When Do Type Structures Contain All Hierarchies of Beliefs?*

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Abstract

Hierarchies of beliefs play a central role in game theory. In practice, they are modeled by type structures. This allows the analyst to describe hierarchies without explicitly specifying an infinite sequence of beliefs. The focus of this paper is type structures that contain all hierarchies of beliefs. Can the analyst identify these structures without explicit reference to hierarchies? That is, does there exist a test, defined on the type structure alone, so that the structure passes this test only if it contains all hierarchies of beliefs? This paper investigates one such test. The test is based on the concept of completeness (Brandenburger [6, 2003]), a concept that has played an important role in epistemic game theory.

Many game-theoretic analyses require specifying players' hierarchies of beliefs. This is true both of the epistemic program and of so-called games of incomplete information.

Harsanyi [15, 1967-1968] introduced the concept of a type structure and argued that it is a model of hierarchies of beliefs. The benefit of this model is that it allows the analyst to describe hierarchies without explicitly specifying an infinite sequence of beliefs. But for a type structure to be a model, it should be able to represent all hierarchies of beliefs. That is, the analyst shouldn't lose any hierarchies by using the model.

When does a type structure contain all hierarchies of beliefs? Fix a given type structure. By explicitly specifying the associated hierarchies of beliefs, the analyst can verify whether or not the structure contains all hierarchies. Yet, the point of using a type structure is to avoid explicit

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reference to hierarchies. This raises the question: Can the analyst identify whether or not a type structure "contains" all hierarchies of beliefs without explicitly specifying the associated hierarchies? Put differently, does there exist a test defined on the type structure alone, so that the structure passes the test only if it contains all hierarchies of beliefs?

The literature has considered one such test, based on Mertens-Zamir's [27, 1985] idea of embedding type structures. This paper considers another test, based on Brandenburger's [6, 2003] idea of completeness. Here are the ideas.

> **The Embedding Test:** A type structure passes this test if any other type structure can be embedded into it in a unique way, via maps called type morphisms.
>
> **The Completeness Test:** A type structure passes this test if it contains all possible types within the given type structure.

We will discuss the embedding test in Sections 1.3 and 1.6. This test has played a crucial role in constructing large type structures. See Heifetz-Samet [17, 1998] and subsequent extensions of their work.

Our focus will be on the completeness test. (See Section 2 for a formal definition.) Like the embedding test, the completeness test does not make explicit reference to hierarchies of beliefs. Unlike the embedding test, the completeness test does not make reference to other type structures.

The main theorem of the paper states:

> **Main Theorem:** If a complete structure is compact and continuous, then it induces all hierarchies of beliefs.

Precise definitions will be given in Section 2. The formal statement of the result is given in Theorem 3.1.

How does this result relate to what is known in the literature? Many papers begin by describing all hierarchies of beliefs and show that their constructions give a type structure as an output. (This can be found in a variety of settings. See, for instance, Armbruster-Böge [2, 1979], Böge-Eisele [5, 1979], Mertens-Zamir [27, 1985], Brandenburger-Dekel [7, 1993], Heifetz [16, 1993], Heifetz-Samet [17, 1998], Battigalli-Siniscalchi [3, 1999], Meier [24, 2001], and Meier [25, 2006].) Indeed, some of these papers get a type structure that is complete (e.g., Mertens-Zamir [27, 1985], Brandenburger-Dekel [7, 1993], Heifetz [16, 1993], Battigalli-Siniscalchi [3, 1999], and Meier [24, 2001]). That is, they start by describing all hierarchies of beliefs and get completeness as an output. So, in a certain sense, they address the question: If a structure contains all hierarchies of beliefs, is it complete? But they do not address the converse: Does a complete structure induce all hierarchies of beliefs? This is the main question of the current paper.

To the best of our knowledge, this question is new. The one notable exception is Di Tillio [10, 2008]. In the course of addressing a very different question, Di Tillio provides conditions under which a complete preference-based type structure contains all hierarchies of preferences. In Section

5e, we return to discuss the relationship between asking this question in the context of hierarchies of preferences vs. beliefs.

Why is this question of interest? Type structures are the predominant model of hierarchies of beliefs. Often, applications—both in the epistemic literature and in the incomplete information literature—require the use of type structures that contain all hierarchies. In such a case, it would be desirable to have a property, based on type structures alone, that delivers this output. Thus, the characterization given here has the potential to simplify analyses.

The completeness condition has already proven important in the epistemic program. Specifically, it is intrinsic to the epistemic analyses of Battigalli-Siniscalchi [4, 2002] and Brandenburger-Friedenberg-Keisler [8, 2008]. Each of these papers models hierarchies of beliefs in terms of type structures, and so each looks for epistemic conditions defined on the type structure alone. Indeed, this is true of the epistemic program more generally. As a consequence, at times, the epistemic conditions involve technical conditions imposed on the type structure. It is not transparent what such conditions mean in terms of players' reasoning. Section 5c argues that the main result here (Theorem 3.1) is important for interpreting one such set of conditions. (Friedenberg [12, 2008] expands on this point.) Section 5d discusses further connections with [4, 2002] and [8, 2008].

The paper proceeds as follows. Section 1 begins with a heuristic treatment of the main result. It is designed to be accessible to both expert and non-expert readers. The formal treatment begins in Section 2. Section 3 proves the main result. Section 4 provides a converse—i.e., conditions under which a type structure that contains all hierarchies of beliefs passes the completeness test. Finally, Section 5 concludes with a discussion.

1 Heuristic Treatment

This section provides the motivation for and an explanation of the main result. It includes some important conceptual material not included in the formal treatment to come. (The expert may want to read this section side-by-side with Sections 2-3, skipping Section 1.1. The non-expert may want to read this section, followed by Sections 2 and 4.1.)

Consider a game situation where Nature can choose one of two actions—namely, Heads or Tails. Suppose there are two players, Ann and Bob, each of whom faces uncertainty about this choice of Nature. Ann (resp. Bob) may also face uncertainty about what Bob (resp. Ann) believes Nature chooses. And so on.

1.1 Type Structure

We want a model to represent the hierarchies of beliefs associated with such a situation. Following Harsanyi [15, 1967-1968], we use a type structure as such a model. Figure 1.1 depicts one possible type structure associated with this situation.

The type structure, written $\mathcal{T}$, has two components. First, for each player, there is a set of types. In our example, this is given by $T^a = \{t^a\}$ and $T^b = \{t^b\}$. Second, there are maps, viz. β^a and β^b, from the types of a given player to measures on the choices of Nature and the types of the other player. In our example, $\beta^a(t^a)$ assigns probability $\frac{1}{2}$ to (Heads, t^b) and probability $\frac{1}{2}$ to (Tails, t^b). Also, $\beta^b(t^b)$ assigns probability 1 to (Heads, t^a). At times, we will be loose and say that "type t^b assigns probability 1 to (Heads, t^a)" instead of referring to the measure $\beta^b(t^b)$.

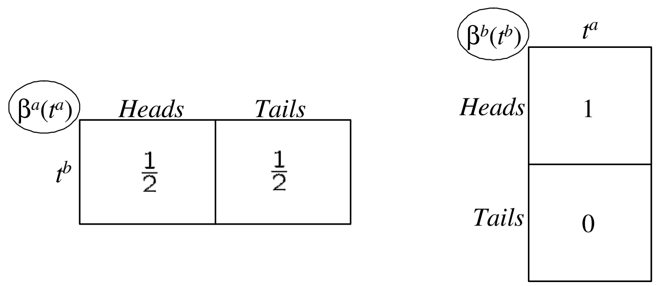

Figure 1.1

The type structure implicitly models hierarchies of beliefs. To see this, notice that type t^a of Ann assigns probability $\frac{1}{2}$: $\frac{1}{2}$ to Heads : Tails. This will be called type t^a's first-order belief. Similarly, type t^b has a first-order belief that assigns probability 1 to Heads. Using this, type t^a assigns probability $\frac{1}{2}$ to the event "Nature chooses Heads, and Bob assigns probability 1 to Nature choosing Heads" and probability $\frac{1}{2}$ to the event "Nature chooses Tails, and Bob assigns probability 1 to Nature choosing Heads." This is type t^a's second-order belief. And so on.

We point to a feature of the hierarchies of beliefs induced by types t^a and t^b. These hierarchies of beliefs are *coherent*: Type t^a's second-order belief agrees with her first-order belief, in the sense that both assign probability $\frac{1}{2}$: $\frac{1}{2}$ to Heads : Tails.

More generally, suppose type t^a is associated with a hierarchy of beliefs $\delta^a(t^a) = (\delta_1^a(t^a), \delta_2^a(t^a), \ldots)$, where δ_m^a is the "m^{th} order belief" on the "m^{th} space of uncertainty." (Section 2 gives a precise definition.) Then, the marginal of $\delta_{m+1}^a(t^a)$ on the "m^{th} space of uncertainty" is $\delta_m^a(t^a)$. Type structures induce hierarchies of beliefs that are coherent, assign probability 1 to coherent beliefs, etc. (See Lemma 3.4.) Following Brandenburger-Dekel [7, 1993], we will say that types induce hierarchies of beliefs that respect *coherency and common belief of coherency.*[1]

1.2 Large Type Structures

This paper asks: Which type structures represent all hierarchies of beliefs (that respect coherency and common belief of coherency)?

Return to the example in Figure 1.1. In that type structure, Ann has one type, and so there

[1] Brandenburger-Dekel [7, 1993] used the term "common knowledge of coherency." Given the modern usage of the terms "knowledge" and "belief," we stick to the latter.

is only one associated hierarchy of belief. For instance, there is no type of Ann, viz. u^a, with $\beta^a(u^a)(\{Heads\} \times T^b) = 1$. As such, there is no type of Ann with a first-order belief that assigns probability 1 to Heads.

In part, the example fails to represent all first-order beliefs (and, so, all hierarchies of beliefs) because Ann's map β^a is not surjective. Of course, in our example, it cannot be surjective since T^a contains only a single point. So, imagine that we expand T^a so that it is now the set of probability measures on $\{Heads, Tails\} \times T^b$ and take β^a to be the identity map. Then, β^a is surjective, and so the new set T^a is associated with all possible first-order beliefs of Ann.

Still, the new expanded structure may fail to induce all hierarchies of beliefs. While β^a is surjective, β^b is not. This causes two problems. First, as above, there are first-order beliefs of Bob that are not associated with Bob's unique type. But because such first-order beliefs of Bob are missing, there are also second-order beliefs of Ann that are missing from this structure. Given this, we will want to expand T^b too, so that the map β^b can be surjective—i.e., take T^b so that it is now the set of probability measures on (the new) $\{Heads, Tails\} \times T^a$. But when we do this, there are now more possible measures of Ann, and so we will need to, once again, expand T^a. And so on.

Such reasoning naturally leads to the notion of a *complete type structure* (Brandenburger [6, 2003]): a type structure where both β^a, β^b are surjective. Section 5c further discusses the (important) role of complete structures in the epistemic analyses of games.

One natural conjecture is that a complete type structure represents all possible hierarchies of beliefs. We will show that this is true, provided that the complete structure satisfies certain conditions. Put differently, under certain conditions, completeness is sufficient for terminality.

To state this result formally, we will need to introduce further terminology.

1.3 Terminal Type Structures

We begin by stating the condition that a type structure represents all hierarchies of beliefs (that respect coherency and common belief of coherency). There are two conditions:

(a) A type structure $\mathcal{T}$ is *finitely terminal* if, for each type u^a (resp. u^b) that occurs in some type structure and, for each m, there is some type t^a_m (resp. t^b_m) in $\mathcal{T}$ so that the hierarchies induced by u^a and t^a_m (resp. u^b and t^b_m) agree up to level m.

(b) A type structure $\mathcal{T}$ is *terminal* if, for each type u^a (resp. u^b) that occurs in some type structure, there is some type t^a (resp. t^b) in $\mathcal{T}$ so that u^a and t^a (resp. u^b and t^b) induce the same hierarchies of beliefs.

Condition *(a)* is new. Section 5c below argues that it plays an important role in certain epistemic analyses. In our setting, it is equivalent to the requirement that a type structure contains all finite order beliefs that respect coherency and common belief of coherency. (See Proposition B1 in

Appendix B.) This seems like a natural condition to consider, in a situation where players reason up to finite levels.[2]

Condition *(b)* has a long history in game theory, going back to Böge-Eisele [5, 1979], who also refer to this property as terminal. The use of the phrase terminal fits with Brandenburger-Keisler's [9, 2006] and Siniscalchi's [31, 2008] recent taxonomy. In our setting, condition *(b)* is equivalent to the requirement that a type structure contains all hierarchies of beliefs that respect coherency and common belief of coherency. (See Result 2.1 in Section 2 and Proposition B1 in Appendix B.) Section 5a discusses the relationship between the terminal property and the so-called "universal type structure." Of course, condition *(b)* implies condition *(a)*.

The terminality property can be thought of as reflecting the following requirement. There exists a hierarchy morphism—i.e., a map that preserves hierarchies of beliefs—from every type structure to the terminal structure. In the terminology of category theory, an object, viz. $\mathcal{O}$, is terminal if, for every object $\mathcal{N}$, there is a unique morphism from $\mathcal{N}$ to $\mathcal{O}$. In our setting, the hierarchy morphisms will be unique if and only if any two types in the terminal structure induce distinct hierarchies of beliefs. Put differently, up to an equivalence class, the hierarchy morphisms are unique. (Here, for a given structure, we put two types in the same equivalence class if they induce the same hierarchies of beliefs.) Indeed, this is the basis for which Böge-Eisele [5, 1979] use the phrase terminal.

Sometimes the phrase terminal is reserved for a type structure $\mathcal{T}$ that satisfies the following condition: For every type structure $\mathcal{T}_*$, there exists a unique type morphism from $\mathcal{T}_*$ to $\mathcal{T}$. (See, for instance, Meier [25, 2006].) This is the alternate embedding test we mentioned in the introduction. Section 1.6 provides a formal definition of a type morphism and records conditions under which this notion of terminality is equivalent to condition *(b)*.

1.4 Main Result

We will show:

> **Theorem:** Fix a complete type structure $\mathcal{T}$ with metrizable type sets T^a, T^b and measurable maps β^a, β^b.
>
> (i) If the type sets T^a, T^b are analytic subsets of a Polish space, then the structure is finitely terminal.
>
> (ii) If the type sets T^a, T^b are compact metrizable and β^a, β^b are continuous, then the structure is terminal.

To understand the need for these assumptions, suppose Ann and Bob face the same underlying set of uncertainty, viz. X, and this set is finite. In our example, X is the set {Heads, Tails}, and so finite. The main text treats a more general case, in which X is Polish.

[2] Many game-theoretic papers study the implications of finite-order reasoning. The idea goes back, at least, to Geanakoplos-Polemarchakis [14, 1982]. See Lipman [21, 2003] and Weinstein-Yildiz [33, 2007] for more recent incarnations. (Much thanks to a referee for pointing out the connection.)

Begin with part (i) of the result. Fix a complete structure. Also fix a hierarchy of beliefs for Ann (that respects coherency and common belief thereof), viz. $h = (\mu_1, \mu_2, \ldots)$. We want to argue that, for each m, there is some type t_m^a in the complete structure so that the hierarchies induced by t_m^a agree with $h = (\mu_1, \mu_2, \ldots)$ up to level m.

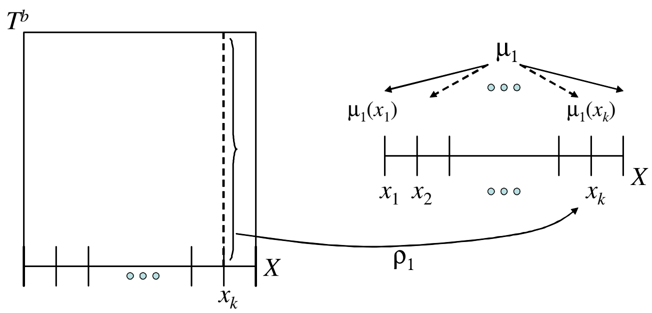

Figure 1.2

First, suppose $m = 1$. A first-order belief, viz. μ_1, is a measure on X. This is the right-hand panel of Figure 1.2. It suffices to show that there is some measure ν_1 on $X \times T^b$, i.e., on the left-hand panel of Figure 1.2, so that μ is the marginal of ν on X. If so, by completeness, there is some type t_1^a with $\beta^a(t_1^a) = \nu_1$, and this type must be associated with the first-order belief μ_1.

To construct the measure ν_1: Refer to Figure 1.2, and consider the projection map $\rho_1 : X \times T^b \to X$. Construct $\nu_1((\rho_1)^{-1}(x_k)) = \nu_1(\{x_k\} \times T^b) = \mu_1(x_k)$, for each point $x_k \in X$. The measure ν_1 is well-defined.[3] So we have established the result for $m = 1$.

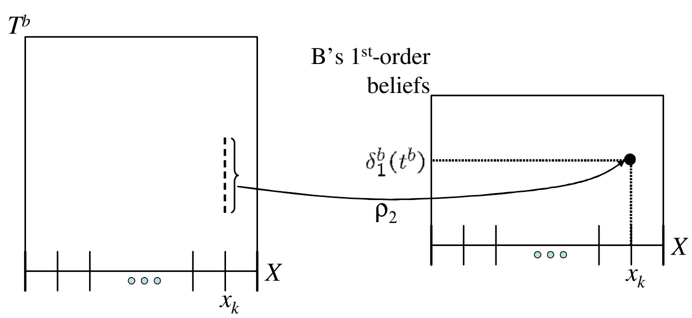

Figure 1.3

Next, suppose $m = 2$. Here, we are concerned with Ann's second-order space of uncertainty. This is X cross the set of Bob's first-order beliefs, i.e., the right-hand panel of Figure 1.3. We can write a map ρ_2 from $X \times T^b$ to Ann's second-order space of uncertainty. Specifically, ρ_2 maps (x_k, t^b) to $(x_k, \delta_1^b(t^b))$, i.e., to x_k and the first-order belief induced by t^b. This mapping is illustrated in Figure 1.3.

[3] Here is the idea. Fix a type $t^b \in T^b$. Also, for each k, fix the Dirac measure centered on (x_k, t^b). By rescaling and combining these measures, we can construct a measure ν_1 on $X \times T^b$ with the desired property.

Now, refer to Figure 1.4. We want to find a measure ν_2 on $X \times T^b$ associated with the same measure on "Ann's second-order space of uncertainty" as μ_2. (If so, by completeness, there is a type whose second-order beliefs coincide with μ_2.) To do so, follow a construction similar to the case of $m = 1$. Now, for any event E in "Ann's second-order space of uncertainty," set $\nu_2((\rho_2)^{-1}(E)) = \mu_2(E)$. Doing this gives a probability to every event in the sub-σ-algebra generated by ρ_2. That is, as constructed, ν_2 is a probability measure on a sub-σ-algebra, not a probability measure on all the Borel sets. So, the question is: Can we extend ν_2 to a probability measure on all the Borel sets?

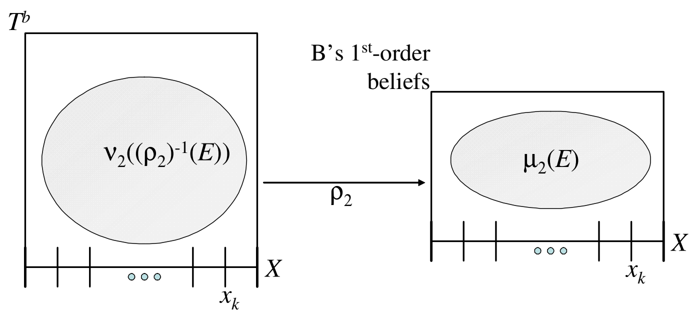

Figure 1.4

This question is an instance of a general mathematical question: Fix a topological space Ω and a sub-σ-algebra of the Borel σ-algebra. Given a measure on the sub-σ-algebra, can the measure be extended to a measure on all the Borel sets? In general, the answer is no. (See Aldaz-Render [1, 2000] for examples.) But when Ω is an analytic subset of a Polish space (and, in particular, Polish spaces are analytic subsets of Polish spaces), any probability measure on the sub-σ-algebra can be extended to a probability measure on the Borel sets. This result is due to Lubin [23, 1974]. (See, also, Landers-Rogge [20, 1974] and Yershov [35, 1974]. Aldaz-Render [1, 2000] provide other topological conditions.)

With this, we complete the analysis for $m = 2$. An induction argument establishes the result more generally.

Now, turn to part (ii) of the result and continue to fix Ann's hierarchy of beliefs, viz. $h = (\mu_1, \mu_2, \ldots)$. We begin by taking the structure to be finitely terminal, as given by part (i) of the result. As such, for each m, there is some type t^a whose induced hierarchy agrees with h up to level m. Let $T[h|m]$ be the set of all such types and note that, by finite terminality, each $T[h|m]$ is non-empty. Also note that $T[h|1], T[h|2], \ldots$ is a decreasing sequence of sets. A type's induced hierarchy agrees with h if and only if it is contained in $\bigcap_m T[h|m]$. So, this raises the question: Is (each) $\bigcap_m T[h|m]$ non-empty?

Of course, in general, a decreasing sequence of non-empty sets may have an empty intersection. But, in the particular case where this decreasing sequence of sets is closed and in a compact space, the intersection is non-empty. This is why the stated result assumes that the set T^a is compact (so that the sets lie in a compact space) and that the map β^a is continuous (to insure that the sets are closed).

This is the idea of the proof. To show that a compact, continuous, complete structure is terminal, we first show that it is finitely terminal and then use compactness and continuity to show that it is also terminal. Section 5b discusses this point further.

1.5 Applying the Test

The general treatment begins with an underlying set of uncertainty, viz. X, that is Polish. One might expect that, in this case, any complete structure is terminal and, so, finitely terminal. But, at least using the method of proof above, this seems to be true only under certain structural conditions—for terminal structures, there are compactness and continuity requirements, and, for finitely terminal structures, there is an analytic requirement. (These requirements are crucial to the proofs here. There are other methods to prove part (ii) that, in somewhat more subtle ways, also use compactness and continuity.)

What are the implications of this result, with an eye toward applying the test? There are known complete structures that fail the compactness requirement. (Such structures exist even when X is compact metrizable. See Section 5b.) These structures will not pass the test suggested by the above theorem. Are these structures nonetheless terminal? This is related to a broader question: Are the results here true, absent compactness and continuity? Or is there an example of a complete structure that is not terminal?

Of course, the mere fact that a complete structure fails the test does not imply that it fails to be terminal. Consider the case in which X is Polish, but not compact. Here, there is no compact and continuous structure. (If β^a is continuous and T^a is compact, then the set of measures on $X \times T^b$ must also be compact. But this would require that X be compact.) That said, there is a complete structure that is terminal. An example is the construction in Brandenburger-Dekel [7, 1993]. There, the maps β^a, β^b are continuous, but T^a, T^b are only Polish and, specifically, not compact. So, there are certainly situations in which the test cannot be applied, despite the fact that the structure may be both complete and terminal. (See, also, Section 4.) Put differently, the completeness test—as formalized by completeness, compactness, and continuity—may deliver false negatives.

1.6 False Negatives

In light of the fact that the completeness test may deliver false negatives, let us ask: Is there an alternate test available? We already mentioned that the literature proposes a second test, based on the idea of embedding type structures (via type morphisms).

To see the idea, fix two type structures, viz. $\mathcal{T} = (X; T^a, T^b; \beta^a, \beta^b)$ and $\mathcal{T}_* = (X; T^a_*, T^b_*; \beta^a_*, \beta^b_*)$, and write $\mathrm{id} : X \to X$ for the identity map. Consider maps $\tau^a : T^a \to T^a_*$ and $\tau^b : T^b \to T^b_*$. We will call $\left(\tau^a, \tau^b\right)$ a type morphism if, for each $t^a \in T^a$, $\beta^a_* \left(\tau^a \left(t^a\right)\right)$ is the image measure of $\beta^a \left(t^a\right)$

under $\mathrm{id} \times \tau^b$, and likewise with a and b reversed. More loosely, (τ^a, τ^b) is a type morphism if τ^a and τ^b preserve the belief maps β^a and β^b. With this, we say that a type structure $\mathcal{T}_*$ passes the embedding test if, for any type structure $\mathcal{T}$, there is a type morphism from $\mathcal{T}$ to $\mathcal{T}_*$. (Usually, papers add on the requirement that the type morphism be unique.)

But, the embedding test may also deliver false negatives. To see this, let us record the relationship between type and hierarchy morphisms. (Refer back to Section 1.3 for the idea of a hierarchy morphism.)

Result: Fix two type structures $\mathcal{T}$ and $\mathcal{T}_*$ and maps $\tau^a : T^a \to T^a_*$ and $\tau^b : T^b \to T^b_*$.

(i) If (τ^a, τ^b) is a type morphism, then τ^a and τ^b are hierarchy morphisms.

(ii) If (τ^a, τ^b) is a hierarchy morphism, T^a_*, T^b_* are Polish, and no two types in $\mathcal{T}_*$ induce the same hierarchies of beliefs, then (τ^a, τ^b) is a type morphism.

Part (i) corresponds to Proposition 5.1 in Heifetz-Samet [17, 1998]. Part (ii) is a special case of Corollary 4.1 in Friedenberg-Meier [13, 2008]. (The result in Friedenberg-Meier [13, 2008] is somewhat more general.)

In light of these results: if a structure passes the embedding test, then it is terminal. But, a terminal structure need not pass the embedding test. (In particular, Section 3 in Friedenberg-Meier [13, 2008] provides two counter examples.) So, much like the completeness test, the embedding test may deliver false negatives.

To the extent that the main theorem here and the main theorem in Friedenberg-Meier [13, 2008] are both tight, it would appear that the completeness and embedding tests are simply distinct criteria. That said, we do not know of a general treatment of the relationship between the two tests. This would be desirable.

2 Type Structures

This section begins the formal treatment. Throughout, we restrict attention to metrizable spaces. We endow a subset of a metrizable space with the induced topology so that it is again metrizable. Likewise, we endow the (countable) product of metrizable spaces with the product topology so that it is again metrizable. A metrizable space, viz. Ω, induces a measurable space $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(\Omega)$ is the Borel σ-algebra on Ω. Note that if Ω_1 and Ω_2 are metrizable and $\Omega_1 \in \mathcal{B}(\Omega_2)$, then $\mathcal{B}(\Omega_1) \subseteq \mathcal{B}(\Omega_2)$. (See Theorem 1.9 in Parthasarathy [29, 2005].) We will make use of this fact repeatedly.

Write $\mathcal{P}(\Omega)$ for the set of (Borel) probability measures on $(\Omega, \mathcal{B}(\Omega))$. Endow $\mathcal{P}(\Omega)$ with the topology of weak convergence so that it is again a metrizable space. Of course, if Ω is Polish (resp. compact metrizable), so is $\mathcal{P}(\Omega)$ (when endowed with the topology of weak convergence). Given a map $f : \Omega \to \Phi$, write $\underline{f} : \mathcal{P}(\Omega) \to \mathcal{P}(\Phi)$ where $\underline{f}(\mu)$ is the image measure of f under μ.

For notational simplicity, we restrict attention to two-player game situations with players a and b. (The results readily follow for game situations with three or more players.) Let X^a (resp. X^b) be a Polish space, interpreted as player a's (primitive) set of uncertainty. We will write c for an arbitrary player amongst a, b. Given an arbitrary player c, we will write d for the other player.

Definition 2.1 *An (X^a, X^b)-**based type structure** is given by $\mathcal{T} = (X^a, X^b; T^a, T^b; \beta^a, \beta^b)$, where T^a and T^b are each metrizable and*

$$\beta^a : T^a \to \mathcal{P}\left(X^a \times T^b\right) \qquad \beta^b : T^b \to \mathcal{P}\left(X^b \times T^a\right)$$

*are Borel measurable. The sets T^a and T^b are **type sets**. A **state** is some quadruple $\left(x^a, t^a, x^b, t^b\right) \in X^a \times T^a \times X^b \times T^b$.*

Definition 2.1 defines a type structure where T^a, T^b are metrizable spaces.[4] We will also be interested in cases where T^a, T^b are more specific topological spaces. Now we mention some spaces that will be of interest. Say that a set is an analytic subset of a Polish space if it is the continuous image of a Polish space.

Definition 2.2 *Call the type structure $\mathcal{T}$ **analytic** (resp. **compact**) if T^a and T^b are each analytic subsets of Polish spaces (resp. compact).*

Definition 2.3 *Call the type structure $\mathcal{T}$ **continuous** if β^a and β^b are continuous.*

We are also interested in a particular "large" type structure:

Definition 2.4 *An (X^a, X^b)-based type structure, viz. $\mathcal{T} = (X^a, X^b; T^a, T^b; \beta^a, \beta^b)$, is **complete** if β^a, β^b are onto.*

We will want to relate a complete type structure to structures that contain all possible hierarchies of beliefs. For that, we will need to specify how types induce hierarchies of beliefs. (This treatment follows Battigalli-Siniscalchi [3, 1999].)

Begin by defining $Z_1^c = X^c$ and $Z_{m+1}^c = Z_m^c \times \mathcal{P}\left(Z_m^d\right)$. For each $m \geq 1$, $Z_{m+1}^c = Z_1^c \times \prod_{n=1}^m \mathcal{P}\left(Z_n^d\right)$. (See Lemma S1 in the Online Supplement.[5]) Note that each of the sets Z_m^c is Polish since X^c is Polish.

Now, define maps $\rho_{m+1}^c : X^c \times T^d \to Z_{m+1}^c$ so that

$$\begin{aligned} \rho_1^c\left(x^c, t^d\right) &= x^c \\ \rho_{m+1}^c\left(x^c, t^d\right) &= \left(\rho_m^c\left(x^c, t^d\right), \underline{\rho}_m^d(\beta^d(t^d))\right). \end{aligned}$$

Note, for each m, $\rho_{m+1}^c\left(x^c, t^d\right) = (x^c, \delta_1^d(t^d), \ldots, \delta_m^d(t^d))$. (See Lemma S1.) Define $\delta_m^c : T^c \to \mathcal{P}\left(Z_m^c\right)$ so that $\delta_m^c = \underline{\rho}_m^c \circ \beta^c$. So, δ_m^c maps each type into its associated m^{th}-order beliefs. Note,

[4] Metrizability is sufficient to guarantee that types induce hierarchies of beliefs.

[5] Results labeled "S" can be found in the Online Supplement. (See www.public.asu.edu/~afrieden.)

the maps ρ^c_m and δ^c_m are measurable. They are continuous when the structure is continuous. (See Lemma S4.)

Define $\delta^c : T^c \to \prod_{m=1}^{\infty} \mathcal{P}(Z^c_m)$ so that $\delta^c(t^c) = (\delta^c_1(t^c), \delta^c_2(t^c), ...)$. Then, $\delta^c(t^c)$ is the hierarchy of beliefs induced by t^c. The map δ^c is measurable and continuous when the structure is continuous.

We can now state the condition that a type structure "contains" all hierarchies of beliefs. In the definitions below, fix (X^a, X^b)-based type structures, viz. $\mathcal{T} = (X^a, X^b; T^a, T^b; \beta^a, \beta^b)$ and $\mathcal{T}_* = (X^a, X^b; T^a_*, T^b_*; \beta^a_*, \beta^b_*)$.

Definition 2.5 *An (X^a, X^b)-based type structure $\mathcal{T}$ is **finitely terminal** if, for each (X^a, X^b)-based type structure $\mathcal{T}_*$, each type $t^c_* \in T^c_*$, and each m, there is a type $t^c \in T^c$ with*

$$(\delta^c_1(t^c), \ldots, \delta^c_m(t^c)) = (\delta_{*}{}^c_1(t^c_*), \ldots, \delta_{*}{}^c_m(t^c_*)).$$

(Note that, in Definition 2.5, t^c can depend on both t^c_* and m.)

Definition 2.6 *An (X^a, X^b)-based type structure $\mathcal{T}$ is **terminal** if, for each (X^a, X^b)-based type structure $\mathcal{T}_*$ and each type $t^c_* \in T^c_*$, there is a type $t^c \in T^c$ with $\delta^c(t^c) = \delta^c_*(t^c_*)$.*

Definition 2.5 says that the structure $\mathcal{T}$ is finitely terminal if, for each type t^c_* that occurs in some structure and each m, there is a type t^c_m in $\mathcal{T}$ whose hierarchy agrees with t^c_* up to level m. Definition 2.6 says that the structure $\mathcal{T}$ is terminal if, for each type t^c_* that occurs in some structure, there is a type t^c in $\mathcal{T}$ with the same hierarchy of beliefs as t^c_*.

Proposition B1 in Appendix B establishes the following related result.[6]

Result 2.1 *A type structure $\mathcal{T}$ is terminal if and only if, for any hierarchy of beliefs consistent with "coherency and common belief of coherency," there is a type in $\mathcal{T}$ associated with that hierarchy of beliefs.*

A similar result holds for finitely terminal structures. The proof makes use of the fact that X^a and X^b are Polish. See Section 5a.

3 Main Theorem

This section is devoted to showing the main result.

Theorem 3.1 *Fix a complete type structure $\mathcal{T} = (X^a, X^b; T^a, T^b; \beta^a, \beta^b)$.*

(i) If $\mathcal{T}$ is analytic, then $\mathcal{T}$ is finitely terminal.

(ii) If $\mathcal{T}$ is compact and continuous, then $\mathcal{T}$ is terminal.

[6] The result is stated formally in Appendix B. At this point in the text, the result can only be stated informally.

Here is the idea of the proof: We begin by defining the notion of an $\left(X^a, X^b\right)$-based belief structure. This is a structure of hierarchies consistent with "coherency and common belief of coherency." (We mentioned this in Section 1.1.) It can be viewed as a topological subset of the set of hierarchies of beliefs, viz. $\prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^a\right) \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^b\right)$. (See Lemma 3.3.) We first show that any type structure can be mapped into this belief structure—intuitively, any type structure induces beliefs that satisfy coherency and common belief of coherency. (Lemma 3.4 gives a finite version of the result, and Lemma 3.7 states the result for the entire hierarchy.) This step is standard. Next, we show that, for any analytic complete type structure and any element of the belief structure, viz. $h = (\mu_1, \mu_2, \ldots)$, there is a type in the complete structure whose hierarchy of beliefs coincides with h up to level m. (See Proposition 3.1.) If, in addition, the structure is compact and continuous, there is a type of the complete structure whose hierarchy of beliefs coincide with h. (See Proposition 3.2.) These last two steps appear to be novel to the literature.

The reader will notice a close parallel between the proofs of Lemma 3.4 and Proposition 3.1 (resp. Lemma 3.7 and Proposition 3.2). This is meant to highlight the role of the various topological assumptions in the proofs of Propositions 3.1-3.2. We think it provides a more complete understanding of the issues that arise.

3.1 Belief Structures

This section inductively constructs the set of all hierarchies of belief that are coherent, assign probability 1 to coherent hierarchies of beliefs, etc.

Fix measurable spaces $\Omega_1, \ldots, \Omega_m$ and a sequence of measures $(\mu_1, \ldots, \mu_m) \in \prod_{k=1}^{m} \mathcal{P}(\prod_{n=1}^{k} \Omega_n)$. Write $\operatorname{marg}_{\Omega_n} \mu_k$ for the marginal of μ_k on Ω_n, where $n \leq k$. If $\operatorname{marg}_{\Omega_n} \mu_k = \mu_n$, for all $1 \leq k \leq m$ and all $n \leq k$, say the sequence $(\mu_1, \ldots, \mu_m)$ is *coherent.*

Inductively define sets X_m^c, Y_m^c, H_m^c as follows: Set $X_1^c = X^c$ and $Y_1^c = H_1^c = \mathcal{P}\left(X_1^c\right)$. Assuming the sets are defined for m, set $X_{m+1}^c = X_1^c \times H_m^d$, $Y_{m+1}^c = Y_m^c \times \mathcal{P}\left(X_{m+1}^c\right)$, and

$$H_{m+1}^c = \left[H_m^c \times \mathcal{P}\left(X_{m+1}^c\right)\right] \cap \left\{\left(\mu_1, \ldots, \mu_{m+1}\right) \in Y_{m+1}^c : \operatorname{marg}_{X_m^c} \mu_{m+1} = \mu_m\right\}.$$

For H_{m+1}^c to be well-defined, we need that, for any measure $\mu_{m+1} \in \mathcal{P}\left(X_{m+1}^c\right)$, $\operatorname{marg}_{X_m^c} \mu_{m+1}$ is indeed well-defined. But this is so since

$$X_{m+1}^c \subseteq X_1^c \times H_{m-1}^d \times \mathcal{P}\left(X_m^d\right) = X_m^c \times \mathcal{P}\left(X_m^d\right).$$

Notice that, if $\left(\mu_1, \ldots, \mu_{m+1}\right) \in H_{m+1}^c$, then $(\mu_1, \ldots, \mu_n) \in H_n^c$ for all $n \leq m$. So, the hierarchy of beliefs up to level $m+1$ is coherent.

Several facts: First, a standard inductive argument gives that, for each m, $X_{m+1}^c \subseteq X_1^c \times \prod_{n=1}^{m} \mathcal{P}\left(X_n^d\right)$ and $H_m^c \subseteq Y_m^c$. Second, for each m, the sets X_m^c, Y_m^c are Polish and H_m^c is a closed subset of a Polish space (and, so, Polish). See Lemma A2 in Appendix A. When X^a, X^b are both compact, each of X_m^c, Y_m^c, and H_m^c is compact metrizable, for all m. (See Remark A1 in

Appendix A.)

Definition 3.1 *The set $H_m^a \times H_m^b$ is called the* $\mathbf{m}^{th}$***-level*** $\left(X^a, X^b\right)$***-based belief structure****. Further, set*

$$\begin{aligned} H^a &= \left\{(\mu_1, \mu_2, \ldots) \in \textstyle\prod_{m=1}^{\infty} \mathcal{P}\left(X_m^a\right) : \textit{for each } m,\ (\mu_1, \ldots, \mu_m) \in H_m^a\right\}, \\ H^b &= \left\{(\mu_1, \mu_2, \ldots) \in \textstyle\prod_{m=1}^{\infty} \mathcal{P}\left(X_m^b\right) : \textit{for each } m,\ (\mu_1, \ldots, \mu_m) \in H_m^b\right\}, \end{aligned}$$

and call $H^a \times H^b$ the $\left(X^a, X^b\right)$***-based belief structure****.*

The $\left(X^a, X^b\right)$-based belief structure consists of all hierarchies of beliefs associated with "coherency and common belief of coherency." The construction of the $\left(X^a, X^b\right)$-based belief structure is akin to the canonical constructions in Mertens-Zamir [27, 1985] and Brandenburger-Dekel [7, 1993]. The $\left(X^a, X^b\right)$-based belief structure is a closed subset of a Polish space and, so, Polish. Similarly, when X^a, X^b are both compact metrizable, it is compact metrizable. (See Lemma A3 in Appendix A.)

The $\left(X^a, X^b\right)$-based belief structure can be viewed as a topological subset of the set of hierarchies of beliefs. And likewise for the m^{th}-level based belief structure. To state this formally, begin by defining maps $\zeta_m^c : X_m^c \to Z_m^c$ and $\eta_m^c : H_m^c \to \prod_{n=1}^{m} \mathcal{P}\left(Z_n^c\right)$ inductively: Let ζ_1^c, η_1^c be the identity maps. Assuming ζ_m^c, η_m^c are defined, take ζ_{m+1}^c and η_{m+1}^c so that

$$\begin{aligned} \zeta_{m+1}^c\left(x_1^c, h_m^d\right) &= \left(\zeta_1^c\left(x_1^c\right), \eta_m^d\left(h_m^d\right)\right) \\ \eta_{m+1}^c\left(h_m^c, \mu_{m+1}^c\right) &= \left(\eta_m^c\left(h_m^c\right), \underline{\zeta}_{m+1}^c(\mu_{m+1}^c)\right). \end{aligned}$$

A standard induction argument gives the following.

Lemma 3.1 *For each $m \geq 1$,*

$$\begin{aligned} \zeta_{m+1}^c\left(x_1^c, \mu_1^d, \ldots, \mu_{m-1}^d, \mu_m^d\right) &= \left(\zeta_m^c\left(x_1^c, \mu_1^d, \ldots, \mu_{m-1}^d\right), \underline{\zeta}_m^d\left(\mu_m^d\right)\right) \\ \eta_m^c\left(\mu_1^c, \ldots, \mu_m^c\right) &= \left(\mu_1^c, \underline{\zeta}_2^c\left(\mu_2^c\right), \ldots, \underline{\zeta}_m^c\left(\mu_m^c\right)\right). \end{aligned}$$

We will repeatedly make use of this fact.

Lemma 3.2 *For each m, the maps $\zeta_m^c : X_m^c \to Z_m^c$ and $\eta_m^c : H_m^c \to \prod_{n=1}^{m} \mathcal{P}\left(Z_n^c\right)$ are embeddings.*

Now define $\eta^c : H^c \to \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^c\right)$ such that, for each $(\mu_1, \mu_2, \ldots) \in H^c$,

$$\eta^c\left(\mu_1, \mu_2, \ldots\right) = \left(\mu_1, \underline{\zeta}_2^c\left(\mu_2^c\right), \ldots\right).$$

Notice that if $\left(\mu_1, \underline{\zeta}_2^c\left(\mu_2^c\right), \ldots, \underline{\zeta}_m^c\left(\mu_m^c\right)\right)$ is an initial segment of $\eta^c\left(\mu_1, \mu_2, \ldots\right)$ of length m, then

$$\eta_m^c\left(\mu_1, \ldots, \mu_m\right) = \left(\mu_1, \underline{\zeta}_2^c\left(\mu_2^c\right), \ldots, \underline{\zeta}_m^c\left(\mu_m^c\right)\right).$$

(This uses the above characterization of η_m^c.) We will make use of this fact repeatedly. Now:

Lemma 3.3 *The map $\eta^c : H^c \to \prod_{m=1}^{\infty} \mathcal{P}(Z_m^c)$ is an embedding.*

3.2 From Type to Belief Structures

Section 2 showed that types induce hierarchies of beliefs, via the mapping δ^c. In this section, we will see that (given any type structure) $\delta^c(T^c)$ can be embedded in the belief structure, via the inverse of η^c. That is, $\delta^c(T^c) \subseteq \eta^c(H^c)$. Begin with a finite version of this result:

Lemma 3.4 *Fix a type structure $\mathcal{T}$. For each m:*

(i) for any $(x^c, t^d) \in X^c \times T^d$, $\rho_m^c(x^c, t^d) \subseteq \zeta_m^c(X_m^c)$;

(ii) for any $t^c \in T^c$, $\delta_m^c(t^c)(\zeta_m^c(X_m^c)) = 1$; and

(iii) for any $t^c \in T^c$, $(\delta_1^c(t^c), \ldots, \delta_m^c(t^c)) \in \eta_m^c(H_m^c)$.

Part (iii) is the result of interest. It says that the m^{th}-level hierarchies of beliefs induced by some type is topologically equivalent to some m^{th}-level hierarchy of beliefs in the belief structure.

To establish this, we will need some preliminary results. First, types induce coherent hierarchies of beliefs.

Lemma 3.5 *For all m and all $n \leq m$, $\operatorname{marg}_{Z_n^c} \delta_{m+1}^c(t^c) = \delta_n^c(t^c)$.*

Next:

Lemma 3.6 *Let Ω, Φ be Polish spaces and let $f : \Omega \to \Phi$ be a continuous injective map. If $\nu \in \mathcal{P}(\Phi)$ with $\nu(f(\Omega)) = 1$, then there exists some $\mu \in \mathcal{P}(\Omega)$ such that ν is the image measure of μ under f.*

Lemma 3.6 can be given an independent proof by the Lusin-Souslin Theorem [19, 1995; Theorem 15.1]. But since it will be a specific case of Lemma 3.9 below, the proof is omitted.

Now we can prove the stated result:

Proof of Lemma 3.4. By induction on m.

$m = 1$: Part (i) is immediate from the fact that $X_1^c = Z_1^c$ and ζ_1^c is the identity map. Part (ii) follows, then, from the fact that $\zeta_1^c(X_1^c) = Z_1^c$. Part (iii) follows again from the fact that $\mathcal{P}(X_1^c) = \mathcal{P}(Z_1^c)$ and η_1^c is the identity map.

$m \geq 2$: Assume that the result holds for m. For Part (i), fix $(x^c, t^d) \in X^c \times T^d$ and recall,

$$\rho_{m+1}^c(x^c, t^d) = (x^c, \delta_1^d(t^d), \ldots, \delta_m^d(t^d)).$$

So, using Part (iii) of the induction hypothesis, there is some $h^d_m \in H^d_m$ with

$$\rho^c_{m+1}\left(x^c, t^d\right) = \left(x^c, \eta^d_m\left(h^d_m\right)\right).$$

By construction, then,

$$\zeta^c_{m+1}\left(x^c, h^d_m\right) = \left(\zeta^c_1\left(x^c\right), \eta^d_m\left(h^d_m\right)\right) = \rho^c_{m+1}\left(x^c, t^d\right),$$

as required.

It follows from Part (i) applied to $(m+1)$ that

$$X^c \times T^d \subseteq \left(\rho^c_{m+1}\right)^{-1}\left(\zeta^c_{m+1}\left(X^c_{m+1}\right)\right) \subseteq X^c \times T^d.$$

So

$$\begin{aligned}
\delta^c_{m+1}\left(t^c\right)\left(\zeta^c_{m+1}\left(X^c_{m+1}\right)\right) &= \beta^c\left(t^c\right)\left((\rho^c_{m+1})^{-1}(\zeta^c_{m+1}(X^c_{m+1}))\right) \\
&= \beta^c\left(t^c\right)\left(X^c \times T^d\right) \\
&= 1,
\end{aligned}$$

establishing Part (ii).

Finally, by the induction hypothesis, there is some $h^c_m \in H^c_m$ with $\left(\delta^c_1\left(t^c\right), \ldots, \delta^c_m\left(t^c\right)\right) = \eta^c_m\left(h^c_m\right)$. Write $h^c_m = \left(\mu^c_1, \ldots, \mu^c_m\right)$. It suffices to show that there is some $\mu^c_{m+1} \in \mathcal{P}\left(X^c_{m+1}\right)$ with $\underline{\zeta}^c_{m+1}\left(\mu^c_{m+1}\right) = \delta^c_{m+1}\left(t^c\right)$ and $\operatorname{marg}_{X^c_m} \mu^c_{m+1} = \mu^c_m$.

First, we construct a measure $\mu^c_{m+1} \in \mathcal{P}\left(X^c_{m+1}\right)$ with $\underline{\zeta}^c_{m+1}\left(\mu^c_{m+1}\right) = \delta^c_{m+1}\left(t^c\right)$. To see this, note that X^c_{m+1} and Z^c_{m+1} are Polish. Also, Part (ii) gives that $\delta^c_{m+1}\left(t^c\right)\left(\zeta^c_{m+1}\left(X^c_{m+1}\right)\right) = 1$. By Lemma 3.6, there is a measure, viz. μ^c_{m+1}, with $\underline{\zeta}^c_{m+1}\left(\mu^c_{m+1}\right) = \delta^c_{m+1}\left(t^c\right)$.

Now, we show that, for this measure μ^c_{m+1}, $\operatorname{marg}_{X^c_m} \mu^c_{m+1} = \mu^c_m$. Fix an event E in X^c_m and note that $G = \left(E \times \mathcal{P}\left(X^d_m\right)\right) \cap X^c_{m+1}$ is Borel. By the Lusin-Souslin Theorem [19, 1995; Theorem 15.1], $\zeta^c_{m+1}(G)$ is Borel. (This uses the fact that X^c_{m+1}, Z^c_{m+1} are Polish and ζ^c_{m+1} is injective continuous.) Then,

$$\delta^c_{m+1}\left(t^c\right)\left(\zeta^c_{m+1}(G)\right) = \mu^c_{m+1}\left(\left(E \times \mathcal{P}\left(X^d_m\right)\right) \cap X^c_{m+1}\right),$$

since ζ^c_{m+1} is injective. Also, by Lemma 3.1,

$$\zeta^c_{m+1}(G) = \left[\zeta^c_m(E) \times \mathcal{P}\left(Z^d_m\right)\right] \cap \zeta^c_{m+1}\left(X^c_{m+1}\right).$$

So, by Part (ii) applied to $m+1$,

$$\delta^c_{m+1}\left(t^c\right)\left(\zeta^c_{m+1}(G)\right) = \delta^c_{m+1}\left(t^c\right)\left(\zeta^c_m(E) \times \mathcal{P}\left(Z^d_m\right)\right).$$

Putting the above together with Lemma 3.5, we have

$$\begin{aligned}\underline{\zeta}^c_{m+1}\left(\mu^c_{m+1}\right)(G) &= \delta^c_{m+1}(t^c)\left(\zeta^c_m(E)\times\mathcal{P}\left(Z^d_m\right)\right)\\ &= \delta^c_m(t^c)\left(\zeta^c_m(E)\right)\\ &= \underline{\zeta}^c_m\left(\mu^c_m\right)(E),\end{aligned}$$

where the last line follows from Lemma 3.1. ■

Lemma 3.7 *For each type $t^c \in T^c$, $\delta^c(t^c) \in \eta^c(H^c)$.*

Proof. Fix a type $t^c \in T^c$. Inductively construct a sequence $(\mu_1, \mu_2, \ldots) \in H^c$ as follows: Choose $\mu_1 \in H^c_1$ so that $\delta^c_1(t^c) = \underline{\zeta}^c_1(\mu^c_1) = \mu_1$. Assuming that we have found measures $(\mu_1, \ldots, \mu_m) \in H^c_m$ with

$$\left(\delta^c_1(t^c), \ldots, \delta^c_m(t^c)\right) = \left(\mu_1, \underline{\zeta}^c_2(\mu^c_2), \ldots, \underline{\zeta}^c_m(\mu^c_m)\right),$$

Lemma 3.4(iii) says that we can find a measure $\mu_{m+1} \in \mathcal{P}\left(X^c_{m+1}\right)$ with $\left(\mu_1, \ldots, \mu_m, \mu_{m+1}\right) \in H^c_{m+1}$ and

$$\left(\delta^c_1(t^c), \delta^c_2(t^c), \ldots, \delta^c_m(t^c), \delta^c_{m+1}(t^c)\right) = \left(\mu_1, \underline{\zeta}^c_2(\mu^c_2), \ldots, \underline{\zeta}^c_m(\mu^c_m), \underline{\zeta}^c_{m+1}\left(\mu^c_{m+1}\right)\right).$$

So, inductively, we identify a sequence $(\mu_1, \mu_2, \ldots) \in H^c$ with $\delta^c_m(t^c) = \underline{\zeta}^c_m(\mu^c_m)$ for each m. This establishes the result. ■

3.3 From Belief to Complete Structures

This section relates complete type structures to belief structures. In particular, we begin by establishing conditions under which the m^{th}-level belief structure is topologically equivalent to a complete type structure. We then go on to show conditions under which the belief structure is topologically equivalent to a complete structure. As discussed in Section 1.3, these results appear to require additional topological structure of the type sets T^a and T^b.

We begin by stating one of the main results of this section:

Proposition 3.1 *Fix a complete analytic type structure. For each m:*

(i) $\rho^c_m\left(X^c \times T^d\right) = \zeta^c_m(X^c_m)$;

(ii) for each $\eta^c_m(\mu_1, \ldots, \mu_m) = (\varpi_1, \ldots, \varpi_m)$, $\varpi_m\left(\zeta^c_m(X^c_m)\right) = 1$; *and*

(iii) for each $(\mu_1, \ldots, \mu_m) \in H^c_m$, *there exists a type* t^c *such that* $\eta^c_m(\mu_1, \ldots, \mu_m) = \left(\delta^c_1(t^c), \ldots, \delta^c_m(t^c)\right)$.

Part (iii) says that, for an analytic complete structure $\mathcal{T}$, each element of the m^{th}-level belief structure is topologically equivalent to some m^{th}-level hierarchy of beliefs associated with a type t^c in $\mathcal{T}$.

Showing this requires two preliminary results. First, coherency is preserved under the map η^c_m.

Lemma 3.8 *Fix* $(\mu_1, \ldots, \mu_{m+1}) \in H^c_{m+1}$. *Then, for each* $n \leq m$, $\operatorname{marg}_{Z^c_n} \underline{\zeta}^c_{m+1}(\mu_{m+1}) = \underline{\zeta}^c_n(\mu_n)$.

Next, as discussed in Section 1.3, given a measure on μ_m on Z^c_m, we will want to guarantee that we can find a measure ν_m on $X^c \times T^d$ so that μ_m is the image measure of ν_m under ρ^c_m. This is given by the following result due to Lubin [23, 1974]. (Again, see, also, Landers-Rogge [20, 1974] and Yershov [35, 1974].)

Lemma 3.9 (Lubin) *Let* Ω, Φ *be analytic subsets of a Polish space and let* $f : \Omega \to \Phi$ *be a measurable map. If* $\nu \in \mathcal{P}(\Phi)$ *with* $\nu(f(\Omega)) = 1$, *then there exists some* $\mu \in \mathcal{P}(\Omega)$ *such that* ν *is the image measure of* μ *under* f.

We now turn to proving Proposition 3.1.

Proof of Proposition 3.1. By induction on m.

$m = 1$: By Lemma 3.4(i), $\rho^c_1(X^c \times T^d) \subseteq \zeta^c_1(X^c_1)$. Conversely, fix some $x^c \in X^c = X^c_1$ and note that, for any $t^d \in T^d$, $\rho^c_1(x^c, t^d) = x^c = \zeta^c_1(x^c)$, establishing Part (i). Part (ii) is immediate from the fact that $\zeta^c_1(X^c_1) = Z^c_1$. For Part (iii), fix some $\mu_1 \in H^c_1 = \mathcal{P}(X^c)$. By Lemma 3.9, there exists a measure $\nu_1 \in \mathcal{P}(X^c \times T^d)$ such that μ_1 is the image measure of ν_1 under ρ^c_1. By completeness, there is a type t^c with $\beta^c(t^c) = \nu_1$. Since $\delta^c_1(t^c)$ is the image measure of $\beta^c(t^c)$ under ρ^c_1, the result is established.

$m \geq 2$: Assume that the result holds for m. We will show that it also holds for $m + 1$.

Begin with Part (i). By Lemma 3.4(i), $\rho^c_{m+1}(X^c \times T^d) \subseteq \zeta^c_{m+1}(X^c_{m+1})$. Fix some $\zeta^c_{m+1}(x^c_1, \mu^d_1, \ldots, \mu^d_m)$. By Part (iii) of the induction hypothesis, there exists a type t^d such that

$$\eta^d_m(\mu^d_1, \ldots, \mu^d_m) = (\delta^d_1(t^d), \ldots, \delta^d_m(t^d)).$$

So, $\rho^c_{m+1}(x^c_1, t^d) = (x^c_1, \delta^d_1(t^d), \ldots, \delta^d_m(t^d))$ as required.

Now, turn to Part (ii). Recall from Lemma 3.1 that $\varpi_{m+1} = \underline{\zeta}^c_{m+1}(\mu_{m+1})$. Using the injectivity of ζ^c_{m+1},

$$\begin{aligned} \varpi_{m+1}(\zeta^c_{m+1}(X^c_{m+1})) &= \mu_{m+1}((\zeta^c_{m+1})^{-1}(\zeta^c_{m+1}(X^c_{m+1}))) \\ &= \mu_{m+1}(X^c_{m+1}) \\ &= 1 \end{aligned}$$

as required.

For Part (iii), fix some $(\mu_1, \ldots, \mu_{m+1}) \in H^c_{m+1}$ with $\eta^c_{m+1}(\mu_1, \ldots, \mu_{m+1}) = (\varpi_1, \ldots, \varpi_{m+1})$.

First, notice that there exists a measure $\nu_{m+1} \in \mathcal{P}(X^c \times T^d)$ such that ϖ_{m+1} is the image measure of ν_{m+1} under ρ^c_{m+1}: This follows from Lemma 3.9 and the fact that $\varpi_{m+1}(\rho^c_{m+1}(X^c \times T^d)) = 1$ (Parts (i)-(ii) of this Lemma).

Now, notice that, for each $n \leq m$, $\mathrm{marg}_{Z_n^c} \varpi_{m+1}$ is the image measure of ν_{m+1} under ρ_n^c. To see this, fix an event E_n in Z_n^c. Then,

$$\begin{aligned}
\mathrm{marg}_{Z_n^c} \varpi_{m+1}(E_n) &= \varpi_{m+1}\left(E_n \times \textstyle\prod_{k=n}^m \mathcal{P}\left(Z_k^d\right)\right) \\
&= \nu_{m+1}((\rho_{m+1}^c)^{-1}(E_n \times \textstyle\prod_{k=n}^m \mathcal{P}(Z_k^d))) \\
&= \nu_{m+1}((\rho_n^c)^{-1}(E_n)),
\end{aligned}$$

as required.

By completeness, there exists a type t^c such that $\beta^c(t^c) = \nu_{m+1}$. Since $\delta_{m+1}^c = \underline{\rho}_{m+1}^c \circ \beta^c$, we also have that $\delta_{m+1}^c(t^c) = \varpi_{m+1}$. Moreover, by Lemma 3.8, $\varpi_n = \mathrm{marg}_{Z_n^c} \varpi_{m+1}$ for all $n \leq m$. Using this and the fact that, for all $n \leq m$, $\delta_n^c(t^c) = \underline{\rho}_n^c(\beta^c(t^c)) = \underline{\rho}_n^c(\nu_{m+1})$, we also have that $\delta_n^c(t^c) = \varpi_n$ for all $n \leq m$. This establishes the result. ■

Now:

Proposition 3.2 *Fix a complete compact continuous type structure. For each $(\mu_1, \mu_2, \ldots) \in H^c$, there exists a type $t^c \in T^c$ such that $\eta^c(\mu_1, \mu_2, \ldots) = (\delta_1^c(t^c), \delta_2^c(t^c), \ldots)$.*

Proof. Fix a type structure. Also fix some $h^c = (\mu_1, \mu_2, \ldots) \in H^c$. Write $h^c|m$ for the initial segment of h^c of length m, i.e., $h^c|m = (\mu_1, \ldots, \mu_m)$. Let $T^c[h^c|m]$ be the set of types $t^c \in T^c$ with $\eta_m^c(h^c|m) = (\delta_1^c(t^c), \ldots, \delta_m^c(t^c))$, i.e., the set of types whose hierarchy induces an initial segment of $h^c|m$.

By Proposition 3.1, the sets $T^c[h^c|m]$ are each non-empty. These sets are decreasing. Moreover, they are closed. To see this last fact, note that each set $\{h^c|m\}$ is closed. Since η_m^c is an embedding, it follows that $\eta_m^c(\{h^c|m\})$ is closed. Now, using the fact that β^c is continuous, each δ_n^c is continuous (for all $n \leq m$). (This is Lemma S4 in the Online Supplement.) Then, the map $t^c \to (\delta_1^c(t^c), \ldots, \delta_m^c(t^c))$ is continuous (see Munkres [28, 1975; Theorem 8.5]), establishing that $T^c[h^c|m]$ is closed.

We have established that $T^c[h^c|1], T^c[h^c|2], \ldots$ is a decreasing sequence of non-empty closed sets in a compact space. It follows that $\bigcap_m T^c[h^c|m] \neq \emptyset$, establishing the result. ■

3.4 Proof of Main Theorem

Now, Theorem 3.1 is a Corollary of the results of Sections 3.2-3.3:

Proof of Theorem 3.1. Part (i) follows from Lemma 3.7 and Proposition 3.1. Part (ii) follows from Lemma 3.7 and Proposition 3.2. ■

4 When a Terminal Structure is Complete

If a structure is terminal, must it pass the completeness test? We have already seen that the answer is no. Recall, from Section 1.3, that a terminal structure need not be compact and continuous.

As such, a structure may be complete and terminal and still fail the completeness test (at least, as encompassed by Theorem 3.1).

In this section, we will see that a terminal structure may fail the completeness test for a more basic reason: A terminal structure need not be complete. We will go on to establish conditions under which a terminal structure is complete.

4.1 Are Terminal Structures Complete?

This section asks about a converse to Theorem 3.1: Given a terminal structure $\mathcal{T}$, is $\mathcal{T}$ also complete? The answer is no (even if $\mathcal{T}$ is compact and continuous).

To see this, consider a game situation with players Ann and Bob. They each face uncertainty about a choice of Nature, but now this uncertainty must be degenerate. Specifically, Nature has only one choice, i.e., $X^a = X^b =$\{Heads\}. It follows that there is only one possible hierarchy of beliefs for each player, i.e., Ann (resp. Bob) assigns probability 1 to Heads, assigns probability 1 to "Heads and 'Bob (resp. Ann) assigns probability 1 to Heads,'" and so on.

Now consider the following type structure: Let $T^a = \{t^a, u^a\}$ and $T^b = \{t^b\}$. Also, let $\beta^a(t^a)(\text{Heads}, t^b) = 1$, $\beta^a(u^a)(\text{Heads}, t^b) = 1$, and $\beta^b(t^b)(\text{Heads}, t^a) = 1$.

This type structure does induce the one hierarchy of beliefs, as it must. So it is a terminal structure. But it is not complete since β^b is not surjective. For instance, there is no type of Bob that assigns probability 1 to (Heads, u^a).

A key feature of this example is that the structure has two types of Ann that induce the same hierarchy of beliefs, namely t^a and u^a. As such, the structure may contain all hierarchies of beliefs for Bob, even though no type of Bob considers u^a possible. One conjecture is that, when no two types induce the same hierarchy of beliefs, a terminal structure must be complete. We will see that a bit more is required—the type sets T^a, T^b must satisfy an additional condition. This requirement will be quite related to the reason we require that any pair of types induce distinct hierarchies of beliefs.

4.2 When Terminal Structures are Complete

In discussing the example of Section 4.1, we argued that the difficulty lies in the fact that the terminal structure is redundant, i.e., has two types associated with the same hierarchy of beliefs. As such, we will be interested in non-redundant type structures:

Definition 4.1 (Mertens-Zamir [27, 1985]) *An (X^a, X^b)-based type structure is **non-redundant** if the associated maps δ^a, δ^b are injective.*

We need one more condition. For this, recall, a measurable space is standard Borel if it is isomorphic to a Borel subset of a Polish space.

Definition 4.2 *Call the type structure* $\mathcal{T} = (X^a, X^b; T^a, T^b; \beta^a, \beta^b)$ ***standard Borel*** *if* $(T^a, \mathcal{B}(T^a))$ *and* $\left(T^b, \mathcal{B}\left(T^b\right)\right)$ *are standard Borel.*

The result is:

Proposition 4.1 *Fix a non-redundant standard Borel type structure* $\mathcal{T}$. *If* $\mathcal{T}$ *is terminal, then it is complete.*

The proof can be found in Appendix C. Here, we give the idea. Doing so will illuminate the role of a standard Borel type structure.

Suppose $\mathcal{T}$ is a structure under which δ^b is injective (so no two types of Bob induce the same hierarchy of beliefs). Further, suppose that the structure is not complete. That is, there is a measure μ^a on $X^a \times T^b$ such that each $\beta^a(t^a) \neq \mu^a$. We want to argue that the hierarchy of beliefs induced by μ^a cannot be associated with any type of Ann. If so, then $\mathcal{T}$ is not terminal (Proposition B1).

To see this, define a map

$$\rho^a : X^a \times T^b \to X^a \times \textstyle\prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^b\right)$$

so that $\rho^a\left(x^a, t^b\right) = (x^a, \delta^b(t^b))$. This induces the map

$$\underline{\rho}^a : \mathcal{P}\left(X^a \times T^b\right) \to \mathcal{P}\left(X^a \times \textstyle\prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^b\right)\right)$$

so that $\underline{\rho}^a(\mu^a)$ is the image measure of $\mu^a \in \mathcal{P}\left(X^a \times T^b\right)$ under ρ^a. Suppose the map $\underline{\rho}^a$ is injective: Then, for each type t^a in $\mathcal{T}$, $\beta^a(t^a)$ and μ^a induce different beliefs on X^a and Bob's hierarchies of beliefs. (See Lemma C5.) It will then follow that each $\beta^a(t^a)$ induces a different hierarchy of beliefs than μ^a, as required.

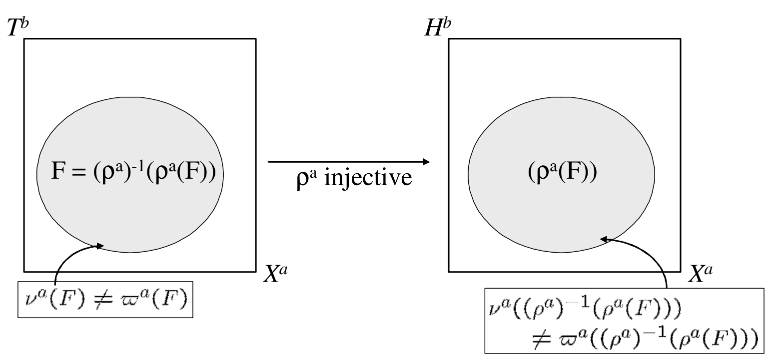

Figure 4.1

But is the map $\underline{\rho}^a$ injective? Since no two types of Bob induce the same hierarchy of beliefs, δ^b–and so ρ^a–must be injective. Now, refer to Figure 4.1. Fix two distinct measures ν^a and

ϖ^a on $X^a \times T^b$. There is some event, viz. F, with $\nu^a(F) \neq \varpi^a(F)$. Since ρ^a is injective, $F = (\rho^a)^{-1}(\rho^a(F))$, so that

$$\nu^a((\rho^a)^{-1}(\rho^a(F))) \neq \varpi^a((\rho^a)^{-1}(\rho^a(F))).$$

Supposing that $\rho^a(F)$ is an event in $X^a \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^b\right)$, it follows that the image measure of ν^a under ρ^a must assign to the event $\rho^a(F)$ and different probability than the image measure of ϖ^a under ρ^a assigns to this event—i.e., that $\underline{\rho}^a(\nu^a) \neq \underline{\rho}^a(\varpi^a)$, as required.

To show that $\rho^a(F)$ is an event: Notice that the map ρ^a is injective and measurable, mapping a metrizable space into a Polish space. Using a result due to Purves [30], $\rho^a(F)$ is measurable provided T^a is standard Borel. This is why we require that $\mathcal{T}$ be standard Borel.

5 Discussion

This section discusses some technical and conceptual aspects of the paper.

a. Large Type Structures: The literature has a number of notions of a "large" type structure. (See Siniscalchi [31, 2008] for a recent survey.) We have established a relationship between two such type structures: terminal type structures and complete type structures. The former goes back to Böge-Eisele [5, 1979]. The latter is more recent, due to Brandenburger [6, 2003].

There is another notion of a "large type structure," namely type structures satisfying a universality property. Here is roughly the idea.

Suppose the analyst constructs a belief structure that contains all hierarchies that are coherent, that assign probability 1 to coherent hierarchies, etc. The first level of these hierarchies reflects players' beliefs about the underlying state of uncertainty. The second level reflects players' beliefs both about the underlying state of uncertainty and other players' beliefs, and so on. Does this reasoning ever stop? That is, does there exist an ordinal α so that the hierarchies of beliefs up to level α determine all subsequent beliefs? A structure is said to satisfy the universality property if such an ordinal exists.

In the context of ordinary probabilities, Armbruster-Böge [2, 1979], Mertens-Zamir [27, 1985], Brandenburger-Dekel [7, 1993] and Heifetz [16, 1993] each address the question of whether there exists a structure satisfying the universality property. To do so, they each construct a "canonical type structure." Specifically, they begin with a belief structure that contains all hierarchies of beliefs that are coherent, assign probability 1 to coherent beliefs, etc., i.e., akin to $H^a \times H^b$ here. They then show that all beliefs up to the first infinite ordinal determine the belief at the first infinite ordinal. With this, they take the type sets to be the hierarchies associated with $H^a \times H^b$ and (using universality) show that this naturally gives bijective continuous maps β^a, β^b. (Notice, a canonical type structure is specified by writing down hierarchies of beliefs.)

Are terminal structures essentially the same as the canonical type structure? The answer appears to be no. Fix a terminal structure $\mathcal{T}$, and ask: For each type in $\mathcal{T}$, does there exist a type in the canonical structure associated with the same hierarchy of beliefs? Heifetz-Samet [18, 1999] show that there is some hierarchy of beliefs (consistent with coherency and common belief of coherency), where beliefs up to the first infinite ordinal do not determine a belief at the first infinite ordinal. As such, this hierarchy is not associated with any type in the canonical type structure.[7]

But in the specific case where X^a and X^b are Polish, the answer is yes—for every type in a terminal structure $\mathcal{T}$, there exists a type in the canonical structure associated with the same hierarchy of beliefs. (See Appendix A.)

What is the relationship between complete structures and the canonical type structure? The canonical type structure is complete. But there is a complete structure that is distinct from the canonical structure. The first such example is in Mertens-Sorin-Zamir [26, page 193, Remark 2; 1993]. The following example is similar in spirit.[8] Let $X^a = X^b = \{x\}$. Then, the canonical construction of a large type structure involves one type for each player, each of which assigns probability 1 to x, to "x and the other player assigning probability 1 to x," etc. But there is another complete structure. In particular, take T^a, T^b to be the Cantor space and note that there exists a continuous surjective map β^a (resp. β^b) from T^a (resp. T^b) to $\mathcal{P}\left(\{x\} \times T^b\right)$ (resp. $\mathcal{P}\left(\{x\} \times T^a\right)$). (See Kechris [19, Theorem 4.18; 1995].) This gives a complete structure that is not canonical. In particular, all types in $T^a = \{0,1\}^{\mathbb{N}}$ induce the same hierarchy of beliefs, while the canonical construction is non-redundant.

This example might suggest that any complete structure that differs from the canonical structure differs by redundancy–i.e., it may have too many types. But Sections 1.3 and 3 suggest that this may not be the case–that a complete type structure may not "contain" all hierarchies of beliefs in $H^a \times H^b$. That said, at this time, such an example is lacking.

b. Main Result and Method of Proof: The main result says that if a complete structure is analytic (resp. compact and continuous), then it is finitely terminal (resp. terminal). We do not know if there is a complete structure that is not analytic. But there is a complete structure that is not compact, even if X^a, X^b are compact. Here is an example, kindly provided by H. Jerome Keisler. Take $T^a = T^b$ to be copies of the Baire space. Then, there exists a continuous surjection from T^a (resp. T^b) onto $\mathcal{P}\left(X^a \times T^b\right)$ (resp. $\mathcal{P}\left(X^b \times T^a\right)$). (See Kechris [19, Theorem 7.9; 1995].) This gives a non-compact complete structure. An open question is whether this structure is terminal.

An important comment about the method of proof is in order. A compact metrizable space is analytic. As such, imposing the compactness requirement on a structure implies that the structure is finitely terminal. In fact, the proof of Theorem 3.1(ii) uses the fact that a compact and continuous

[7] When X^a and X^b are not Polish, some care is needed in defining a terminal structure. Implicit in this discussion is one suggestion. Of course, others may be of interest too.

[8] The example follows Proposition 7.2 in Brandenburger-Friedenberg-Keisler [8, 2007] and Theorem 10.4 in Brandenburger-Keisler [9, 2006].

complete structure is finitely terminal. Put differently, a natural first step to showing that a structure is terminal is showing that it is finitely terminal, and this is exactly how the proof works. We do not know of another proof that avoids this step.

c. Application to Game Theory: The main result here sheds light on the epistemic conditions for the iteratively undominated (IU) strategies. (Here, IU means maximal simultaneous deletion.) To see this, consider the following characterization theorem, which, at the time of this writing, is the state of the art in the field.

> **A Characterization Theorem:** Fix a game and an associated $\left(S^b, S^a\right)$-based type structure.
>
> (i) If the structure is complete, then the set of strategies consistent with rationality and m^{th}-order belief of rationality (RmBR) is (exactly) the $(m+1)$-undominated strategies.
>
> (ii) If the structure is complete, compact, and continuous, then the set of strategies consistent with rationality and common belief of rationality (RCBR) is (exactly) the IU strategy set.

It is well understood that parts (i)-(ii) may fail absent the completeness condition. Likewise, compactness and continuity are important requirements for part (ii). (See Friedenberg [12, 2008] on both points.)

The compactness and continuity requirements are integral components of the epistemic conditions. Yet, at the conceptual level, it is unclear what they mean in terms of players' reasoning. The main theorem here suggests an answer: Completeness, compactness, and continuity imply that the structure is terminal. Thus, part (ii) says that, for particular terminal structures, the strategies consistent with RCBR are exactly the IU strategies. This raises the question: Do the RCBR strategies correspond (exactly) to the IU strategies for any terminal structure, or simply for terminal structures that are also complete, compact, and continuous? The answer is that it holds more generally, and, indeed, we can make a corresponding point for the $(m+1)$-undominated strategies. Specifically:

> **New Characterization Theorem:** Fix a game and an associated $\left(S^b, S^a\right)$-based type structure.
>
> (i) If the structure is finitely terminal, then the set of strategies consistent with RmBR is (exactly) the $(m+1)$-undominated strategies.
>
> (ii) If the structure is terminal, then the set of strategies consistent with RCBR is (exactly) the IU strategy set.

The proof can be found in Friedenberg [12, 2008]. Note, part (i) says that, for any finitely terminal structure, the $(m+1)$-undominated strategies characterize RmBR. This appears to be a distinct requirement from part (i) of the first characterization theorem: For that result, we need not require that the complete structure be analytic. (See the discussion in [12, 2008].) So, if there exists a complete structure that is not finitely terminal, the two conditions are different. Of course, whether such a structure exists remains an open question.

d. Beyond Ordinary Probabilities: In the introduction, we mentioned that the completeness condition has played an instrumental role in certain game theoretic analyses—namely, Battigalli-Siniscalchi's [4, 2002] analysis of extensive-form rationalizability and Brandenburger-Friedenberg-Keisler's [8, 2008] analysis of iterated admissibility. Each of these analyses move away from ordinary probabilities. Battigalli-Siniscalchi [4, 2002] makes use of complete type structures, where types are mapped into conditional probability systems (CPS's). Brandenburger-Friedenberg-Keisler [8, 2008] makes use of complete type structures, where types are mapped into lexicographic probability systems (LPS's).

Is there an analog to Theorem 3.1, for CPS-based and LPS-based type structures? We do not know. In the specific case of LPS's, there are some non-trivial issues in formalizing the idea of a terminal structure—so we cannot state any conjectures, at this time. In this case of CPS's, there is a conjecture that there is indeed an analog—at least for the particular CPS's used in Battigalli-Siniscalchi's [4, 2002] game theoretic analysis.

Recall a key step in the proof of Theorem 3.1. We fix an m^{th}-level belief and push it back to a belief on $X^c \times T^d$. But, doing so makes use of the fact that both the $X^c \times T^d$ and the m^{th}-order space of uncertainty are analytic. (The fact that the m^{th}-order space of uncertainty is analytic, follows from the fact that it is Polish. See Lemma A2.) Is the m^{th}-order space of uncertainty analytic, for the case of CPS's?

Let $(\Omega, \mathcal{B}(\Omega))$ be a Polish and write $\mathcal{C}(\Omega, \mathcal{E})$ for the set of CPS's on $(\Omega, \mathcal{B}(\Omega))$ whose conditioning events are $\mathcal{E} \subseteq \mathcal{B}(\Omega)$. In the case where $\mathcal{E}$ is at most countable and each event in $\mathcal{E}$ is clopen, $\mathcal{C}(\Omega, \mathcal{E})$ is Polish. (See Lemma 2.1 in Battigalli-Siniscalchi [3, 1999].) So, if we start with a primitive set of uncertainty that is Polish and we look at a fixed set of clopen conditioning events, the set of first-order beliefs is Polish. Indeed, in this case, the m^{th}-order space of uncertainty will be Polish and so analytic. (See Sections 2.3-2.4 in Battigalli-Siniscalchi [3, 1999].) In the game theoretic analysis of Battigalli-Siniscalchi [4, 2002], the conditioning events correspond to information sets in the tree. So, there $\mathcal{E}$ satisfies the countability and clopenness conditions, as required.

e. Related Literature: Di Tillio [10, 2008] constructs a universal structure for "hierarchies of preferences." In the course of proving this result, he establishes a preference-based analog to Theorem 3.1(ii) here. Specifically, Proposition 4 in Di Tillio [10, 2008] shows that any "simple" complete preference structure satisfies a preference-based notion of terminality. (A set is simple if its events form a countable base for a compact Hausdorff topology.) Di Tillio (private communication) also conjectures that there is a preference-based analog to Proposition 4.1.

There are a number of important differences between asking this question in the context of "hierarchies of preferences" vs. "hierarchies of beliefs." First, in the context of "hierarchies of preferences," the canonical construction of a type structure is simple when the underlying space of uncertainty is finite. (See Di Tillio [10, Proposition 1; 2008].) In that context, it is natural to ask if a simple complete structure is terminal. By contrast, in the context of "hierarchies of beliefs" the canonical construction of a type structure is not simple (even when the underlying space of uncertainty is finite). Here, compactness and continuity replace the role of simple structures–a simple structure is compact and continuous.

For the second difference, return to Section 1.3 and recall a key mathematical step: Given a measurable map $f : \Omega \to \Phi$ and a measure $\nu \in \mathcal{P}(\Phi)$ with $\nu(f(\Omega)) = 1$, does there exist a measure $\mu \in \mathcal{P}(\Omega)$ such that ν is the image measure of μ under f? In the context of preference structures, this question becomes: Given a measurable surjective map $f : \Omega \to \Phi$ and a preference relation on Φ, does there exist a preference relation on Ω that induces the preference relation on Φ? Constructing such a preference relation is easier when there are fewer axioms on the set of preference relations. In particular, because Di Tillio [10, 2008] does not require that preference relations be complete, it is easier to find the required preference relation on Ω.[9] By analogy: It would be easier to find a measure μ that induces ν, if we allowed μ to be only finitely additive. In fact, for this, Ω need not be analytic–only measurable. (See Łos'-Marczewski [22, 1949] and Meier [25, 2006].)

Appendix A Proofs for Section 3.1

This appendix proves some properties of the (X^a, X^b)-based belief structure (resp. the m^{th}-level (X^a, X^b)-based belief structure) that were mentioned in the text. It goes on to use belief structures to show that a finitely terminal structure (resp. terminal structure) contains all finite order beliefs (resp. all hierarchies of beliefs) that respect coherency and common belief of coherency.

Lemma A1 *Let I be (at most) a countable collection of integers $1, 2, \ldots$. For each integer $m \in I$, let Ω_m, Φ_m be Polish spaces where Φ_m is measurable in $\prod_{n \leq m} \Omega_n$. Then,*

$$\{(\mu_1, \mu_2, \ldots) \in \textstyle\prod_{m \in I} \mathcal{P}(\Phi_m) : \text{ for all } m \text{ and all } n \leq m, \ \operatorname{marg}_{\Phi_n} \mu_m = \mu_n\}$$

is closed in $\prod_{m \in I} \mathcal{P}(\Phi_m)$.

Proof. Fix a sequence $(\mu_1^k, \mu_2^k, \ldots)$ in $\prod_{m \in I} \mathcal{P}(\Phi_m)$ with $\operatorname{marg}_{\Phi_n} \mu_m^k = \mu_n^k$ for all m and all $n \leq m$. Fix, also, $(\mu_1, \mu_2, \ldots)$ with $(\mu_1^k, \mu_2^k, \ldots) \to (\mu_1, \mu_2, \ldots)$. We'd like to show that $\operatorname{marg}_{\Phi_n} \mu_m = \mu_n$ for all m and all $n \leq m$. To show this, fix some m and note that it suffices to show $\operatorname{marg}_{\Phi_n} \mu_m^k \to$

[9] The beauty of Di Tillio [10, 2008] is that he can construct a universal structure while imposing few axioms on preferences. Of course, his question is quite different from the one in this paper. Here, the question is about countably additive probability measures.

$\text{marg}_{\Phi_n}\mu_m$ whenever $n \leq m$. If so, then $\mu_n^k \to \text{marg}_{\Phi_n}\mu_m$ whenever $n \leq m$. Since each $\mu_n^k \to \mu_n$, it follows that $\mu_n = \text{marg}_{\Phi_n}\mu_m$ for all $n \leq m$, as required.

Now, we turn to show that $\text{marg}_{\Phi_n}\mu_m^k \to \text{marg}_{\Phi_n}\mu_m$. By the Portmanteau Theorem, it suffices to show that

$$\liminf \text{marg}_{\Phi_n}\mu_m^k(U) \geq \text{marg}_{\Phi_n}\mu_m(U),$$

for any open set U in Φ_n. Fix such an open set and note that $\left(U \times \prod_{l=n+1}^m \Omega_l\right) \cap \Phi_m$ is open in the relative topology. Using the fact that $\mu_m^k \to \mu_m$ and the Portmanteau Theorem,

$$\liminf \mu_m^k\left(\left(U \times \textstyle\prod_{l=n+1}^m \Omega_l\right) \cap \Phi_m\right) \geq \mu_m\left(\left(U \times \textstyle\prod_{l=n+1}^m \Omega_l\right) \cap \Phi_m\right).$$

Also, for each k,

$$\text{marg}_{\Phi_n}\mu_m^k(U) = \mu_m^k\left(\left(U \times \textstyle\prod_{l=n+1}^m \Omega_l\right) \cap \Phi_m\right)$$

and

$$\text{marg}_{\Phi_n}\mu_m(U) = \mu_m\left(\left(U \times \textstyle\prod_{l=n+1}^m \Omega_l\right) \cap \Phi_m\right).$$

Now, applying the Portmanteau Theorem again, the result is established. ■

Lemma A2 *For each c and each m,*

(i) X_m^c, Y_m^c are Polish; and

(ii) H_m^c is a closed subset of Y_m^c and so Polish.

Proof. By induction on m. For $m = 1$, the result is immediate. Assume that the result holds for m. Then, it is immediate that X_{m+1}^c and Y_{m+1}^c are Polish. By the induction hypothesis, H_m^c is a closed subset of Y_m^c, so that $H_m^c \times \mathcal{P}\left(X_{m+1}^c\right)$ is a closed subset of Y_{m+1}^c. As such, to show that H_{m+1}^c is a closed subset of Y_{m+1}^c, it suffices to show that

$$\left\{\left(\mu_1, \ldots, \mu_{m+1}\right) \in Y_{m+1}^c : \text{marg}_{X_m^c}\mu_{m+1} = \mu_m\right\}$$

is closed. This follows from Lemma A1. ■

Remark A1 *The proof of Lemma A2 can readily be amended to cover the following case. Suppose X^a, X^b are compact metrizable. Then, for each m, X_m^c, Y_m^c, and H_m^c are compact metrizable.*

Lemma A3 *The set H^c is a closed subset of a Polish space and so Polish. If X^a, X^b are compact metrizable, then H^c is a closed subset of a compact metrizable space and so compact metrizable.*

Proof. First, note that H^c is a subset of $\prod_m \mathcal{P}(X_m^c)$, a Polish space. To see that H^c is a closed subset of $\prod_m \mathcal{P}(X_m^c)$, fix a sequence $\left(\mu_1^k, \mu_2^k, \ldots\right) \in H^c$ with $\left(\mu_1^k, \mu_2^k, \ldots\right) \to (\mu_1, \mu_2, \ldots)$. If $(\mu_1, \mu_2, \ldots)$ is not contained in H^c then there exists some initial segment of $(\mu_1, \mu_2, \ldots)$, namely $(\mu_1, \ldots, \mu_m)$, where $(\mu_1, \ldots, \mu_m) \notin H_m^c$. But then, by Lemma A2, $\left(\mu_1^k, \ldots, \mu_m^k\right)$ does not converge to $(\mu_1, \ldots, \mu_m)$, a contradiction. Finally, if X^a, X^b are compact metrizable, then $\prod_m \mathcal{P}(X_m^c)$ is compact metrizable, establishing that H^c is compact metrizable. ■

Appendix B Terminal Structures

Here, we show that a structure is finitely terminal (resp. terminal) if and only if it contains all hierarchies of beliefs available in the m^{th}-level belief structure (resp. belief structure). Specifically:

Proposition B1 *Fix an (X^a, X^b)-based type structure $\mathcal{T} = (X^a, X^b; T^a, T^b; \beta^a, \beta^b)$.*

(i) The structure $\mathcal{T}$ is finitely terminal if and only if, for each m and each $(\mu_1^c, \ldots, \mu_m^c) \in H_m^c$, there is a type $t^c \in T^c$ with $(\delta_1^c(t^c), \ldots, \delta_m^c(t^c)) = \eta_m^c(\mu_1^c, \ldots, \mu_m^c)$.

(ii) The structure $\mathcal{T}$ is terminal if and only if, for each each $(\mu_1^c, \mu_2^c, \ldots) \in H^c$, there is a type $t^c \in T^c$ with $\delta^c(t^c) = \eta^c(\mu_1^c, \mu_2^c, \ldots)$.

To show this, we provide a construction of a particular (X^a, X^b)-based type structure. We build off the (X^a, X^b)-based belief structure $H^a \times H^b$. The construction here is very similar to the constructions in Mertens-Zamir [27, 1985] and Brandenburger-Dekel [7, 1993]. (See, also, Battigalli-Siniscalchi [3, 1999].)

Begin by defining type sets so that $T^c = \eta^c(H^c) \subseteq \prod_{m=1}^{\infty} \mathcal{P}(Z_m^c)$. Recall, we endow T^a with the relative topology. Since η^c is an embedding and H^c is closed, $\eta^c(H^c)$ is a closed subset of a Polish space and so Polish. Lemma B1, below, will establish that each element of $T^c = \eta^c(H^c)$ is coherent. By the Kolmogorov Extension Theorem, there is an injective map

$$\beta^c : T^c \to \mathcal{P}\left(Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)\right)$$

such that $\beta^c(t^c)$ extends t^c to $Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)$, i.e., if $t^c = \eta^c(\mu_1^c, \mu_2^c, \ldots) = (\varpi_1^c, \varpi_2^c, \ldots)$ then

$$\mathrm{marg}_{Z_1^c} \beta^c(t^c) = \varpi_1^c$$

and

$$\mathrm{marg}_{Z_1^c \times \prod_{n=1}^{m} \mathcal{P}(Z_n^d)} \beta^c(t^c) = \varpi_{m+1}^c.$$

Let $\widetilde{T}^c$ be the set of coherent sequences in $\prod_{m=1}^{\infty} \mathcal{P}(Z_m^c)$.

Lemma B1 *The set $\eta^c(H^c)$ is a closed subset of $\widetilde{T}^c$.*

Proof. By Lemmata 3.1-3.8, for each m, $\eta_m^c(\mu_1^c, \ldots, \mu_m^c)$ is coherent. So, $\eta^c(H^c) \subseteq \widetilde{T}^c$. Given this, to show $\eta^c(H^c)$ is a closed subset of $\widetilde{T}^c$, it suffices to show that $\widetilde{T}^c$ closed in $\prod_{m=1}^{\infty} \mathcal{P}(Z_m^c)$ and $\eta^c(H^c)$ closed in $\prod_{m=1}^{\infty} \mathcal{P}(Z_m^c)$. If so, then Theorem 6.2 of Chapter 2 in Munkres [28, 1975] says that $\eta^c(H^c)$ is closed in $\widetilde{T}^c$. But, the former follows from Lemma A1 and the latter from Lemma 3.3. ■

Lemma B2 *The map β^c is closed and continuous.*

Proof. First, we show that β^c is a closed map. To see this, it will be convenient to introduce another map $\widetilde{\beta}^c$. Let $\widetilde{T}^c$ be the set of coherent sequences in $\prod_{m=1}^{\infty} \mathcal{P}(Z_m^c)$. By the Kolmogorov Extension Theorem, we can choose

$$\widetilde{\beta}^c : \widetilde{T}^c \to \mathcal{P}\left(Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)\right)$$

such that it is injective and $\widetilde{\beta}^c(\varpi_1, \varpi_2, \ldots)$ extends $(\varpi_1, \varpi_2, \ldots)$ to $\mathcal{P}\left(Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)\right)$. Lemma 1 in Brandenburger-Dekel [7, 1992] says that $\widetilde{\beta}^c$ is a homeomorphism.

In fact, since the extension is uniquely determined and $\eta^c(H^c) \subseteq \widetilde{T}^c$ (Lemma B1), β^c is $\widetilde{\beta}^c$ restricted to $\eta^c(H^c)$. Moreover, $\eta^c(H^c)$ is a closed subset of $\widetilde{T}^c$ (Lemma B1). So, for any closed C in $\eta^c(H^c)$, we have C is closed in $\widetilde{T}^c$ (see Theorem 6.2 of Chapter 2 in Munkres [28, 1975]) and so $\beta^c(C) = \widetilde{\beta}^c(C)$ is closed. Conversely, fix a closed set F in $\mathcal{P}\left(Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)\right)$. Then

$$(\beta^c)^{-1}(F) = (\widetilde{\beta}^c)^{-1}(F) \cap \eta^c(H^c),$$

and so closed (again using Theorem 6.2 of Chapter 2 in Munkres [28, 1975]). ■

There is also an embedding $\theta^c : \mathcal{P}\left(Z_1^c \times T^d\right) \to \mathcal{P}\left(Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)\right)$ with

$$\theta^c\left(\mathcal{P}\left(Z_1^c \times T^d\right)\right) = \left\{\nu \in \mathcal{P}\left(Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)\right) : \nu\left(Z_1^c \times T^d\right) = 1\right\}.$$

See Lemma C6. In particular, $\theta^c(\mu)\left(E \cap \left(Z_1^c \times T^d\right)\right) = \mu(E)$ for any event E in $Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)$.

Lemma B3 *For each m,*

$$\eta_m^c(H_m^c) = \mathrm{proj}_{\prod_{k=1}^{m} \mathcal{P}(Z_k^c)} \eta^c(H^c).$$

Proof. It is immediate that

$$\mathrm{proj}_{\prod_{k=1}^{m} \mathcal{P}(Z_k^c)} \eta^c(H^c) \subseteq \eta_m^c(H_m^c).$$

To show the opposite inclusion, fix some $\eta_m^c(\mu_1^c, \ldots, \mu_m^c) = (\varpi_1^c, \ldots, \varpi_m^c)$. It suffices to show that there is some pair of measures $\mu_{m+1}^c \in \mathcal{P}\left(X_{m+1}^c\right)$ and $\varpi_{m+1}^c \in \mathcal{P}\left(Z_{m+1}^c\right)$ such that $\mathrm{marg}_{X_m^c} \mu_{m+1}^c = \mu_m^c$ and $\zeta_{m+1}^c\left(\mu_{m+1}^c\right) = \varpi_{m+1}^c$. If so, a standard induction argument establishes that there is some $\eta^c\left(\mu_1^c, \ldots, \mu_m^c, \mu_{m+1}^c, \ldots\right) = \left(\varpi_1^c, \ldots, \varpi_m^c, \varpi_{m+1}^c, \ldots\right)$.

To see this, first find a measure $\mu_{m+1}^c \in \mathcal{P}\left(X_{m+1}^c\right)$ with $\mathrm{marg}_{X_m^c} \mu_{m+1}^c = \mu_m^c$. (This can be done: Consider the projection map from X_{m+1}^c to X_m^c. Then, by Lemma 3.9, there is a measure μ_{m+1}^c such that μ_{m+1}^c is the image measure of μ_m^c under the projection map. It follows that $\mathrm{marg}_{X_m^c} \mu_{m+1}^c = \mu_m^c$.) Finally, note that we can simply take ϖ_{m+1}^c to be the image measure of μ_{m+1}^c under ζ_{m+1}^c. This establishes the result. ■

Lemma B4 *If $\left(\varpi_1^c, \ldots, \varpi_{m+1}^c\right) \in \eta_{m+1}^c\left(H_{m+1}^c\right)$ then $\varpi_{m+1}^c\left(X^c \times \eta_m^d\left(H_m^d\right)\right) = 1$.*

Proof. Fix some $\eta_{m+1}^c\left(\mu_1^c,\ldots,\mu_{m+1}^c\right)=\left(\varpi_1^c,\ldots,\varpi_{m+1}^c\right)$. Then,

$$\begin{aligned}\varpi_{m+1}^c\left(X^c\times\eta_m^d\left(H_m^d\right)\right) &= \varpi_{m+1}^c(\zeta_{m+1}^c((\zeta_{m+1}^c)^{-1}(X^c\times\eta_m^d(H_m^d))))\\ &= \mu_{m+1}^c((\zeta_{m+1}^c)^{-1}(X^c\times\eta_m^d(H_m^d)))\\ &= \mu_{m+1}^c\left(X^c\times H_m^d\right)\\ &= 1,\end{aligned}$$

as required. ■

Lemma B5 $\beta^c\left(T^c\right)\subseteq\theta^c\left(\mathcal{P}\left(Z_1^c\times T^d\right)\right)$.

Proof. Fix $t^c=(\varpi_1^c,\varpi_2^c,\ldots)\in\eta^c\left(H^c\right)$. It suffices to show that $\beta^c\left(t^c\right)\left(Z_1^c\times\eta^d\left(H^d\right)\right)=1$. To see this, fix an event E in $\prod_{m=1}^\infty\mathcal{P}\left(Z_m^d\right)$ with $\eta^d\left(H^d\right)\cap E=\emptyset$. Suppose, contra hypothesis, that $\beta^c\left(t^c\right)\left(Z_1^c\times E\right)>0$. Then, there exists some m with

$$\begin{aligned}\operatorname{marg}_{Z_1^c\times\prod_{n=1}^m\mathcal{P}(Z_n^d)}\beta^c\left(t^c\right)\left(Z_1^c\times\operatorname{proj}_{\prod_{k=1}^m\mathcal{P}(Z_k^c)}E\right) &= \varpi_{m+1}^c(Z_1^c\times\operatorname{proj}_{\prod_{k=1}^m\mathcal{P}(Z_k^c)}E)\\ &> 0.\end{aligned}$$

But this contradicts Lemmata B3-B4. ■

Lemma B6 *Let $\mathcal{T}=(X^a,X^b;T^a,T^b;\beta^a,\beta^b)$, be such that T^c is given as above and $\beta^c\left(t^c\right)=\left(\theta^c\right)^{-1}\left(\beta^c(t^c)\right)$. Then, $\mathcal{T}$ is a continuous $\left(X^a,X^b\right)$-based type structure, where each T^c is Polish.*

Proof. It is immediate that T^c is metrizable, in fact Polish. To see that β^c is well-defined, first notice that $\beta^c\left(t^c\right)$ is a single point contained in $\theta^c\left(\mathcal{P}\left(Z_1^c\times T^d\right)\right)$ (Lemma B5). So, $(\theta^c)^{-1}(\beta^c(t^c))$ is non-empty. Since θ^c is injective, $(\theta^c)^{-1}(\beta^c(t^c))$ must be a single point.

Now, we show that β^c is continuous (and so measurable): Fix a closed set C in $X^c\times T^d$. Then,

$$\begin{aligned}\left(\beta^c\right)^{-1}(C) &= \{t^c\in T^c:\left(\theta^c\right)^{-1}\left(\beta^c(t^c)\right)\in C\}\\ &= \{t^c\in T^c:\beta^c\left(t^c\right)\in\theta^c\left(C\right)\}\\ &= \left(\beta^c\right)^{-1}\left(\theta^c\left(C\right)\right).\end{aligned}$$

Since θ^c is an embedding, $\theta^c\left(C\right)$ is closed. Now, use the fact that β^c is continuous (Lemma B2), to get that $\left(\beta^c\right)^{-1}(C)$ is closed, as required. ■

Lemma B7 *Consider the $\left(X^a,X^b\right)$-based type structure $\mathcal{T}=(X^a,X^b;T^a,T^b;\beta^a,\beta^b)$, as defined above. For each $t^c\in T^c$, $\delta^c\left(t^c\right)=t^c$.*

Proof. Fix $t^c=\eta^c\left(\mu_1,\mu_2,\ldots\right)$. It suffices to show that, for each m, $\left(\delta_1^c\left(t^c\right),\ldots,\delta_m^c\left(t^c\right)\right)=\eta_m^c\left(\mu_1,\ldots,\mu_m\right)$. The proof is by induction on m.

$m = 1$: Fix an event E_1 in X^c. Then

$$\begin{aligned} \delta_1^c(t^c)(E_1) &= \beta^c(t^c)\left(E_1 \times T^d\right) \\ &= \beta^c(t^c)\left(E_1 \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)\right) \\ &= \mu_1(E_1). \end{aligned}$$

Now, using the fact that η_1^c is the identity map, $\delta_1^c(t^c) = \eta_1^c(\mu_1)$, as required.

$m \geq 2$: Assume that the result holds for m. Then, by the induction hypothesis,

$$\begin{aligned} \left(\delta_1^c(t^c), \ldots, \delta_m^c(t^c)\right) &= \eta_m^c(\mu_1, \ldots, \mu_m) \\ &= \left(\mu_1, \underline{\zeta}_2^c(\mu_2^c), \ldots, \underline{\zeta}_m^c(\mu_m^c)\right). \end{aligned}$$

It suffices to show that $\delta_{m+1}^c(t^c) = \underline{\zeta}_{m+1}^c\left(\mu_{m+1}^c\right)$.

Write $\varpi_{m+1}^c = \underline{\zeta}_{m+1}^c\left(\mu_{m+1}^c\right)$ and fix E_{m+1} in Z_{m+1}^c. It suffices to show that

$$\left(E_{m+1} \times \prod_{n=m+1}^{\infty} \mathcal{P}\left(Z_n^d\right)\right) \cap \left(X^c \times T^d\right) = \left(\rho_{m+1}^c\right)^{-1}(E_{m+1}). \tag{B1}$$

If so, then

$$\begin{aligned} \varpi_{m+1}^c(E_{m+1}) &= \beta^c(t^c)\left(E_{m+1} \times \prod_{n=m+1}^{\infty} \mathcal{P}\left(Z_n^d\right)\right) \\ &= \beta^c(t^c)\left(\left(E_{m+1} \times \prod_{n=m+1}^{\infty} \mathcal{P}\left(Z_n^d\right)\right) \cap \left(X^c \times T^d\right)\right) \\ &= \beta^c(t^c)\left((\rho_{m+1}^c)^{-1}(E_{m+1})\right), \end{aligned}$$

as required.

We now turn to showing Equation B1. Fix some

$$\left(x^c, t^d\right) = \left(x^c, \varpi_1^d, \varpi_2^d, \ldots\right) \in \left(E_{m+1} \times \prod_{n=m+1}^{\infty} \mathcal{P}\left(Z_n^d\right)\right) \cap \left(X^c \times T^d\right).$$

Then, $\left(x^c, \varpi_1^d, \ldots, \varpi_m^d\right) \in E_{m+1}$. By the induction hypothesis, $\left(\varpi_1^d, \ldots, \varpi_m^d\right) = (\delta_1^d(t^d), \ldots, \delta_m^d(t^d))$, so that $\left(x, t^d\right) \in \left(\rho_{m+1}^c\right)^{-1}(E_{m+1})$. Conversely, fix some $\left(x, t^d\right) \in \left(\rho_{m+1}^c\right)^{-1}(E_{m+1})$. Then, $(x^c, \delta_1^d(t^d), \ldots, \delta_m^d(t^d)) \in E_{m+1}$. Again, by the induction hypothesis, $\left(x, \varpi_1^d, \ldots, \varpi_m^d\right)$ where $\left(\varpi_1^d, \ldots, \varpi_m^d\right)$ is an initial segment of t^d. It follows that $\left(x, t^d\right) \in E_{m+1} \times \prod_{n=m+1}^{\infty} \mathcal{P}\left(Z_n^d\right)$ as required. ■

Proof of Proposition B1.

Let $\mathcal{T} = (X^a, X^b; T^a, T^b; \beta^a, \beta^b)$ be as constructed above. For any $(\mu_1, \ldots, \mu_m) \in H_m^c$ (resp. $(\mu_1, \mu_2, \ldots) \in H^c$), there is a type $t^c \in T^c$ with $(\delta_1^c(t^c), \ldots, \delta_m^c(t^c)) = \eta_m^c(\mu_1, \ldots, \mu_m)$ (resp. $\delta^c(t^c) = \eta^c(\mu_1, \mu_2, \ldots)$). (See Lemma B7.) So, if an X-based type structure

$$\mathcal{T}_* = (X^a, X^b; T_*^a, T_*^b; \beta_*^a, \beta_*^b)$$

is finitely terminal (resp. terminal) then, for each $(\mu_1, \ldots, \mu_m) \in H_m^c$ (resp. $(\mu_1, \mu_2, \ldots) \in H^c$), there is a type $t_*^c \in T_*^c$ with $(\delta_{*1}^c(t_*^c), \ldots, \delta_{*m}^c(t_*^c)) = \eta_m^c(\mu_1, \ldots, \mu_m)$ (resp. $\delta_*^c(t_*^c) = \eta^c(\mu_1, \mu_2, \ldots)$).

The converse follows from Lemma 3.4 (resp. 3.7). ■

Appendix C Results for Section 4

In this section, we prove Proposition 4.1. Fix a type structure $\mathcal{T}$ and define maps ρ^a, ρ^b as in the text, i.e., for each player c, let

$$\rho^c : X^c \times T^d \to Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)$$

be a map with $\rho^c\left(x^c, t^d\right) = (x^c, \delta^d(t^d))$. Note, if $\mathcal{T}$ is non-redundant, then ρ^c is injective. Also note that ρ^c is measurable since it is the product of two measurable maps.

Lemma C1 *Let Ω be metrizable. If $(\Omega, \mathcal{B}(\Omega))$ is standard Borel, then there is a Polish space, viz. $(\Phi, \mathcal{B}(\Phi))$, so that $\Omega \in \mathcal{B}(\Phi)$ and $\mathcal{B}(\Omega)$ is generated by the induced topology.*

Proof. This follows from Proposition 3.3.7(ii) and Remark 3.38 in Srivastava [32, 1998]. ■

Lemma C2 *Fix a metrizable space Ω, so that $(\Omega, \mathcal{B}(\Omega))$ is standard Borel. Also, fix a Polish space $(\Phi, \mathcal{B}(\Phi))$. If the map $f : \Omega \to \Phi$ is injective and measurable, then the map $\underline{f} : \mathcal{P}(\Omega) \to \mathcal{P}(\Phi)$ is also injective.*

Proof. Suppose $\underline{f}(\mu) = \underline{f}(\nu)$. By Lemma C1 and Purves' Theorem [30, 1966], $\{f(E) : E \in \mathcal{B}(\Omega)\} \subseteq \mathcal{B}(\Phi)$. So, for each E in $\mathcal{B}(\Omega)$,

$$\mu(E) = \mu\left(f^{-1}(f(E))\right) = \nu\left(f^{-1}(f(E))\right) = \nu(E),$$

where the first and third equalities follow from the fact that f is injective and the second from the fact that $\underline{f}(\mu) = \underline{f}(\nu)$. As such, $\mu = \nu$. ■

Corollary C1 *Fix a non-redundant and standard Borel structure $\mathcal{T}$. The maps $\underline{\rho}^a$ and $\underline{\rho}^b$ are injective.*

Another fact about the map ρ^c:

Lemma C3 *For any event E_m in $\mathcal{B}(Z_1^c \times \prod_{n=1}^{m-1} \mathcal{P}(Z_n^d))$, $(\rho^c)^{-1}(E_m \times \prod_{n=m}^{\infty} \mathcal{P}\left(Z_n^d\right)) = \left(\rho_m^c\right)^{-1}(E_m)$.*

Proof. We have that

$$\begin{aligned}(\rho^c)^{-1}(E_m \times \textstyle\prod_{n=m}^{\infty} \mathcal{P}\left(Z_n^d\right)) &= \left\{\left(x^c, t^d\right) : \left(x^c, \delta_1^d\left(t^d\right), \ldots, \delta_{m-1}^d\left(t^d\right)\right) \in E_m\right\} \\ &= \left(\rho_m^c\right)^{-1}(E_m),\end{aligned}$$

establishing the result. ■

Now, define a map $\varphi^c : \mathcal{P}\left(X^c \times T^d\right) \to \prod_m \mathcal{P}\left(Z_m^c\right)$ so that

$$\varphi^c(\mu) = (\underline{\rho}_1^c(\mu), \underline{\rho}_2^c(\mu), \ldots).$$

Extend the definition of a coherent sequence to a sequence of countable length. The following Remark follows immediately from the proof of Lemma 3.5.

Remark C1 *For each $\mu \in \mathcal{P}\left(X^c \times T^d\right)$, $\varphi^c(\mu)$ is coherent.*

We now state a relationship between the maps $\underline{\rho}^c$ and φ^c. As usual, say $\underline{\rho}^c(\mu)$ *extends* $\varphi^c(\mu) = (\underline{\rho}_1^c(\mu), \underline{\rho}_2^c(\mu), \ldots)$ *to* $Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)$ if (i) $\underline{\rho}_1^c(\mu) = \operatorname{marg}_{Z_1^c} \underline{\rho}^c(\mu)$ and (ii) for each $m \geq 1$, $\underline{\rho}_{m+1}^c(\mu) = \operatorname{marg}_{Z_1^c \times \prod_{n=1}^{m} \mathcal{P}(Z_n^d)} \underline{\rho}^c(\mu)$.

Lemma C4 *For each $\mu \in \mathcal{P}\left(X^c \times T^d\right)$, $\underline{\rho}^c(\mu)$ extends $\varphi^c(\mu)$ to $Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)$.*

Proof. Fix some $\mu \in \mathcal{P}\left(X^c \times T^d\right)$. First, note that, for any event E_1 in Z_1^c,

$$\begin{aligned} \underline{\rho}_1^c(\mu)(E_1) &= \mu((\rho_1^c)^{-1}(E_1)) \\ &= \mu\left(E_1 \times T^d\right) \\ &= \mu((\rho^c)^{-1}(E_1 \times \textstyle\prod_{m=1}^{\infty} \mathcal{P}(Z_m^d))) \\ &= \operatorname{marg}_{Z_1^c} \underline{\rho}^c(\mu)(E_1), \end{aligned}$$

establishing that $\underline{\rho}_1^c(\mu) = \operatorname{marg}_{Z_1^c} \underline{\rho}^c(\mu)$. Similarly, fix some $m \geq 1$ and some E_{m+1} in $Z_1^c \times \prod_{n=1}^{m} \mathcal{P}\left(Z_n^d\right)$. Then,

$$\begin{aligned} \underline{\rho}_{m+1}^c(\mu)(E_{m+1}) &= \mu((\rho_{m+1}^c)^{-1}(E_{m+1})) \\ &= \mu((\rho^c)^{-1}(E_{m+1} \times \textstyle\prod_{n=m+1}^{\infty} \mathcal{P}(Z_n^d))) \\ &= \operatorname{marg}_{Z_1^c \times \prod_{n=1}^{m} \mathcal{P}(Z_n^d)} \underline{\rho}^c(\mu)(E_{m+1}), \end{aligned}$$

where the second line follows from Lemma C3. ■

Lemma C5 *Given a non-redundant standard Borel structure, the map φ^c is injective.*

Proof. Fix a non-redundant structure $\mathcal{T}$ and some $\mu, \nu \in \mathcal{P}(X^c \times T^c)$ with $\mu \neq \nu$. By Corollary C1, $\underline{\rho}^c(\mu) \neq \underline{\rho}^c(\nu)$. Now note that, if $\varphi^c(\mu) = \varphi^c(\nu)$, then we would have (at least) two extensions of $\varphi^c(\mu)$ to $Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)$. (See Lemma C4.) But, there is a unique extension of $\varphi^c(\mu)$ to $Z_1^c \times \prod_{m=1}^{\infty} \mathcal{P}\left(Z_m^d\right)$. (See, e.g., Theorem 21.10d in [34, 2006]). So, $\varphi^c(\mu) \neq \varphi^c(\nu)$. ■

Lemma C6 *Let Ω_1 be a metrizable space, where Ω_2 is closed in Ω_1. Take Φ to be the set of measures on Ω_1 that assign probability 1 to Ω_2.*

(i) The set Φ is closed in $\mathcal{P}(\Omega_1)$.

(ii) Define $f : \mathcal{P}(\Omega_2) \to \Phi$ so that $f(\mu)(E) = \mu(E \cap \Omega_2)$. The map f is well-defined and continuous.

(iii) Given a measurable map $g_2 : \Delta \to \mathcal{P}(\Omega_2)$, define $g_1 : \Delta \to \mathcal{P}(\Omega_1)$ so that $g_1 = f \circ g_2$. Then g_1 is measurable.

Proof. For Part (i), fix a sequence of measures μ_n in Φ with $\mu_n \to \mu$. Since Ω_2 is closed in Ω_1, the Portmanteau Theorem (stated in Parthasarathy [29, Theorem 6.1; 2005] for metrizable spaces), $1 = \limsup \mu_n(\Omega_2) \le \mu(\Omega_2)$. So, $\mu \in \Phi$, as required.

Now turn to Part (ii). First, to see that $f(\mu)$ defines a probability measure on Ω_1, note that $f(\mu)(\Omega_1) = \mu(\Omega_2) = 1$ and $f(\mu)(\emptyset) = 0$. Also, given disjoint events $E_1, E_2, \ldots$ in Ω_1, $E_1 \cap \Omega_2, E_2 \cap \Omega_2, \ldots$ are disjoint events in Ω_2. From this, countable additivity follows. Now also note that $f(\mu) \in \Phi$ by construction.

To show that f is continuous, it suffices to establish that if $\mu_n \to \mu$ in $\mathcal{P}(\Omega_2)$, then $f(\mu_n) \to f(\mu)$ in Φ. Suppose $f(\mu_n)$ does not converge to $f(\mu)$ in Φ. Again using the Portmanteau Theorem, we can find a closed set C in Ω_1 so that $\limsup f(\mu_n)(C) > f(\mu)(C)$. Then $\limsup \mu_n(C \cap \Omega_2) > \mu(C \cap \Omega_2)$. But, since $C \cap \Omega_2$ is closed in Ω_1, again using the Portmanteau Theorem, this says that μ_n does not converge to μ in $\mathcal{P}(\Omega_2)$, establishing the result.

Part (iii) is immediate. ■

Proof of Proposition 4.1. Fix a non-redundant, standard Borel, terminal (X^a, X^b)-based structure $\mathcal{T} = (X^a, X^b; T^a, T^b; \beta^a, \beta^b)$. Suppose, contra hypothesis, that the structure is not complete. That is, there is a measure $\mu \in \mathcal{P}(X^a \times T^b)$ where $\beta^a(t^a) \neq \mu$ for all types $t^a \in T^a$. By injectivity of φ^a (Lemma C5), for all types $t^a \in T^a$, we then have that

$$\delta^a(t^a) = \varphi^a(\beta^a(t^a)) \neq \varphi^a(\mu).$$

We will now show that there is an (X^a, X^b)-based type structure $\mathcal{T}_* = (X^a, X^b; T^a_*, T^b_*; \beta^a_*, \beta^b_*)$ and a type $\widehat{t^a}$ with $\delta^a_*(\widehat{t^a}) = \varphi^a(\mu)$, contradicting that $\mathcal{T}$ is terminal.

Take $T^a_* = T^a \cup \{\widehat{t^a}\}$ with $T^a \cap \{\widehat{t^a}\} = \emptyset$. Write $\mathcal{U}(T^a)$ for the set of open sets on T^a and endow T^a_* with the topology generated by $\mathcal{U}(T^a) \cup \{\{\widehat{t^a}\}\}$. (This fits with restricting attention to the induced topology.) Then, T^a_* is metrizable, as we require. (See Engelking [11, Proposition 2.2.4 and Theorem 4.2.1; 1977].) Construct a metrizable space T^b_* analogously.

Choose $\beta^a_*(t^a) = \beta^a(t^a)$ for all $t^a \in T^a$ and $\beta^a_*(\widehat{t^a}) = \mu$. Then, for any event E^a in $\mathcal{P}(X^a \times T^b_*)$,

$$(\beta^a_*)^{-1}(E^a) = \begin{cases} (\beta^a)^{-1}(E^a) & \text{if } \mu \notin E^a \\ (\beta^a)^{-1}(E^a) \cup \{\widehat{t^a}\} & \text{if } \mu \in E^a. \end{cases}$$

Since β^a is measurable, β^a_* is also measurable. For each $t^b \in T^b$,

$$\beta^b_*(t^b)(F^b) = \beta^b(t^b)(F^b \cap (X^b \times T^a)) \quad \text{for all } F^b \text{ in } X^b \times T^a_*.$$

By Lemma C6, each $\beta^b_*\left(t^b\right)$ is a probability measure (uses Part (ii)) and β^b_* is measurable (uses Parts (i) and (iii)).

It is immediate that, for each $t^a \in T^a$ and each $t^b \in T^b$, $\delta^a_*\left(t^a\right) = \delta^a\left(t^a\right)$ and $\delta^b_*\left(t^b\right) = \delta^b\left(t^b\right)$. From this, it is immediate that $\delta^a_*(\widehat{t}^a) = \varphi^a_*\left(\mu\right) = \varphi^a\left(\mu\right)$, as required. ■

References

[1] Aldaz, J., and H. Render, "Borel Measure Extensions of Measures Defined on Sub-σ-Algebras," *Advances in Mathematics*, 50, 2000, 233-263.

[2] Armbruster, W. and W. Böge, "Bayesian Game Theory," in *Game Theory and Related Topics*, edited by O. Moeschlin and D. Pallaschke, North-Holland, 1979.

[3] Battigalli, P., and M. Siniscalchi, "Hierarchies of Conditional Beliefs and Interactive Epistemology in Dynamic Games," *Journal of Economic Theory*, 88, 1999, 188-230.

[4] Battigalli, P., and M. Siniscalchi, "Strong Belief and Forward-Induction Reasoning," *Journal of Economic Theory*, 106, 2002, 356-391.

[5] Böge, W., and Th. Eisele, "On Solutions of Bayesian Games," *International Journal of Game Theory*, 8, 1979, 193-215.

[6] Brandenburger, A., "On the Existence of a 'Complete' Possibility Structure," in *Cognitive Processes and Economic Behavior*, edited by Marcello Basili, Nicola Dimitri, and Itzhak Gilboa, Routledge, 2003, 30-34.

[7] Brandenburger, A., and E. Dekel, "Hierarchies of Beliefs and Common Knowledge," *Journal of Economic Theory*, 59, 1993, 189-198.

[8] Brandenburger, A., A. Friedenberg, and H.J. Keisler, "Admissibility in Games," *Econometrica*, 76, 2008, 307-352.

[9] Brandenburger, A., and H.J. Keisler, "An Impossibility Theorem on Beliefs in Games," *Studia Logica*, 84, 2006, 211-240.

[10] Di Tillio, A., "Subjective Expected Utility in Games," *Theoretical Economics*, 3, 2008, 287-343.

[11] Engelking, R., *General Topology*, PWN-Polish Scientific Publishers, 1977.

[12] Friedenberg, A., "A Note on Epistemic Conditions for the Iteratively Undominated Strategies," 2008, available at www.public.asu.edu/~afrieden.

[13] Friedenberg, A., and M. Meier, "On the Relationship Between Hierarchy and Type Morphisms," 2008, available at www.public.asu.edu/~afrieden.

[14] Geanakoplos, J.D., and H.M. Polemarchakis, "We Can't Disagree Forever," *Journal of Economic Theory*, 28, 1982, 192-200.

[15] Harsanyi, J., "Games of Incomplete Information Played by 'Bayesian' Players, I-III," *Management Science*, 14, 1967-68, 159-182, 320-334, 486-502.

[16] Heifetz, A., "The Bayesian Formulation of Incomplete Information—The Non-Compact Case," *International Journal of Game Theory*, 21, 1993, 329-338.

[17] Heifetz, A., and D. Samet, "Topology-Free Typology of Beliefs," *Journal of Economic Theory*, 82, 1998, 324-341.

[18] Heifetz, A., and D. Samet, "Coherent Beliefs are Not Always Types," *Journal of Mathematical Economics*, 32, 1999, 475-488.

[19] Kechris, A., *Classical Descriptive Set Theory*, Springer-Verlag, 1995.

[20] Landers, D., and L. Rogge, "On the Extension Problem for Measures," *Z. Wahrscheinlichkeitstheorieverw Gebiete*, 30, 1974, 168-169.

[21] Lipman, B., "Finite Order Implications of Common Priors," *Econometrica*, 71, 2003, 1255-1267.

[22] Łos', J. and E. Marczewski, "Extensions of Measure," *Fundamenta Mathematicae*, 36, 1949, 267–276.

[23] Lubin, A., "Extensions of Measures and the Von Neumann Selection Theorem," *Proceedings of the American Mathematical Society*, 1974, 43, 118-122.

[24] Meier, M., "An Infinitary Probability Logic for Type Spaces," CORE Discussion Paper, 2001.

[25] Meier, M., "Finitely Additive Beliefs and Universal Type Spaces," *The Annals of Probability*, 34, 2006, 286-422.

[26] Mertens, J-F., S. Sorin, and S. Zamir, *Repeated Games, Part A: Background Material*, CORE Discussion Paper, 1993.

[27] Mertens, J-F., and S. Zamir, "Formulation of Bayesian Analysis for Games with Incomplete Information," *International Journal of Game Theory*, 14, 1985, 1-29.

[28] Munkres, J., *Topology: A First Course*, Prentice-Hall, 1975.

[29] Parthasarathy, K.R., *Probability Measures on Metric Spaces*, AMS Chelsea Publishing, 2005.

[30] Purves, R., "Bimeasurable Functions," *Fundamenta Mathematicae*, 58, 1966, 149-157.

[31] Siniscalchi, M., "Epistemic Game Theory: Beliefs and Types," *The New Palgrave Dictionary of Economics*, Second Edition, edited by L. Blume and S.N. Durlauf, Palgrave Macmillan, 2008.

[32] Srivastava, S.M., *A Course on Borel Sets*, Springer, 1998.

[33] Weinstein, J. and M. Yildiz, "Impact of Higher-Order Uncertainty," *Games and Economic Behavior*, 60, 2007, 200-212.

[34] Yeh, J. *Real Analysis: Theory of Measure and Integration*, 2nd Edition, World Scientific, 2006.

[35] Yershov, M., "Extensions of Measures and Stochastic Equations," *Theoretical Probability Applications*, 19, 1974, 431-444.