Caps and Pants: Topological Quantum Field Theories In Dimension 2

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Outline

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The Category 2COB

Frobenius Algebra

Defining a Topological Quantum Field Theory

Two dimensional Topological Quantum Field Theories
  Two dimensional TQFTs with Closed Strings
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Relevance

Topological Quantum Field Theory → Area Field Theory → Quantum Field Theory

Topological Quantum Field Theory ← Conformal Field Theory ← Quantum Field Theory
Applications

- A basic model for Quantum Field Theory
- Knot Theory
- String Theory
Conventions

- All manifolds are real, smooth, oriented and compact.
- Space-time $\Sigma$ is a 2 dimensional manifold.
- Spaces $X$ and $Y$ are 1 dimensional manifolds.
- Morphisms between manifolds are smooth maps.
- For $M$ a manifold, $\overline{M}$ is the orientation reversal of $M$.
- View the empty set as a closed manifold of a given dimension.
Definition

A cobordism is a triple \((\Sigma, X, Y)\) such that \(\Sigma\) is a manifold with an orientation preserving isomorphism

\[
\partial \Sigma \rightarrow \overline{X} \sqcup Y.
\]

Orientation determines the direction in which the cobordism propagates:

- \((\Sigma, X, Y)\) propagates from \(X\) to \(Y\).
- \((\overline{\Sigma}, Y, X)\) propagates from \(Y\) to \(X\).
Example
Example

Right-cap

Left-cap
Example

Left-pair of pants

Right-pair of pants
Definition

For $\Sigma'$, $\Sigma''$, 2 manifolds with common boundary $X$, form a manifold $\Sigma = \Sigma' \cup_X \Sigma'' = \Sigma'' \circ \Sigma'$ by sewing $\Sigma'$ and $\Sigma''$ along $X$. 
The 2-category pre-2COB

- **Objects:** 1 dimensional manifolds.
- **1-morphisms** \(\Sigma := X \to Y\): 2 dim'l cobordisms \((\Sigma, X, Y)\).
- **2-morphisms:** orientation preserving isomorphisms \(\alpha : \Sigma \to \Sigma'\) for cobordisms \((\Sigma, X, Y)\) and \((\Sigma', X, Y)\) such that the following diagram commutes:
Definition

**2COB**, the category of 2 dimensional isomorphic cobordisms:

- **Objects**: 1 dimensional manifolds up to isomorphism.
- **Morphisms**: equivalence classes of 1-morphisms defined in terms of pre-2COB. Two 1-morphisms are equivalent if there exists a 2-morphism between them.
- **Composition**: the sewing of cobordisms.
- **The identity morphism**: \( X \times I, X \in |2COB|, I \) a compact interval.

A \((\Sigma, X, Y)\) cobordism is a representative of an equivalence class.
Theorem

Given $\mathcal{A}$, a finite dimensional $\mathbb{C}$ algebra, TFAE:

1. There exists an $\mathcal{A}$ module isomorphism

   \[ \lambda := \mathcal{A} \rightarrow (\mathcal{A})^{*}. \]

   where $\mathcal{A}^{*} = \text{Hom}_{\mathbb{C}}(\mathcal{A}, \mathbb{C})$ has diagonal $\mathcal{A}$-action.

2. There exists a non-degenerate linear form $\eta : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \rightarrow \mathbb{C}$

   which is “associative”:

   \[ \eta(ab \otimes c) = \eta(a \otimes bc) \quad \text{for } a, b, c \in \mathcal{A}. \]
Definition

- An algebra that satisfies either condition in the previous theorem is a **Frobenius algebra**.
- A **symmetric Frobenius algebra** is a Frobenius algebra such that $\eta$ is induced by a linear map $\theta: A \to \mathbb{C}$, called a trace map, such that $\theta(ab) = \theta(ba)$ and

$$\eta(x \otimes y) = \theta(xy).$$

Thus $\eta(a \otimes b) = \eta(b \otimes a)$.

Theorem

*If $A$ is a Frobenius algebra, then $A^*$ is a Frobenius algebra.*
Given a Frobenius algebra $\mathcal{A}$, define a multiplication map $\beta$ on $\mathcal{A} \otimes \mathcal{A}$:

$$\beta := \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$$

$$\sum a_i \otimes b_i \longmapsto \sum a_i b_i.$$

This consequently defines a comultiplication map $\beta^*$ on $\mathcal{A}^*$:

$$\beta^* \mathcal{A}^* \longrightarrow \mathcal{A}^* \otimes \mathcal{A}^* \cong (\mathcal{A} \otimes \mathcal{A})^*$$

$$f \longmapsto f \circ \beta.$$
As $\mathcal{A}$ and $\mathcal{A}^*$ are isomorphic, $\mathcal{A}$ is prescribed a coalgebra structure by defining the comultiplication map

$$\alpha := (\lambda^{-1} \otimes \lambda^{-1}) \circ \beta^* \circ \lambda$$

in terms of the following commutative diagram:
General Idea

A TQFT $Z$ assigns complex vector spaces $Z(X)$ and $Z(Y)$ to $X$ and $Y$. Based on the cobordism from $X$ to $Y$, an element $Z(\Sigma) \in \text{Hom}_C(Z(X), Z(Y))$ is assigned to $\Sigma$. 

\[
\begin{array}{ccc}
Z(X) & \xrightarrow{Z(\Sigma)} & Z(Y)
\end{array}
\]
Definition

A **Topological Quantum Field Theory** in dimension $d$ is a functor $Z : d\text{COB} \to \text{Vect}/\mathbb{C}$ such that disjoint unions in $d\text{COB}$ map to tensor products.

A TQFT is defined for a fixed dimension.
Property

\[ Z(\Sigma) \in Z(\partial \Sigma) \]

Proof.

\[
\begin{align*}
Z(\partial \Sigma) &= Z(\overline{X} \amalg Y) \\
&= Z(\overline{X}) \otimes Z(Y) \\
&= Z(X) \ast \otimes Z(Y) \\
&\cong \text{Hom}_C(Z(X), Z(Y)) \\
&\ni Z(\Sigma) \text{ (by functoriality)}
\end{align*}
\]

It is essential that \( Z(\cdot) \) be finite dimensional.
Property

\[ Z(\emptyset) \cong \mathbb{C} \] if the empty set is considered a closed 1 manifold.

Proof.

\[ Z(X) = Z(X \amalg \emptyset) = Z(X) \otimes_{\mathbb{C}} Z(\emptyset). \]

Thus

\[ \dim(Z(X)) = \dim(Z(X) \otimes_{\mathbb{C}} Z(\emptyset)) = \dim(Z(X)) \cdot \dim(Z(\emptyset)). \]

Therefore \( Z(\emptyset) \) must be a 1 dim’l vector space over \( \mathbb{C} \). [\qed]
Property

\[ Z(\emptyset \times I) \cong 1_{Z(\emptyset)} \in \mathbb{C} \text{ for } I \text{ a compact interval.} \]

Proof.

\[ Z(\emptyset \times I) \cong 1_{Z(\emptyset)} \in \text{Hom}_\mathbb{C}(Z(\emptyset), Z(\emptyset)) \cong \text{Hom}_\mathbb{C}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C} \]
Property

\[ Z(\Sigma) \in Z(\emptyset) = \mathbb{C} \text{ if } \Sigma \text{ is a closed manifold.} \]

Proof.

\[ Z(\Sigma) \in Z(\partial \Sigma) = Z(\emptyset) = \mathbb{C}. \]
Definition

For $M$ an object in $2\text{COB}$, a connected component of $M$ is a closed (open) string if it is isomorphic to a circle (compact line segment).

Remark

Objects in $2\text{COB}$ are composed of open and closed strings. The sub-category $2\text{COB}|_C$ is restricted to objects composed exclusively of closed strings.
Classifying $2\text{COB}|_C$

Let $\Sigma$ be a two dimensional manifold whose boundary is a disjoint union of closed strings. $\Sigma$ can be characterized according to its genus and the number of closed strings in its boundary.

We say $\Sigma$ is a $(\Sigma, g, k)$ manifold with genus $g$ and number of closed strings $k$. 
Generating $2\text{COB}|_c$

A $(\Sigma, g, k)$ manifold can be formed by sewing copies of cups and pairs of pants.
Proof.

Utilizing a cup and pairs of pants, construct a \((\Sigma, g, k)\) manifold by induction on \(g\) and \(k\). This construction is not unique.

If \(g = 0\), sew \(k - 2\) pants.
Proof con’t.

If $k < 3$ sew the necessary cups.
Proof con’t.

If \( g = 1 \), sew pants together in the following manner

then sew a \( g = 0 \) piece with the appropriate number of \( k \)’s.
Proof con’t.

If $g > 1$ sew the appropriate number of $g = 1$ pieces then sew a $g = 0$ piece with the necessary $k$.

If $(\Sigma, g, k) = \coprod_{i=1}^{n}(\Sigma_i, g_i, k_i)$ where $\sum_{i=1}^{n} g_i = g$, $\sum_{i=1}^{n} k_i = k$ then construct $(\Sigma, g, k)$ by constructing each $(\Sigma_i, g_i, k_i)$.

$\square$
Relations
Theorem

In $2\text{COB}|_C$, a 2 dimensional TQFT is equivalent to a symmetric, commutative Frobenius algebra $\mathcal{A}$. 
Proof.

TQFT \implies \text{symmetric, commutative Frobenius algebra } \mathcal{A}

\$S^1\$ is the only non-empty closed connected oriented 1 manifold.
Define the vector space \( \mathcal{A} := \mathbb{Z}(S^1) \).
Proof con’t.

A right-cap $R$ is a cobordism from $S^1$ to $\emptyset$.

Define the trace map as $Z(R) \theta A \rightarrow \mathbb{C}$.

\[ A \overset{\theta}{\rightarrow} \mathbb{C} \]
Proof con’t.

A left-cap $L$ is a cobordism from $\emptyset$ to $S^1$.

Associate the unit map $\mathbb{C} \rightarrow \mathcal{A}$ to the element $Z(L) \in \text{Hom}_{\mathbb{C}}(\mathbb{C}, \mathcal{A})$ such that $1_{\mathbb{C}} \mapsto 1_{\mathcal{A}}$.

$\emptyset \amalg S^1 \cong S^1$, and the following objects are equivalent in $2\text{COB}|_{\mathbb{C}}$: 

![Diagram](image-url)
Proof con’t.

A left-pair of pants is a cobordism from \((S^1 \sqcup S^1)\) to \(S^1\).

Associate to it the product \(\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}\).
Proof cont.'

Remark

A right-pair of pants is a cobordism from $\mathbb{S}^1$ to $(\mathbb{S}^1 \sqcup \mathbb{S}^1)$. Associate to it comultiplication $\alpha \vec{v} \mapsto \vec{v} \otimes \vec{v}$, defined earlier.
Proof con’t.

Commutativity

Consider the cobordism

\[
\left[ S^1 \sqcup S^1 \to S^1 \right] \equiv \left[ S^1 \sqcup S^1 \to S^1 \right].
\]

There exists a homeomorphism from a pair of pants onto itself giving the necessary relations.
Proof con’t.

Frobenius Property

Consider the following cobordism where the topology of the manifold does not change under the given diffeomorphism.

\[
\left( \mathbb{S}^1 \amalg \mathbb{S}^1 \right) \amalg \mathbb{S}^1 \to \mathbb{S}^1 \equiv \left[ \mathbb{S}^1 \amalg \left( \mathbb{S}^1 \amalg \mathbb{S}^1 \right) \to \mathbb{S}^1 \right].
\]
Proof con’t.
Symmetric Frobenius algebra $\Rightarrow$ TQFT

The above correspondences are bijections, thus the properties of a symmetric Frobenius algebra manifest themselves in a TQFT.

Left and right-pairs of pants, left and right-caps and cylinders generate an element in $2\text{COB}|_C$. Equivalent relations manifest isomorphic algebraic structures.
Definition

- A **cobordism** is a 4-tuple \((\Sigma, X, Y, \partial_{cstr}\Sigma)\). \(\partial\Sigma = \overline{X} \sqcup Y\).
- \(\partial_{cstr}\Sigma\) is the **constrained boundary**, a cobordism from \(\partial X\) to \(\partial Y\).

When \(\partial X = \partial Y = \emptyset\), as before, refer to the 3-tuple \((\Sigma, X, Y)\).

Again, the orientation of \(\Sigma\) determines the direction in which the cobordism propagates.

Subsequently, consider surfaces whose boundaries are open and closed strings, i.e. objects and morphisms in 2COB.
Example

\((\Sigma, 1 \amalg 1 \amalg 1, 1 \amalg 1, 1 \amalg 1)\) is a cobordism from two line segments to one.
Definition

Let $B_0$ be a set of elements of an $R$–module, $R$ a commutative ring with unity.

- A **$B_0$ decorated string** is an oriented compact 1-manifold with a labeling of the boundary components by elements of $B_0$.

- A **$B_0$ decorated morphism** is a cobordism with labeling of the connected components of the constrained boundary by elements of $B_0$. 
The category $B_0$-decorated 2COB

- **Objects**: $B_0$-decorated strings.
- **Morphisms**: $B_0$-decorated morphisms with labeling consistent with the labeling of the unconstrained part of the boundary.
Theorem

Let $\Sigma$ be a two dimensional manifold whose boundary is a disjoint union of closed and open strings. $\Sigma$ can be categorized according to its genus and the number of closed and open strings in the boundary.

$\Sigma$ is a $(\Sigma, g, k, l)$ manifold with genus $g$, number of closed strings $k$ and number of open strings $l$. 
Proof

Construct a \((\Sigma, g, k + l)\) manifold as before, then add \(l\) copies of the following piece.

![Diagram of a manifold with a piece added](image-url)
In a TQFT, a cobordism with open strings gives a linear map:

\[ \mathcal{O}_{ab} \otimes \mathcal{O}_{bc} \rightarrow \mathcal{O}_{ac} \]

Line segment \( ba \) is a cobordism from \( b \) to \( a \). The set of morphisms from \( b \) to \( a \) is a vector space.

\[ Z(ba) = \text{Hom}_{\mathbb{C}}(b, a) = \mathcal{O}_{ab} \]
Theorem

In the undecorated category $2COB$, a two dimensional TQFT $Z: 2COB \to \text{Complex Vector Spaces}$ is equivalent to the following algebraic structure:

- A finite dimensional, symmetric and commutative Frobenius algebra over $\mathbb{C}$ with a non-degenerate trace map $\theta: A \to \mathbb{C}$.
- A finite dimensional, symmetric, not necessarily commutative Frobenius algebra over $\mathbb{C}$ with a non-degenerate trace map $\theta: Z(I) \to \mathbb{C}$.
- A homomorphism $\mathfrak{i}: A \to Z(I)$ such that $\mathfrak{i}(1_A) = 1_{Z(I)}$ and the image of $A$ is in the center of $Z(I)$.
Property

There exists linear maps

\[ \iota^a : \mathcal{O}_{aa} \rightarrow \mathcal{A} \]

and

\[ \iota_a : \mathcal{A} \rightarrow \mathcal{O}_{aa}. \]
Property

\[ \iota_a \text{ is an algebra homomorphism since for } \phi_i \in A, \]

\[ \iota_a (\phi_1 \phi_2) = \iota_a (\phi_1) \iota_a (\phi_2). \]
Property

\( \iota_a \) preserves the identity in the sense that \( \iota_a(1_A) = 1_a \).
Property

\( \tau_a \) is central, it maps into the center of \( \mathcal{O}_{aa} \). \( \tau_a(\phi)\psi = \psi \tau_a(\phi) \) for all \( \phi \in \mathcal{A} \), \( \psi \in \mathcal{O}_{aa} \).
Property

\( \iota_a \) and \( \iota^a \) are adjoints \( \theta(\iota^a(\psi)\phi) = \theta_a(\psi\iota_a(\phi)) \) for all \( \psi \in \mathcal{O}_{aa}, \phi \in A. \)
Property

The Cardy Condition

For a space $\mathcal{O}_{ab}$ with basis $\psi_\mu$, $\mathcal{O}_{ba}$ is its dual space with basis $\psi^\nu$. Let $\pi^a_b \mathcal{O}_{aa} \rightarrow \mathcal{O}_{bb}$ be defined in the following manner.

$$\pi^a_b(\psi) = \sum_p \psi_\mu \psi^\nu.$$ 

This implies the Cardy condition

$$\pi^a_b = \iota_b \circ \iota^a.$$
The Cardy condition can be understood using this diagram.

The association is as follows:

\[ O_{aa} \rightarrow O_{ab} \otimes O_{ba} \rightarrow O_{ba} \otimes O_{ab} \rightarrow O_{bb} \]

An element \( \psi \) maps to:

\[ \psi \rightarrow c_{\mu \nu} \psi_\mu \otimes \psi^\nu \rightarrow c_{\mu \nu} \psi^\nu \otimes \psi_\mu \rightarrow c_{\mu \nu} \psi^\nu \psi_\mu \]

where \( c_{\mu_0 \nu_0} = \theta_b(\psi_{\mu_0} \psi_{\nu_0}) \).
Moving Forward

One can consider a TQFT over closed strings as a restriction of a TQFT over both open and closed strings. What is unclear is whether the converse is true.

Moore and Segal’s work explore this question using K-theory.
References


