Astr 545 – Astrophysics of Stars and Accretion – Fall 2015

Homework #2

Due: Tues. Sept. 22, 1:59PM in class, or earlier in SO mailbox (Youdin)

1. Surface vs. Observed Flux:

(a) *Review of Surface Flux:*
(i) What is the flux per unit frequency, F_ν, at the surface of an object with only outgoing radiation, i.e. no irradiation. Assume that the specific intensity, $I_\nu(\mu)$ is axisymmetric and depends only on $\mu = \cos(\theta)$ at the surface, where θ is the angle between the radiation and the outward unit normal. (ii) Apply this result to the case of outgoing isotropic radiation, which includes the case of a blackbody, $I_\nu = B_\nu(T)$ the Planck function. (iii) For a blackbody calculate the net (frequency integrated) flux in terms of σ and T. (iv) What is the definition of effective temperature for an arbitrary net flux F?

(b) *Observed Flux, Basics:*
Consider an observer at a distance d from (the center of) some luminous object From the definition of specific intensity show that:

$$dF_\nu^{\rm obs} = I_\nu d\Omega \tag{1}$$

where I_ν is the specific intensity at the object's surface, in a direction pointed towards the observer, $d\Omega$ is the solid angle subtended by an element of the emitting surface, as seen by the observer and $d_\nu^{\rm obs}$ is the incremental flux received by the observer from that element. Assume that intervening space is empty.

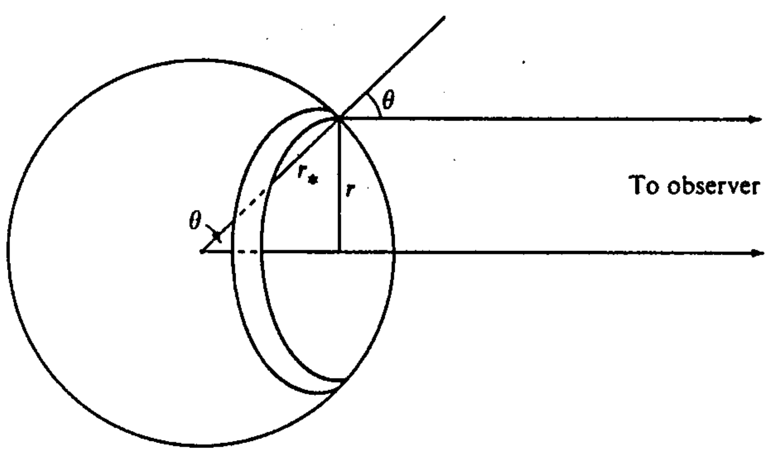

FIGURE 1-3
Geometry of measurement of stellar flux. The annulus on the surface of the star has an area $dS = 2\pi r\, dr = 2\pi_*^2 \sin\theta \cos\theta\, d\theta$ normal to the line of sight; this area subtends a solid angle $d\omega = dS/D^2$ as seen by the observer.

Figure 1: When using this figure, you need to figure out what the typo π_*^2 means. Your answer should also clarify the orientation of dS relative to the observer or surface.

(c) *Observed Flux, Isotropic Case:*
Assume (here and part d) that the emitting object is a sphere with radius R_*. For an isotropic flux, simply integrate over solid angles to get the total observed flux (per frequency). Express your final result in terms of F_ν (at the surface) R_* and d.

(d) *Observed Flux, Anisotropic Axisymmetric Case:*
Again calculate $F_\nu^{\rm obs}$ in terms of F_ν at the surface, but this time allow for the angular variation of $I_\nu(\mu)$ across the surface. For this derivation, assume that the observer is far away, so that all rays from the surface of the star to the observer can be approximated as parallel. In this case, the observer sees rays from $\mu = 1$ all the way to $\mu = 0$. The drawing from Mihalas is useful for visualizing and calculating the area and solid angle elements to be integrated over. You should end up with the same final result as part (c), so the issue here is to justify each step.

2. SED of a grey atmosphere vs. a blackbody:
In class, we worked out some properties of a grey atmosphere, including the relation

$$T(\tau) = T_{\rm eff}\left(\frac{3\tau}{4} + \frac{1}{2}\right)^{1/4} \equiv T_{\rm eff} f(\tau) \tag{2}$$

and that the specific intensity at the surface of a grey atmosphere in LTE can be calculated as

$$I_\nu = \int_0^\infty B_\nu[T(\tau)] \exp[-\tau/\mu] \frac{d\tau}{\mu}\,, \tag{3}$$

where the (matter) LTE limit of the source function with no scattering $S_\nu = B_\nu$ holds. Since the atmosphere is grey, the optical depth is frequency independent.

The goal of this problem is to compare the SEDs (spectral energy distributions) of a blackbody and a grey atmosphere. We will follow the convention of plotting SEDs as νF_ν (and not simply F_ν) since this quantity gives the flux per logarithmic interval of frequency, $\nu F_\nu d\ln\nu$. This problem will require performing numerical integrals. You can do these integrals with any software you choose. Options include python (scipy.integrate module, with quad, dquad, etc. functions), Mathematica (NIntegrate function), Numerical Recipes or writing your own integrator in your language of choice. **Include your code as a part of the solution to this problem,** (either printed or as a working link) whether this code is a script or a notebook. You do not need to include plotting scripts.

(a) *Blackbody SED and non-dimensionalization.*
You should have already shown that $F_\nu = \pi B_\nu(T)$ above. For this subproblem, we merely introduce the non-dimensional variables which make plotting and numerical calculations easier. The dimensionless frequency is

$$x \equiv h\nu/(kT_{\rm eff}) \tag{4}$$

(and $T_{\rm eff} = T$ simply for a blackbody). We also normalize the SED as the quantity $\nu F_\nu/(\sigma T_{\rm eff}^4)$. For a blackbody, show that

$$\frac{\nu F_\nu}{\sigma T^4} = \frac{15}{\pi^4}\frac{x^4}{\exp(x) - 1} \tag{5}$$

Verify that the net flux is

$$\int_0^\infty \nu F_\nu \frac{dx}{x} = \sigma T^4 \tag{6}$$

justifying the LHS above.

(b) Starting with equations 2 and 3 show that the SED for a grey atmosphere can be written as the double integral

$$\frac{\nu F_\nu}{\sigma T_{\rm eff}^4}(x) = \frac{30x^4}{\pi^4}\int_0^1 d\mu \int_0^\infty d\tau \left\{\frac{\exp(-\tau/\mu)}{\exp[x/f(\tau)]-1}\right\} \tag{7}$$

(c) Perform the integral in (b) numerically for a dense sampling of x values between 0 and 20. Note that $x = 0$ will not evaluate directly (though clearly $F_\nu = 0$ here). More seriously, a divergence occurs at $\mu = 0$ and a computer can't directly integrate to $\tau =$ infinity. Starting with a very small (but non-zero) μ and ending with a large (but not-infinite) τ gives accurate results. Experiment as needed to be sure that this choice isn't appreciably affecting your results (and see d below).

Plot the results $\nu F_\nu/\sigma T_{\rm eff}^4$ vs. $x = h\nu/(kT_{\rm eff})$. Overplot the SED for the blackbody case (where $T = T_{\rm eff}$). Also make a plot of the difference of the SEDs (grey minus blackbody). Attempt to explain the differences physically.

(d) Show that the total (frequency integrated) flux $F_{\rm r}$ obeys

$$\frac{F_{\rm r}}{\sigma T_{\rm eff}^4} = \frac{30}{\pi^4}\int_0^\infty dx \int_0^1 d\mu \int_0^\infty d\tau \left\{\frac{x^3\exp(-\tau/\mu)}{\exp[x/f(\tau)]-1}\right\} \tag{8}$$

We know from the theory of grey atmospheres that this ratio is unity. How close to unity can your numerical integration get, as $1 - F_{\rm r}/(\sigma T_{\rm eff}^4)$?

3. Eddington Luminosity:

(a) Show that if the radiation pressure satisfies

$$-dP_{\rm rad}/dr > \rho g \tag{9}$$

then the luminosity exceeds the Eddington luminosity, $L_{\rm Edd}$. Assume that radiative diffusion holds up to the photosphere. Does the "derived" result (which is not rigorous) agree with the standard expression? Explain why this derivation is intuitive.

(b) For an electron scattering opacity, compute the $L_{\rm Edd}/L_\odot$ in terms of the stellar mass in Solar units.