Economic Theory* 88(1):188–230.

Battigalli, P. and M. Siniscalchi. 2002. "Strong Belief and Forward Induction Reasoning." *Journal of Economic Theory* 106(2):356–391.

Bernheim, B.D. 1984. "Rationalizable Strategic Behavior." *Econometrica* 52(4):1007–1028.

Billingsley, P. 1968. *Convergence of probability measures.* John Wiley & Sons Inc.

Böge, W. and T. Eisele. 1979. "On Solutions of Bayesian Games." *International Journal of Game Theory* 8(4):193–215.

Brandenburger, A. 2003. "On the Existence of a "Complete" Possibility Structure." *Cognitive Processes and Economic Behaviour* p. 30.

Brandenburger, A. and A. Friedenberg. 2008. "Intrinsic Correlation in Games." *Journal of Economic Theory* 141(1):28–67.

Brandenburger, A., A. Friedenberg and H.J. Keisler. 2008. "Admissibility in Games." *Econometrica* 76(2):307.

Brandenburger, A. and E. Dekel. 1987. "Rationalizability and Correlated Equilibria." *Econometrica* 55(6):1391–1402.

Brandenburger, A. and E. Dekel. 1993. "Hierarchies of Beliefs and Common Knowledge." *Journal of Economic Theory* 59:189–189.

Brandenburger, A. and H.J. Keisler. 2006. "An impossibility theorem on beliefs in games." *Studia Logica* 84(2):211–240.

Fremlin, David H. 2008. *Measure Theory, Volume 5.* Torres Fremlin.

Friedenberg, A. 2010. "When do Type Structures Contain all Hierarchies of Beliefs?" *Games and Economic Behavior* 68(1):108–129.

Friedenberg, A. and M. Meier. 2010. "On the Relationship Between Hierarchy and Type Morphisms." *Economic Theory* pp. 1–23.

Harsanyi, J.C. 1967. "Games with Incomplete Information Played by "Bayesian" Players, I-III. Part I. The Basic model." *Management Science* pp. 159–182.

Heifetz, A. and D. Samet. 1998. "Topology-free typology of beliefs." *Journal of Economic Theory* 82(2):324–341.

Heifetz, A. and D. Samet. 1999. "Coherent beliefs are not always types." *Journal of Mathematical Economics* 32(4):475–488.

Kechris, A.S. 1995. *Classical descriptive set theory.* Springer.

Keisler, H.J. and B.S. Lee. 2010. "Common Assumption of Rationality.".

Kets, W. 2010. "Bounded Reasoning and Higher-Order Uncertainty.".

Marx, L.M. and J.M. Swinkels. 1997. "Order Independence for Iterated Weak Dominance* 1." *Games and Economic Behavior* 18(2):219–245.

Meier, M. 2006. "Finitely Additive Beliefs and Universal Type Spaces." *The Annals of Probability* 34(1):386–422.

Mertens, J.F. and S. Zamir. 1985. "Formulation of Bayesian Analysis for Games with Incomplete Information." *International Journal of Game Theory* 14(1):1–29.

Nash, J. 1951. "Non-Cooperative Games." *Annals of mathematics* 54(2):286–295.

Pearce, D.G. 1984. "Rationalizable Strategic Behavior and the Problem of Perfection." *Econometrica* 52(4):1029–1050.

Tan, T.C.C. and S.R. Werlang. 1988. "The Bayesian Foundations of Solution Concepts of Games." *Journal of Economic Theory* 45(2):370–391.

Yildiz, M. 2009. "Consistent Equilibrium Selection.".