# Online Supplement: When Do Type Structures Contain All Hierarchies of Beliefs?

Amanda Friedenberg

October 2008

This note provides certain technical steps omitted from the main text. The reader is asked to consult the main text for notation.

**Lemma S1** *For each $m$:*

*(i)* $Z^c_{m+1} = Z^c_1 \times \prod_{n=1}^{m} \mathcal{P}\left(Z^d_n\right)$;

*(ii)* $\rho^c_{m+1}\left(x^c, t^d\right) = (x^c, \delta^d_1(t^d), \ldots, \delta^d_m(t^d))$.

**Proof.** For $m = 1$, both parts are immediate. Assume that the result holds for $m \geq 2$. Then

$$\begin{aligned} Z^c_{m+2} &= Z^c_{m+1} \times \mathcal{P}\left(Z^d_{m+1}\right) \\ &= Z^c_1 \times \textstyle\prod_{n=1}^{m} \mathcal{P}\left(Z^d_n\right) \times \mathcal{P}\left(Z^d_{m+1}\right). \end{aligned}$$

Also,

$$\begin{aligned} \rho^c_{m+2}\left(x^c, t^d\right) &= (\rho^c_{m+1}(x^c, t^d), \delta^d_{m+1}(t^d)) \\ &= (x^c, \delta^d_1(t^d), \ldots, \delta^d_m(t^d), \delta^d_{m+1}(t^d)), \end{aligned}$$

establishing the result. ■

**Lemma S2** *Let $\Omega$ be metrizable and $\Phi$ be Polish. If $f : \Omega \to \Phi$ is measurable (resp. continuous), then $\underline{f} : \mathcal{P}(\Omega) \to \mathcal{P}(\Phi)$ is measurable (resp. continuous).*

**Proof.** The case where $f$ is continuous is found in Aliprantis-Border [1, 1999; Theorem 14.14]. We treat the case where $f$ is measurable.

First, note that, since $\Phi$ is Polish, $\mathcal{B}(\mathcal{P}(\Phi))$ is generated by all the sets of the form $\{\nu \in \mathcal{P}(\Phi) : \nu(E) \in K\}$, ranging over $E$ Borel in $\Phi$ and $K$ measurable in $[0, 1]$ (see Kechris [2, 1995; Theorem 17.24]). It suffices to show that each set $\underline{f}^{-1}(\{\nu \in \mathcal{P}(\Phi) : \nu(E) \in K\})$ is in $\mathcal{B}(\mathcal{P}(\Omega))$. If so, then $\underline{f}$ is measurable. (See Aliprantis-Border [1, 1999; Corollary 4.23].)

Note that

$$\underline{f}^{-1}(\{\nu \in \mathcal{P}(\Phi) : \nu(E) \in K\}) = \{\mu \in \mathcal{P}(\Omega) : \underline{f}(\mu)(E) \in K\} = \{\mu \in \mathcal{P}(\Omega) : \mu(f^{-1}(E)) \in K\}.$$

Since $f$ is measurable, $f^{-1}(E)$ is Borel in $\Omega$. Now, by Aliprantis-Border [1, 1999; Lemma 14.16], $\underline{f}^{-1}(\{\nu \in \mathcal{P}(\Phi) : \nu(E) \in K\})$ is in $\mathcal{B}(\mathcal{P}(\Omega))$. ■

**Lemma S3** *Let $I$ be (at most) a countable collection of integers $1, 2, \ldots$. Let $\Omega$ be a topological space and, for each $i \in I$, let $\Phi_i$ be a Polish space. Fix measurable (resp. continuous) maps $f_i : \Omega \to \Phi_i$ and define $f : \Omega \to \prod_i \Phi_i$ so that*

$$f(\omega) = (f_1(\omega), f_2(\omega), \ldots)$$

*for all $\omega \in \Omega$. Then, $f$ is measurable (resp. continuous).*

**Proof.** Theorem 8.5 in Chapter 2 of Munkres [3, 1975] establishes the case where each $f_i$ is continuous. Suppose each $f_i$ is measurable. Let $E$ be a Borel set on $\prod_i \Phi_i$. Then, $f^{-1}(E) = \bigcap_i f_i^{-1}(E)$. Each $f_i^{-1}(E)$ is measurable, so that $f^{-1}(E)$ is measurable. ■

**Lemma S4** *The maps $\rho_m^c$ and $\delta_m^c$ are measurable. Moreover, if the structure is continuous, the maps $\rho_m^c$ and $\delta_m^c$ are continuous.*

**Proof.** First, note that $\rho_1^c = \text{proj}_{X^c}$, and so is certainly continuous. So, by Lemma S2, $\underline{\rho}_1^c$ is continuous. Also using this Lemma, we have that $\delta_1^c$ is measurable (and continuous if $\beta^c$ is continuous).

Now, assume that the result holds for $m$. By the induction hypothesis and Lemma S3, $\rho_{m+1}^c$ is measurable. (When $\rho_m^c, \delta_m^d$ are continuous, Lemma S3 says that $\rho_{m+1}^c$ is continuous.) Now, by Lemma S2, $\underline{\rho}_{m+1}^c$ is measurable (resp. continuous when $\rho_{m+1}^c$ is continuous). So, each $\delta_{m+1}^c$ is measurable (and continuous when $\beta^c$ is continuous). ■

**Lemma S5** *The map $\delta^c$ is measurable. Moreover, $\delta^c$ is continuous if the structure is continuous.*

**Proof.** Immediate from Lemmas S3 and S4. ■

**Proof of Lemma 3.1.** By induction on $m$.

$m = 1$: Since $\eta_1^c$ is the identity map, $\eta_1^c(\mu_1^c) = \mu_1^c$, as required. Now note that that since $\zeta_1^d$ is the identity map, so is $\underline{\zeta}_1^d$, and so $\underline{\zeta}_1^d = \eta_1^d$. This establishes that $\zeta_2^c(x_1^c, \mu_1^d) = (\zeta_1^c(x_1^c), \underline{\zeta}_1^d(\mu_1^d))$, as required.

$m \geq 2$: Assume that the result holds for $m$. It is immediate from the induction hypothesis that

$\eta^c_{m+1}\left(\mu^c_1,\ldots,\mu^c_{m+1}\right)=\left(\mu^c_1,\underline{\zeta}^c_2\left(\mu^c_2\right),\ldots,\underline{\zeta}^c_{m+1}\left(\mu^c_{m+1}\right)\right)$. Now,

$$\begin{aligned}\zeta^c_{m+2}\left(x^c_1,\mu^d_1,\ldots,\mu^d_m,\mu^d_{m+1}\right) &= \left(\zeta^c_1\left(x^c_1\right),\eta^d_{m+1}\left(\mu^d_1,\ldots,\mu^d_m,\mu^d_{m+1}\right)\right)\\ &= \left(\zeta^c_1\left(x^c_1\right),\eta^d_m\left(\mu^d_1,\ldots,\mu^d_m\right),\underline{\zeta}^d_{m+1}\left(\mu^d_{m+1}\right)\right)\\ &= \left(\zeta^c_{m+1}\left(x^c_1,\mu^d_1,\ldots,\mu^d_m\right),\underline{\zeta}^d_{m+1}\left(\mu^d_{m+1}\right)\right),\end{aligned}$$

where the second line follows from the fact that $\eta^c_{m+1}\left(\mu^c_1,\ldots,\mu^c_{m+1}\right)=\left(\mu^c_1,\underline{\zeta}^c_2\left(\mu^c_2\right),\ldots,\underline{\zeta}^c_{m+1}\left(\mu^c_{m+1}\right)\right)$, already established. ■

**Lemma S6** *Let $I=\{1,2,\ldots\}$ be (at most) a countable set of integers. For each $i$, let $f_i:\Omega_i\to\Phi_i$ be an embedding. Define $f:\prod_{i\in I}\Omega_i\to\prod_{i\in I}\Phi_i$ so that $f\left(\omega_1,\omega_2,\ldots\right)=\left(f_1\left(\omega_1\right),f_2\left(\omega_2\right)\ldots\right)$. Then, $f$ is also an embedding.*

**Proof.** Let $g_i:\Omega_i\to f_i\left(\Omega_i\right)$ be such that $g_i\left(\omega_i\right)=f_i\left(\omega_i\right)$. We have that each $g_i$ is a homeomorphism. Define $f$ as in the statement of the Lemma and $g:\prod_i\Omega_i\to f\left(\prod_i\Phi_i\right)$ so that $g\left(\omega\right)=f\left(\omega\right)$ for all $\omega\in\prod_i\Omega_i$. We will show that $g$ is also a homeomorphism. The injectivity of $g$ is immediate from the injectivity of each of the maps $g_i$. The surjectivity of $g$ is immediate.

To show that $g$ is continuous, fix closed sets $C_i$ in $f_i\left(\Omega_i\right)$ where $C_i=f_i\left(\Omega_i\right)$ for all but finitely many $i$. Under the product and relative topologies, these sets form a basis for $\prod_{i\in I}f_i\left(\Omega_i\right)$. So, it suffices to show that $g^{-1}\left(\prod_{i\in I}C_i\right)$ is closed. Let $J$ be the set of $i$ with $C_i\neq f_i\left(\Omega_i\right)$. Then,

$$\begin{aligned}g^{-1}\left(\prod_{i\in I}C_i\right) &= \prod_{i\in J}\left[g_i^{-1}\left(C_i\right)\right]\times\prod_{i\in I\setminus J}\left[g_i^{-1}\left(C_i\right)\right]\\ &= \prod_{i\in J}\left[g_i^{-1}\left(C_i\right)\right]\times\prod_{i\in I\setminus J}\Omega_i.\end{aligned}$$

Since each $g_i$ is continuous, then each $g_i^{-1}\left(C_i\right)$ is closed. It follows that $g^{-1}\left(\prod_{i\in I}C_i\right)$ is indeed closed, as required.

To show that $g$ is closed, fix closed sets $F_i$ in $\Omega_i$, where $F_i=\Omega_i$ for all but finitely many $i$. Again, these sets form a basis for $\prod_{i\in I}\Omega_i$ in the product topology, so that it suffices to show that $g\left(\prod_{i\in I}F_i\right)$ is closed. Let $J$ be the subset of $i$ with $F_i\neq\Omega_i$. Then

$$\begin{aligned}g\left(\prod_{i\in I}F_i\right) &= \prod_{i\in J}g_i\left(F_i\right)\times\prod_{i\in I\setminus J}g_i\left(F_i\right)\\ &= \prod_{i\in J}g_i\left(F_i\right)\times\prod_{i\in I\setminus J}f_i\left(\Omega_i\right),\end{aligned}$$

where the last line follows from the fact that each $g_i$ is surjective. Since each $g_i\left(F_i\right)$ is closed, it follows that $g\left(\prod_{i\in I}F_i\right)$ is closed. ■

**Proof of Lemma 3.2.** By induction on $m$. For $m=1$, the result is immediate since $\zeta^c_1$ and $\eta^c_1$ are the identity maps. Assume that the result holds for $m$. We will show that it also holds for $m+1$. By the induction hypothesis, $\zeta^c_1$ and $\eta^d_m$ are embeddings. Since $\zeta^c_{m+1}$ is the product of $\zeta^c_1$ and $\eta^d_m$, Lemma S6 in Appendix A gives that $\zeta^c_{m+1}$ is an embedding. Similarly, $\eta^c_{m+1}$ is the product of $\eta^c_m$ and $\underline{\zeta}^c_{m+1}$. The former is an embedding by the induction hypothesis. Moreover, the induction hypothesis gives that $\zeta^c_{m+1}$ is an embedding so that $\underline{\zeta}^c_{m+1}$ is an embedding (Kechris [2, 1995; Exercise 17.28]). Again, by Lemma S6, $\eta^c_{m+1}$ is an embedding. ■

**Proof of Lemma 3.3.** First, note that $\eta^c$ is injective since each of the maps $\eta^c_m$ is injective: Fixing $(\mu_1, \mu_2, \ldots) \neq (\varpi_1, \varpi_2, \ldots)$ in $H^c$, we can find some initial segment of these sequences that are distinct, i.e., some $m$ with $(\mu_1, \ldots, \mu_m) \neq (\varpi_1, \ldots, \varpi_m)$. Then, by Lemma 3.2, $\eta^c_m(\mu_1, \ldots, \mu_m) \neq \eta^c_m(\varpi_1, \ldots, \varpi_m)$ so that $\eta^c(\mu_1, \mu_2, \ldots) \neq \eta^c(\varpi_1, \varpi_2, \ldots)$.

Now, we turn to showing the continuity of $\eta^c$. For each $m$, fix closed sets $C_m$ in $\mathcal{P}(Z^c_m)$ with $C_m = \mathcal{P}(Z^c_m)$ for all but finitely many $m$. Since these sets form a basis for the product topology, it suffices to show that $(\eta^c)^{-1}(\prod_m C_m)$ is closed. Note, there is some $M$ such that

$$(\eta^c)^{-1}\left(\prod_{m=1}^{\infty} C_m\right) = \left[(\eta^c_M)^{-1}\left(\prod_{m=1}^{M} C_n\right) \times \prod_{m=M+1}^{\infty} \mathcal{P}(Z^c_m)\right] \cap H^c.$$

By Lemma 3.2, $(\eta^c_m)^{-1}(\prod_{n=1}^{M} C_n)$ is closed, and by Lemma A3, $H^c$ is closed. It follows that $(\eta^c)^{-1}(\prod_{m=1}^{\infty} C_m)$ is closed.

Next, we show that $\eta^c$ is a closed map. For this, fix closed sets $F_m$ in $\mathcal{P}(X^c_m)$ with $F_m = \mathcal{P}(X^c_m)$ for all but finitely many $m$. Since these sets form the basis for the product topology for $\prod_{m=1}^{\infty} \mathcal{P}(X^c_m)$, it suffices to show that $\eta^c(H^c \cap (\prod_{m=1}^{\infty} F_m))$ is closed. Note that there exists some $M$ with

$$\eta^c\left(H^c \cap \prod_{m=1}^{\infty} F_m\right) = \left[\eta^c_M\left(H^c_M \cap \prod_{m=1}^{M} F_m\right) \times \prod_{m=M+1}^{\infty} \mathcal{P}(X^c_m)\right] \cap H^c.$$

By Lemma A2, $H^c_M$ is closed. So, again using Lemmas 3.2 and A3, $\eta^c_M(H^c \cap \prod_{m=1}^{\infty} F_m)$ must be closed, as required. ■

**Proof of Lemma 3.5.** It suffices to show that, for each $m$, $\delta^c_{m+1}(t^c) = \delta^c_m(t^c)$. If so, a standard inductive argument completes the proof. Fix an event $E$ in $Z^c_m$ and notice that

$$\begin{aligned}
\left(\rho^c_{m+1}\right)^{-1}\left(E \times \mathcal{P}\left(Z^d_m\right)\right) &= [(\rho^c_m)^{-1}(E)] \cap [X^c \times (\delta^d_m)^{-1}\left(\mathcal{P}\left(Z^d_m\right)\right)] \\
&= (\rho^c_m)^{-1}(E).
\end{aligned}$$

From this,

$$\begin{aligned}
\delta^c_{m+1}(t^c)\left(E \times \mathcal{P}\left(Z^d_m\right)\right) &= \beta^c(t^c)((\rho^c_{m+1})^{-1}(E \times \mathcal{P}(Z^d_m))) \\
&= \beta^c(t^c)((\rho^c_m)^{-1}(E)) \\
&= \delta^c_m(t^c)(E),
\end{aligned}$$

as required. ■

**Proof of Lemma 3.8.** Fix some $n \leq m$ and some event $G$ in $Z^c_n$. Then $\underline{\zeta}^c_n(\mu_n)(G) = \mu_n((\zeta^c_n)^{-1}(G))$. We know that $\text{marg}_{X^c_n} \mu_{m+1} = \mu_n$, so that

$$
\begin{aligned}
\mu_n((\zeta_n^c)^{-1}(G)) &= \mu_{m+1}(\{(\nu_1,\ldots,\nu_{m+1}) \in H_{m+1}^c : (\nu_1,\ldots,\nu_n) \in (\zeta_n^c)^{-1}(G)\}) \\
&= \mu_{m+1}((\zeta_{m+1}^c)^{-1}(G \times \textstyle\prod_{k=n+1}^{m+1} \mathcal{P}(Z_k^d))) \\
&= \underline{\zeta}_{m+1}^c(\mu_{m+1})(G \times \textstyle\prod_{k=n+1}^{m+1} \mathcal{P}(Z_k^d)) \\
&= \operatorname{marg}_{Z_n^c} \underline{\zeta}_{m+1}^c \left(\mu_{m+1}\right)(G)\,,
\end{aligned}
$$

as required. ■

## References


[1] Aliprantis, C., and K. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer, 1999.

[2] Kechris, A., *Classical Descriptive Set Theory*, Springer-Verlag, 1995.

[3] Munkres, J., *Topology: A First Course*, Prentice-Hall, 1975.