CAPS AND PANTS: TOPOLOGICAL QUANTUM FIELD THEORIES IN DIMENSION 2

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Abstract. A two-dimensional Topological Quantum Field Theory is introduced and defined in an intuitive manner. The foundational concepts including Frobenius Algebras, cobordisms, open and closed strings are defined in detail thus allowing a student with no prior knowledge of the subject the ability to gain a strong grasp of basic aspects of a TQFT. The Dijkgraaf-Witten Toy Model is worked out as an example.

1. Introduction

The notion of a Topological Quantum Field Theory (TQFT) can be introduced in a concise, rigorous and mathematically appealing way, without any reference to quantum field theory, the form this subject normally takes in Physics. For example, in two dimensions, a TQFT is an equivariant functor from the category of oriented compact surfaces to the category of linear algebra where disjoint unions correspond to tensor products.

By considering a pair of pants, one immediately sees that a 2-d TQFT gives rise to an algebra, more precisely a commutative symmetric Frobenius algebra. In an analogous way, when the boundaries are allowed to include closed intervals, a 2-d TQFT gives rise to a symmetric Frobenius algebra, which is not necessarily commutative. This geometric point of view turns out to greatly enlighten the study of these algebras in the same way, for example, that the realization of a group as automorphisms of a covering space can shed light on the structure of a group. This will be the main theme of this general study.

The purpose of this paper is to provide a gentle introduction to the basics of TQFT in a simple, but non-trivial manner. The attempt is to introduce the subject to any interested party with a basic knowledge of algebra, topology and category theory. Segal’s lecture notes [14] are a beautiful and streamlined introduction to the subject. However, to ensure the general picture is effectively delivered to the audience, Segal refrains from including many important but elementary details and connections. At times, every sentence in Segal’s lecture notes as well as in the introduction of [13] read as a theorem. Our purpose here is to work through the material and provide the intermediate steps that help draw the given conclusions. Much of the formal content is incorporated from Abrams [1] and Lauda and Pfeiffer [9]. These sources should be consulted if the reader is interested in the formal content and structure of the presented material.

This paper is an attempt at a thorough but conceptual introduction to 2-dimensional TQFTs and their properties. We begin by defining the 2-category
$d$COB in section 2. Algebras with a Frobenius structure will be explored in section 3 along with examples that will resurface later in the discussion. In the following section we define a TQFT, a functor from $d$COB to linear algebra. Properties of TQFTs are discussed and proven. From here, we can focus on the 2-category 2COB in section 5 and prove our first main theorem - a 2-dimensional TQFT over closed strings is essentially equivalent to a commutative finite-dimensional symmetric Frobenius algebra. Properties of 2-dimensional TQFTs are given. In a parallel development, we explore 2-dimensional TQFTs over open strings, an expansion of the closed string case. We conclude the section by proving our second main theorem, namely that a 2-dimensional TQFT over open strings is equivalent to a not necessarily commutative Frobenius algebra. Finally, in section 6, the Dijkgraaf-Witten Toy Model will be the illustrative example of a TQFT. Many illustrations and examples are provided to ensure the reader has a full understanding of the material.

Finally, to put the subject into perspective, the reader is encouraged to refer to [15] to explore the properties of a Quantum Field Theory (QFT) and how they inspired Segal to form the mathematical axioms of a conformal field theory. Segal’s ideas consequently paved the way to define a Topological Quantum Field Theory (it was Witten who initially coined the term [17]). Segal used the properties of a QFT to establish the simple axioms we will explore. The main difference between a QFT and a TQFT lies in what is emphasized. A QFT is a functor to an infinite-dimensional space and depends on the structure of the manifold and the definition of a metric, whereas a TQFT, a functor to a finite-dimensional space, only depends on the structure of the manifold. TQFTs are very useful since they are a simplification of QFTs; they provide an arena to test conjectures and calculations that are more difficult or possibly impractical to compute for a general QFT.

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1.1. **Conventions.** All manifolds are smooth, oriented and compact. $\Sigma$, representing space-time, is $d$-dimensional; $X$ and $Y$, representing space, are $(d - 1)$-dimensional. At this time, $X$ and $Y$ will not have a boundary. When boundaries are introduced, it will be evident whether or not $X$ or $Y$ are manifolds with boundary. All maps from one manifold to another are smooth. If $M$ is a manifold with a fixed orientation, $\overline{M}$ will denote the orientation reversal of $M$. The empty set is a manifold of arbitrary dimension.

2. **The Category $d\text{COB}$**

We begin with a definition that will be generalized as we expand our initial results.

**Definition 1.** A cobordism is a triple $(\Sigma, X, Y)$ where $\Sigma$ is a manifold with $\partial \Sigma = \overline{X} \amalg Y$, the disjoint union of $X$ and $Y$.

In a $(\Sigma, X, Y)$ cobordism, orientation must be fixed to determine the direction in which the cobordism propagates. Therefore, let $(\Sigma, X, Y)$ be a cobordism such that $\Sigma$ propagates from $X$ to $Y$; thus $(\Sigma, Y, X)$ is a cobordism such that $\Sigma$ propagates from $Y$ to $X$.

**Definition 2.** A $d - 1$ manifold $X$ is null-cobordant if there is a cobordism that propagates from $X$ to the empty manifold. Observe that in this case, $X$ is the boundary of a $d$ manifold.

The following are examples of cobordisms and null-cobordant manifolds:

**Example 1.** A cylinder $C = S^1 \times [0, 1]$ is a cobordism $(C, S^1, S^1)$ from $S^1$ to $S^1$ where the $S^1$s are the antipodal boundaries of the cylinder.

![Cylinder](image)

**Example 2.** $S^{n-1}$ is null-cobordant since $S^{n-1}$ is the boundary of $D^n$, the $n$-dimensional disk. Thus, $D^2$ is a cobordism from the empty manifold to $S^1$. Refer to this cobordism as a cap where $(D^2, S^1, \emptyset)$ is a right-cap and $(D^2, \emptyset, S^1)$ is a left-cap.

![Cap](image)

**Example 3.** Let $\Sigma$ be a genus 0, 2-dimensional manifold such that $\partial \Sigma = (S^1 \amalg S^1) \amalg S^1$, as intended in the diagram below. Then $\Sigma$ is a cobordism.
between $S^1 \amalg S^1$ and $S^1$. We will call $\Sigma$ the pair of pants. Again, we distinguish between a left and right pair of pants. $(\Sigma, S^1 \amalg S^1, S^1)$ is the left-pair of pants. $(\Sigma, S^1, S^1 \amalg S^1)$ is the right-pair of pants.

**Definition 3.** Let $\Sigma', \Sigma''$ be $d$ manifolds where $\partial \Sigma' = X_1 \amalg X_2$ and $\partial \Sigma'' = X_2 \amalg X_3$. Then by sewing $\Sigma'$ and $\Sigma''$ along $X_2$ we form a new manifold $\Sigma = \Sigma' \cup_{X_2} \Sigma'' = \Sigma' \circ \Sigma'$ with boundary $X_1 \amalg X_3$.

We begin our study with the 2-category pre-$d$COB. Generally speaking, a 2-category is a category $C$ such that for two objects $A, B \in |C|$, the set Hom $(A, B)$ is a category. See the appendix for a detailed exposition on 2-categories. Objects in pre-$d$COB are comprised of the class of $(d-1)$-dimensional compact manifolds and 1-morphisms are cobordisms $\Sigma$ in dimension $d$. 2-morphisms are orientation preserving homeomorphisms $\alpha : \Sigma \rightarrow \Sigma'$ for cobordisms $(\Sigma, X, Y)$ and $(\Sigma', X, Y)$ such that the following diagram commutes:

\[ \begin{array}{ccc}
\partial \Sigma & \xrightarrow{=} & X \amalg Y \\
\alpha|_{\partial \Sigma} & \downarrow & \\
\partial \Sigma' & \xrightarrow{=} & \\
\end{array} \]

From this, one can state that in pre-$d$COB, two 1-morphisms $\Sigma$ and $\Sigma'$ are equivalent if there is a 2-morphism $\alpha : \Sigma \rightarrow \Sigma'$.

**Definition 4.** $d$COB is the category of $d$-dimensional homeomorphic cobordisms. The objects are $(d-1)$-dimensional compact manifolds up to isomorphism. Morphisms are the equivalence classes of $d$-dimensional 1-morphisms defined in the 2-category framework of pre-$d$COB where two 1-morphisms are equivalent if there exists a 2-morphism $\alpha : \Sigma \rightarrow \Sigma'$. The composition of morphisms is identified with the sewing of cobordisms. The identity morphism for an object $X$ is the manifold $X \times I$ where $I$ is a compact interval.

The reader should note that this notation is used for convenience. When the two-dimensional case is considered and boundaries are introduced in
section 5, this category will be denoted $2\text{COB}|_C$, a subcategory of $2\text{COB}$. $2\text{COB}$ will therefore be an expansion of the category we are currently focusing on.

The perspective that two elements in pre-$d\text{COB}$ that are diffeomorphic under certain conditions are considered equivalent will be helpful in our development. We can see that a $(\Sigma, X, Y)$ cobordism is actually a representative of an equivalence class, where two manifolds are equivalent if and only if there exists a diffeomorphism between the surfaces (observe $\partial \Sigma \approx \partial \Sigma'$).

We have now found the “Topology” in Topological Quantum Field Theory.

3. Frobenius Algebra

**Definition 5.** If $\mathbb{K}$ is a field, then an algebra $\mathcal{A}$ over $\mathbb{K}$ is a ring and a $\mathbb{K}$-vector space over $\mathbb{K}$ such that $(ka)b = k(ab) = a(kb)$ for all $k \in \mathbb{K}$, $a, b \in \mathcal{A}$. We will assume that all algebras $\mathcal{A}$ have an identity element $1_{\mathcal{A}}$.

The following are examples of algebras. It is clear that these examples satisfy the axioms of an algebra.

- Examples of commutative algebras:
  - **Example 4.** A field $L$ such that $\mathbb{K} \subset L$.
  - **Example 5.** Polynomial algebra $\mathbb{K}[X_1, \ldots, X_n]$. Observe that a polynomial algebra satisfies the requirements of an algebra, but is an infinite-dimensional vector space over $\mathbb{K}$. Thus, there is no requirement that the algebra $\mathcal{A}$ be finite-dimensional.
  - **Example 6.** $\mathbb{K}$ valued functions on a nonempty set $S$.

- Examples of non-commutative algebras:
  - **Example 7.** Matrix algebra $M_n(\mathbb{K})$ where matrix multiplication is the product operation.
  - **Example 8.** $\text{Hom}_\mathbb{K}(V, V)$ for any vector space $V$ with composition being the product operation. Observe that this example and example 7 are equivalent with respect to a choice of basis when $V$ is finite dimensional.

**Definition 6.** Let $\mathcal{A}$ be a $\mathbb{K}$ algebra, and $M$ a vector space over $\mathbb{K}$. Then $M$ is a left $\mathcal{A}$-module if each $a, a' \in \mathcal{A}$, $m, m' \in M$ and $k \in \mathbb{K}$ a product $am \in M$ is defined such that
  - $a(m + m') = am + am'$
  - $(a + a')m = am + a'm$
• \((aa')m = a(a'm)\)
• \(1_{\mathcal{A}}m = m\) where \(1_{\mathcal{A}}\) is the identity element in \(\mathcal{A}\)
• \((ka)m = k(am) = a(km)\)

A right \(\mathcal{A}\)-module is defined analogously.

**Remark 1.** \(\mathcal{A}\) is itself a left \(\mathcal{A}\)-module, denoted \(\mathcal{A}\mathcal{A}\), with canonical left multiplication. This is called the left regular \(\mathcal{A}\)-module. Again, we can define the right regular \(\mathcal{A}\)-module, \(\mathcal{A}\mathcal{A}\) as a right \(\mathcal{A}\)-module with the canonical right multiplication.

**Definition 7.** Let \(\mathcal{A}\) be a \(\mathbb{K}\) algebra and \(M\) a left \(\mathcal{A}\)-module. Let \(M^*\) be the dual of \(M\). Then \(M^*\) becomes a right \(\mathcal{A}\)-module if for \(\psi \in M^*\), \(a \in \mathcal{A}\), \(m \in M\),

\[(\psi a)(m) = \psi(am)\]

The right \(\mathcal{A}\) module \(M^*\) is the dual of \(M\). Similarly, the dual of a right \(\mathcal{A}\)-module is a left \(\mathcal{A}\)-module.

**Theorem 1.** [4] Let \(\mathcal{A}\) be a finite-dimensional algebra over a field \(\mathbb{K}\), then the following are equivalent:

1. For \(\mathcal{A}\mathcal{A}\) and \((\mathcal{A}\mathcal{A})^*\), two left \(\mathcal{A}\)-modules, there exists an \(\mathcal{A}\) algebra isomorphism \(\lambda: \mathcal{A}\mathcal{A} \rightarrow (\mathcal{A}\mathcal{A})^*\).
2. There exists a non-degenerate linear form \(\eta: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathbb{K}\) which is “associative” in the following manner
   \[\eta(ab \otimes c) = \eta(a \otimes bc)\quad \text{for } a, b, c \in \mathcal{A}\.\]
3. There exists a linear form \(f \in \mathcal{A}^*\) whose kernel contains no non-trivial left or right ideals.

**Definition 8.** An algebra that satisfies any of the conditions in Theorem 1 is a Frobenius algebra.

It is important to note that a Frobenius algebra is not a “type” of algebra. Rather, it is an algebra endowed with a given structure. It is an abuse of language to state that “\(\mathcal{A}\) is a Frobenius algebra”. One should rather state that “\(\mathcal{A}\) is an algebra endowed with a Frobenius structure \(\eta\) or \(f\)” In general, we will use the former semi-correct language for the sake of brevity.

**Proof.** **Part 1** \((1) \Rightarrow (2)\):

For \(\lambda: \mathcal{A}\mathcal{A} \rightarrow (\mathcal{A}\mathcal{A})^*\), \(\lambda(ab) = a\lambda(b)\) for all \(a, b \in \mathcal{A}\). Thus for all \(x \in \mathcal{A}\),

\[
\lambda(ab)(x) = (a\lambda(b))(x) = \lambda(b)(xa).
\]

Thus define the linear form \(\eta: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathbb{K}\) where

\[
\eta(x \otimes y) := \lambda(y)(x).
\]
Observe that $\eta$ is non-degenerate since $\lambda$ is an $\mathbb{K}$ isomorphism. In other words, if $\eta(\cdot \otimes y) = 0$ then $\lambda(y) = 0$ and thus $y = 0$. Analogously, $\eta(x \otimes \cdot) = 0$ implies $x = 0$. Associativity $\eta(xy \otimes z) = \eta(x \otimes yz)$ follows from $\lambda(ab)(x) = (a\lambda(b))(xa):

$$
\eta(xy \otimes z) \equiv \lambda(z)(xy) = y\lambda(z)(x) = \lambda(yz)(x) = \eta(x \otimes yz)
$$

**Part 2** ($2 \Rightarrow 1$):

Let $\eta$ be a non-degenerate linear form on $\mathcal{A}$. Define $\lambda : \mathcal{A} \rightarrow \mathcal{A}$ as

$\lambda(y)x := \eta(x \otimes y)$.

$\lambda$ is an $\mathbb{K}$ isomorphism since $\eta$ is non-degenerate and is an $\mathcal{A}$ algebra isomorphism by the associativity of $\eta$.

**Part 3** ($2 \Rightarrow 3$):

Given a linear form $\eta(x \otimes y)$, define a linear function $f \in \mathcal{A}^*$ as

$f(x) := \eta(x \otimes 1_\mathcal{A})$.

Then $f(x\mathcal{A}) = 0$ implies $\eta(x\mathcal{A} \otimes 1_\mathcal{A}) = \eta(x \otimes \mathcal{A}) = 0$. Thus $x = 0$ since $\eta$ is non-degenerate. Likewise, $f(\mathcal{A}x) = 0$ implies $x = 0$. We conclude that the kernel of $f$ contains no non-trivial left or right ideals.

**Part 4** ($3 \Rightarrow 2$):

Given a linear function $f \in \mathcal{A}^*$ with no non-trivial left or right ideals, define

$\eta(x \otimes y) = \eta(x, y) := f(xy)$.

It is clear that $\eta$ satisfies the conditions of (2).

The hypothesis that $f$ does not have a non-trivial left or right ideal is necessary. Suppose that $y$ is in an ideal $I$ contained in the kernel of $f$, then $xy \in I$ implies $f(xy) = 0 = \eta(x \otimes y)$ and $\eta$ is degenerate.

**Definition 9.** A symmetric Frobenius algebra is a Frobenius algebra such that the non-degenerate linear form $\eta$ defined in theorem 1 is induced by a trace map $\theta : \mathcal{A} \rightarrow \mathbb{C}$ where $\eta(x, y) = \theta(xy)$ thus implying $\eta(a, b) = \eta(b, a)$.

**Remark 2.** The term symmetric Frobenius algebra is not the universal term used in defining this structure. Moore, Segal [13], [14], Atiyah [2] and other authors refer to this structure simply as a Frobenius algebra when discussing TQFT. Yet it is clear that a Frobenius algebra differs from a
symmetric Frobenius algebra precisely based on the trace condition which we will see is essential in defining a TQFT. Although symmetric Frobenius algebras are contained in Frobenius algebras, it would be a disservice to the reader if the exact terms used were not clear and if there was a failure to establish the containment of the two types of structures. It should be noted further that Curtis and Reiner [4] among other sources refer to a symmetric Frobenius algebra simply as a symmetric algebra. So as to prevent any possibility of confusion, we will remain abundantly clear by referring to this structure as a symmetric Frobenius algebra, as found in [8].

Clearly if $A$ is a commutative algebra, i.e. $ab = ba$ for all $a, b \in A$ then a commutative Frobenius algebra is a symmetric Frobenius algebra.

There are several examples of a symmetric Frobenius algebra essential to our needs:

**Example 9.** Consider the algebra $M_n(\mathbb{K})$ where $\theta(a) = \text{tr}(a)$, the trace function. $\eta(a \cdot b, c) = \text{tr}((ab)c) = \text{tr}(ab)c = \eta(a, b \cdot c)$

**Example 10.** The field $\mathbb{C}$ is a Frobenius algebra over $\mathbb{R}$ with one possible form $\theta(a + bi) := a$. Another form could be based on a different map. For example consider the form $\theta(2 + 3i) := 7; \theta(1 - i) := 4.[8]$

**Example 11.** Let $G$ be a group (for our purposes, a finite group), $R$ a ring. By considering the set of all $R$ linear combinations of elements of $G$, we can define the group ring $R[G]$. If $a, b$ are elements of $R[G]$, then $a + b$ is defined pointwise, and $a \cdot b$ is defined by the distributive law of the ring.

$\theta(a \cdot b)$ is the coefficient of the identity element of $a \cdot b$. This clearly defines a Frobenius algebra since $\theta(a \cdot b, c)$ is the coefficient of the identity element of $(a \cdot b) \cdot c = a \cdot b \cdot c = a \cdot (b \cdot c)$. The coefficient of the identity element of $a \cdot (b \cdot c)$ is $\theta(a, b \cdot c)$.

Furthermore, $\theta$ is non-degenerate. Suppose for $a \in R[G]$, $\theta(R[G]a) = 0$. Then $\theta(g^{-1}a) = 0$ for each $g \in G$. Thus, $\theta(g^{-1}a)$ is the coefficient of $g$ in $a$, implying that $a = 0$. Similarly, $\theta(aR[G]) = 0$ implies $a = 0$.

**Example 12.** Let $R[G]$ be a group ring, define the center of a group ring as the set of elements in $R[G]$ which commute with all other elements in $R[G]$.

**Claim 1.** $p \in Z(R[G]) \Leftrightarrow p$ is constant over the conjugacy classes of $G$. 
Proof. If \( p = \lambda_1 g_1 + \ldots + \lambda_n g_n \) then for \( h \in G \),
\[
    h \cdot p \cdot h^{-1} = h(\lambda_1 g_1 + \ldots + \lambda_n g_n)h^{-1} \\
    = h\lambda_1 g_1 h^{-1} + \ldots + h\lambda_n g_n h^{-1} \\
    = \lambda_1 hg_1 h^{-1} + \ldots + \lambda_n hg_n h^{-1} \\
    = p \iff \lambda_i = \lambda_j \text{ when } g_j = hg_i h^{-1}
\]

\[ \square \]

**Theorem 2.** [1] If \( \mathcal{A} \) is a Frobenius algebra over a field \( \mathbb{K} \) with form \( f \), then every other Frobenius form on \( \mathcal{A} \) is given by \( u \cdot f \) for \( u \) an invertible element in \( \mathcal{A} \).

Proof. Let \( u \in \mathcal{A} \) be a unit. Then for any \( a \in \mathcal{A} \) where \( u \cdot f(ax) = f(uax) = 0 \) for all \( a \in \mathcal{A} \), \( ua = 0 \) and therefore \( a = 0 \). Thus there are no non-trivial ideals in the kernel and \( u \cdot f \) is a Frobenius form.

Now consider \( g \in \mathcal{A}' \) a Frobenius form not equal to \( f \). Then by the proof in theorem 1, \( g = \lambda(u) = u \cdot f \) for some \( u \in \mathcal{A} \). Since \( g \) is a Frobenius form, the map \( \lambda' := g\beta \) is an isomorphism \( \mathcal{A} \to \mathcal{A}' \), where \( \beta : \mathcal{A} \to \text{End}(\mathcal{A}) \), the map which takes an element \( a \in \mathcal{A} \) to the map “multiplication by \( a \)”. Thus, there is a \( v \in \mathcal{A} \) such that \( f = \lambda'(v) = v \cdot g = vu \cdot f \). Then we see that \( \lambda(1_\mathcal{A}) = f = uv \cdot f = \lambda(vu) \) implies that \( 1_\mathcal{A} = vu \). Since \( \lambda \) is an isomorphism, \( u \) is a unit in \( \mathcal{A} \).

\[ \square \]

**Theorem 3.** If \( \mathcal{A} \) is a Frobenius algebra, then \( \mathcal{A}' \) is a Frobenius algebra.

Proof. Let \( \mathcal{A} \) be a Frobenius algebra with form \( f \in \mathcal{A}' \). By theorem 1, all elements of \( \mathcal{A}' \) are of the form \( a \cdot f \) for \( a \in \mathcal{A} \). As \( \mathcal{A} \) and \( \mathcal{A}' \) are isomorphic, define multiplication in \( \mathcal{A}' \) by \((a \cdot f)(b \cdot f) := (ab \cdot f) \). Now define \( \tau : \mathcal{A}' \to \mathbb{K} \) to be “evaluation at \( 1_\mathcal{A} \)”. Then the identity \( \tau(ax \cdot f) = f(ax) \) lets \( \mathcal{A}' \), with structure \( \tau \), be a Frobenius algebra.

\[ \square \]

**Definition 10.** A coalgebra \( \mathcal{A} \) over a field \( \mathbb{K} \) is a vector space \( \mathcal{A} \) with two \( \mathbb{K} \)-linear maps:
\[
\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \text{ and } \delta : \mathcal{A} \to \mathbb{K}
\]
such that the following diagrams commute
\[
\begin{align*}
    \mathcal{A} \xrightarrow{\alpha} \mathcal{A} \otimes \mathcal{A} \\
    \alpha \downarrow \quad \downarrow \alpha \otimes \text{Id} \\
    \mathcal{A} \otimes \mathcal{A} \xrightarrow{\text{Id} \otimes \alpha} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}
\end{align*}
\]
The map \( \alpha \) is the **comultiplication** map, \( \delta \) is the **counit** map with axioms coassociativity \( [(a \otimes (b \otimes c)) = (a \otimes b) \otimes c] \) and the counit condition \( \mathcal{A} \otimes \mathbb{K} \cong \mathbb{K} \otimes \mathcal{A} \) is a natural isomorphism.

Next, we define a multiplication map \( \beta \) on \( \mathcal{A} \otimes \mathcal{A} \):

\[
\beta : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \\
\sum a_i \otimes b_i \mapsto \sum a_i b_i
\]

which defines a comultiplication map \( \beta^* \) on \( \mathcal{A}^* \):

\[
\beta^* : \mathcal{A}^* \rightarrow \mathcal{A}^* \otimes \mathcal{A}^* \cong (\mathcal{A} \otimes \mathcal{A})^* \\
\sum f \mapsto f \circ \beta
\]

As a Frobenius algebra \( \mathcal{A} \) and \( \mathcal{A}^* \) are isomorphic, \( \mathcal{A} \) is prescribed a coalgebra structure by defining \( \alpha \), the comultiplication map \( (\lambda^{-1} \otimes \lambda^{-1}) \circ \beta^* \circ \lambda \):

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\alpha} & \mathcal{A} \otimes \mathcal{A} \\
\lambda & \downarrow & \downarrow \lambda^{-1} \otimes \lambda^{-1} \\
\mathcal{A}^* & \xrightarrow{\beta^*} & \mathcal{A}^* \otimes \mathcal{A}^*
\end{array}
\]

\( \mathcal{A} \) is coassociative and cocommutative by the definition of \( \alpha \). \( \alpha \) can be also used to define multiplication in \( \mathcal{A}^* \) since \((a \cdot f)(b \cdot f) = [a \cdot f \otimes b \cdot f] \circ \alpha = ab \cdot f \).

Let \( g : \mathbb{K} \rightarrow \mathcal{A} \) denote the unit map. The commutativity of

\[
\begin{array}{ccc}
\mathcal{A}^* & \xrightarrow{\lambda^*} & \mathcal{A}^* \otimes \mathcal{A}^* \\
\lambda & \downarrow & \downarrow \lambda^{-1} \otimes \lambda^{-1} \\
\mathcal{A} & \xrightarrow{f} & \mathcal{K}
\end{array}
\]

\[
\begin{array}{ccc}
f \circ \overline{\beta}(a) & \xrightarrow{a} & \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{K} \\
f(a) = f \circ \overline{\beta}(a)(1_{\mathcal{A}})
\end{array}
\]

guarantees the commutativity of

\[
\begin{array}{ccc}
\mathcal{A}^* & \xrightarrow{\beta^*} & \mathcal{A}^* \otimes \mathcal{A}^* \\
\lambda & \downarrow & \downarrow \lambda^{-1} \otimes \lambda^{-1} \\
\mathcal{A} & \xrightarrow{\alpha} & \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{K} \otimes \mathcal{A}
\end{array}
\]

Since the top row is \( l_{\mathcal{A}^*} \), the bottom row is \( l_{\mathcal{A}} \). Thus \( f \) is the counit in \( \mathcal{A} \).
4. Defining a Topological Quantum Field Theory

We are now in a position to define a Topological Quantum Field Theory. The model to keep in mind is the following: consider a $(\Sigma, X, Y)$ cobordism. A Topological Quantum Field Theory $Z$ assigns complex vector spaces $Z(X)$ and $Z(Y)$ to $X$ and $Y$ respectively. Furthermore, based on the structure of the cobordism from $X$ to $Y$, an element $Z(\Sigma) \in \text{Hom}(Z(X), Z(Y))$ is assigned to $\Sigma$. In its full generality, a TQFT is a functor from $dCOB$ to $B$, a subcategory of $R$ modules over a commutative ring $R$ with unity where disjoint unions correspond to tensor products. The following diagram illustrates the concept.

Consider $S$, the surface of $\Sigma$ as a “function” of time, traveling from $X$ to $Y$. The element $Z(\Sigma) \in \text{Hom}(Z(X), Z(Y))$ can be determined based on the interpretation of information as $S$ propagates from $X$ to $Y$.

Note that a TQFT is defined on a manifold with a fixed dimension.

**Definition 11.** A Topological Quantum Field Theory in dimension $d$ is a monoidal functor $Z: dCOB \rightarrow Vect/K$.

In this case, the monoidal structure that is preserved is the correspondence between disjoint unions in $dCOB$ and tensor products in the category of Linear Algebra. In its fullest generality, a TQFT is a functor to $B$, a subcategory of $R$ modules, where $R$ is a commutative ring with unity.

**Definition 12.** A TQFT is reduced if for some interval $I$, $Z(X \times I) = \text{id}|_{Z(X)}$ for each object $X$.

**Remark 3.** For $\Sigma = X \times I$ for some interval $I$, $Z(\Sigma)^2 = Z(\Sigma)$ since $\Sigma \circ \Sigma \cong \Sigma$. Thus we can suppose that a TQFT is reduced by denoting $\text{id}|_{Z(X)}$ as the image of $X \times I$. Henceforth, we will always assume that a TQFT is reduced.
We will elaborate on the meaning of the definition of a TQFT by showing that this is equivalent to a more expanded definition given by Atiyah [2].

Atiyah [2] states that a Topological Quantum Field Theory in dimension $d$ consists of the following two components and three axioms:

- a finite, positive-dimensional complex vector space $Z(X)$ associated to each oriented $(d - 1)$-dimensional manifold $X$. Under orientation preserving diffeomorphisms of $X$, the vector space behaves functorially.

- an element $Z(\Sigma) \in Z(\partial \Sigma)$ associated to each $d$-dimensional manifold $\Sigma$ with boundary $\partial \Sigma$. Again, under orientation preserving diffeomorphisms of $\Sigma$, $Z(\Sigma)$ behaves functorially.

TQFTs satisfy the axioms laid out by Atiyah[2]:

1. Since $\overline{X}$ is $X$ with opposite orientation, let $Z(\overline{X}) \cong Z(X)^*$ where $Z(X)^*$ is the dual of $Z(X)$. Again, this isomorphism behaves functorially under isomorphisms of $X$.

2. $Z(X_1 \amalg X_2) = Z(X_1) \otimes \mathbb{C} Z(X_2)$.

3. Given $\Sigma'$ and $\Sigma''$ with $\partial \Sigma' = U \amalg V \partial \Sigma'' = V \amalg W$ so that $\Sigma'$ and $\Sigma''$ are sewn along the common boundary $V$ to form $\tilde{\Sigma}$, then $Z(\tilde{\Sigma}) = \langle Z(\Sigma'), Z(\Sigma'') \rangle$ where $\langle \ , \ \rangle$ denotes the natural pairing

$$Z(U) \otimes Z(V) \otimes Z(V)^* \otimes Z(W) \rightarrow Z(U) \otimes Z(W).$$

Thus if $V = \emptyset$, and $\Sigma = \Sigma' \amalg \Sigma''$ then $Z(\Sigma) = Z(\Sigma') \otimes Z(\Sigma'')$.

The functoriality requirement is based on the orientation preserving diffeomorphisms that act on the spaces. If $f: X_1 \rightarrow X_2$ is an orientation preserving diffeomorphism, $f$ induces an isomorphism $Z(f): Z(X_1) \rightarrow Z(X_2)$. Under composition, $Z$ is covariant: for $g: X_2 \rightarrow X_3$ an orientation preserving diffeomorphism, $g \circ f$ induces the composition

$$Z(g \circ f) = Z(g) \circ Z(f): Z(X_1) \rightarrow Z(X_3).$$

If $X_1$ and $X_2$ are boundaries of two $d$ manifolds $\Sigma_1$ and $\Sigma_2$ respectively, and $f$ can be extended to an orientation preserving diffeomorphism from $\Sigma_1$ to $\Sigma_2$, then $Z(f)$ can be extended such that $Z(f): Z(\Sigma_1) \rightarrow Z(\Sigma_2)$.

Several properties can be deduced from Atiyah’s definition that begin to synchronize it with the given definition. These properties are immediate consequences of Atiyah’s definition and illustrate the motivation behind reduced theories. In this regard, remark 3 can actually be seen as a property in Atiyah’s definition as well.

**Property 4.0.1.** For a $(\Sigma, X, Y)$ cobordism, $Z(\Sigma) \in \text{Hom}(Z(X), Z(Y))$. 

Lemma 1. If $V$ and $W$ are finite-dimensional vector spaces over a field $\mathbb{K}$ and $V^*$ is the dual space of $V$, then $V^* \otimes_{\mathbb{K}} W \cong \text{Hom}_{\mathbb{K}}(V, W)$. The isomorphism is via the map

$$T : V^* \otimes W \longrightarrow \text{Hom}(V, W)$$

$$\phi \otimes w \longmapsto T_{\phi \otimes w}$$

where

$$T_{\phi \otimes w}(v) = \phi(v)w$$

Proof. (of Lemma 1): Let $\{v_i\}$ and $\{w_j\}$ be the bases for $V$ and $W$ respectively and let $\{v^*_i\}$ be the basis for $V^*$. Then $V^* \otimes_{\mathbb{K}} W$ has basis $\{v^*_i \otimes_{\mathbb{K}} w_j\}$ with

$$\dim (V^* \otimes_{\mathbb{K}} W) = \dim (V^*) \dim (W).$$

Next, observe that

$$\dim (\text{Hom}_{\mathbb{K}}(V, W)) = \dim (V) \dim (W) = \dim (V^*) \dim (W).$$

As $\dim (V^* \otimes_{\mathbb{K}} W) = \dim (\text{Hom}_{\mathbb{K}}(V, W))$ and both are vector spaces over $\mathbb{K}$, these vector spaces are isomorphic since all vector spaces over a given field with the same dimension are isomorphic.

Proof. (of Property 4.0.1):

$$Z(\partial \Sigma) = Z(\overline{X} \cup Y)$$

$$= Z(\overline{X}) \otimes Z(Y)$$

$$= Z(X)^* \otimes Z(Y)$$

$$\cong \text{Hom}_{\mathbb{C}}(Z(X), Z(Y))$$

Observe, it is essential that $Z(\cdot)$ be a finite-dimensional complex vector space.

Property 4.0.2. If the empty set is considered as a closed $(d-1)$-dimensional manifold, then $Z(\emptyset) = \mathbb{C}$.

Proof.

$$Z(X) = Z(X \cup \emptyset) = Z(X) \otimes_{\mathbb{C}} Z(\emptyset).$$

Thus

$$\dim(Z(X)) = \dim(Z(X) \otimes_{\mathbb{C}} Z(\emptyset)) = \dim(Z(X)) \cdot \dim(Z(\emptyset)).$$

Therefore $Z(\emptyset)$ must be a 1-dimensional vector space over $\mathbb{C}$.
Remark 4. The two previous properties show that if $\Sigma$ is a closed manifold a TQFT calculates an invariant $\Psi_\Sigma$ associated to a closed manifold $\Sigma$. This invariant can be computed by decomposing $\Sigma$ into a series of manifolds that are easier to compute.

Property 4.0.3. If the empty set is considered as a closed $d$-dimensional manifold, then writing the empty set as $\emptyset \times I$, where the empty set has dimension $(d-1)$ and $I$ is an interval, we get $Z(\emptyset \times I) = 1_{Z(\emptyset)} = 1 \in \mathbb{C}$.

The notation, while at the outset may seem ambiguous, is standard in Atiyah’s work. There are attempts to clarify the notation but those endeavors lead to confusion in other discussions on TQFTs.

Proof. 
\[
Z(\emptyset) \in Z(\partial \emptyset) = \text{Hom}_C(\mathbb{C}, \mathbb{C}).
\]
So let $Z(\emptyset) = c \in \mathbb{C}$. This yields the following relation:
\[
c = Z(\emptyset) = Z(\emptyset \circ \emptyset) = Z(\emptyset) \cdot Z(\emptyset) = c^2
\]
Since we have a reduced TQFT based on Remark 3, we exclude the trivial case, and let $c = 1$. □

Property 4.0.4. If $\partial \Sigma = (\bigcup_{i=1}^k X_i) \sqcup (\bigcup_{j=1}^l X'_j)$ then
\[
Z(\Sigma) \in \text{Hom}_C(\bigotimes_{i=1}^k Z(X_i), \bigotimes_{j=1}^l Z(X'_j)).
\]
Proof. Follows inductively from the axioms. □

Property 4.0.5. If $\Sigma$ is a closed manifold (a manifold with empty boundary), then $Z(\Sigma) \in Z(\partial \Sigma) = \mathbb{C}$.

Proof. If $\Sigma$ is a closed manifold, then $Z(\Sigma) \in Z(\partial \Sigma) = Z(\emptyset) = \mathbb{C}$. □

4.1. Connecting to Atiyah. Aside from not using the power of category theory to prove the above properties, Atiyah’s definition is distinct from the given one in one crucial manner. In terms of associating sewing with composition, a priori, Atiyah’s definition provides us with more freedom than the given definition. Resolving the differences allows us to bring the two definitions in line, and we can thus understand how they are equivalent.

Consider the following situation in which one would want to sew a right-pair of pants to one leg of a right-pair of pants. Atiyah’s definition does not restrict this action whereas the given definition does. In order to resolve the foregoing impasse of our definitions, one would sew a cylinder to the other “leg” of the pants. This acts as an identity on the other leg, which does not effect the calculation of the TQFT.
In the general case, given a \((\Sigma, X, Y)\) cobordism, sewing a surface to \(Y'\), a subset of \(Y\), can be achieved by attaching identity cobordisms to \(Y \setminus Y'\).

5. **Two-Dimensional Topological Quantum Field Theories**

We now prove that a two-dimensional TQFT is equivalent to a commutative symmetric Frobenius algebra.

**Definition 13.** For \(M\) an object in \(2\text{COB}\), a component of \(M\) is a closed string if it is homeomorphic to a circle and an open string if it is homeomorphic to a compact line segment.

The objects in the category \(2\text{COB}\) are therefore composed of both open and closed strings\(^1\). We can define a sub-category \(2\text{COB}|_C\) to be the subcategory whose objects are restricted exclusively to closed strings, earlier defined as \(2\text{COB}\). It should be emphasized that what was earlier defined to be \(d\text{COB}\) is now \(d\text{COB}|_C\), while the expansion of the category to include open strings is now \(d\text{COB}\). All results from section 4 still hold for \(d\text{COB}\).

The reader should note that objects with open strings in \(2\text{COB}\) are equivalence classes of manifolds with corners. The structure and detail in using the term will be suppressed for the sake of brevity and simplicity without losing much rigor in this work.

5.1. **Two-Dimensional TQFTs with Closed Strings.** At this time, we will consider surfaces (two manifolds) whose boundary is a disjoint union of closed strings. By expanding the Classification Theorem found below, we

\(^1\)The objects are isomorphism classes of 1-dimensional compact manifolds which reduce to two equivalence classes represented by an open string (compact line segment) and a closed string \((S^1)\).
follow with a generalization. This paves the way for the main theorem of the section.

**Theorem 4.** Classification Theorem. Let $\Sigma$ be a compact two-dimensional closed manifold. Then $\Sigma$ can be classified, up to diffeomorphism, according to its genus and orientability up to diffeomorphism.

*Proof.* The proof can be found in [12]. \hfill $\Box$

**Theorem 5.** Let $\Sigma$ be a two-dimensional compact orientable manifold whose boundary is a disjoint union of closed strings, then $\Sigma$ can be categorized according to its genus and the number of closed strings in its boundary. Thus we say that $\Sigma$ is a $(\Sigma, g, k)$ manifold (with genus $g$ and number of closed strings $k$).

*Proof.* Fix an orientation on $\Sigma$, a closed 2 manifold with genus $g$, i.e. a $(\Sigma, g, 0)$ manifold. Then given a $(\Sigma', g, k)$ 2 manifold, form $\Sigma'$ from $\Sigma$ by removing $k$ open disks.

If $(\Sigma', g, k)$ is a 2 manifold and $(\Sigma, g, l)$ is a 2 manifold where $k \neq l$, then $\Sigma'$ and $\Sigma$ are not equivalent as they have non-isomorphic fundamental groups.

Conversely, given an $(\Sigma', g, k)$ 2 manifold, by sewing $k$ open disks to each closed string (essentially closing each boundary) we form an oriented closed 2 manifold $\Sigma$ of genus $g$, which can be classified. \hfill $\Box$

**Theorem 6.** Every $(\Sigma, g, k)$ manifold can be formed from sewing copies of cups and pairs of pants.

*Proof.* Utilizing a cup and pair of pants, we proceed to construct any $(\Sigma, g, k)$ manifold by induction on $g$ and $k$. Observe that this construction is not unique.

- If $g = 0$, sew $k - 2$ pants as follows

  ![Diagram](https://via.placeholder.com/150)

  If $k < 3$ sew the necessary cups.

  ![Diagram](https://via.placeholder.com/150)

- If $g = 1$, sew pants together in the following manner
then sew a $g = 0$ piece with the appropriate number of $k$’s.

- If $g > 1$ sew the appropriate number of $g = 1$ pieces then sew a $g = 0$ piece with the necessary $k$.

- If $(\Sigma, g, k) = \bigcup_{i=1}^{n} (\Sigma_i, g_i, k_i)$ where $\sum_{i=1}^{n} g_i = g$, $\sum_{i=1}^{n} k_i = k$ then construct $(\Sigma, g, k)$ by constructing each $(\Sigma_i, g_i, k_i)$.

\[ \square \]

**Remark 5.** Although a $(\Sigma, g, k)$ manifold can be made in this manner, we have seen for example that a $(\Sigma, 0, 3)$ manifold can be realized as a left or right pair of pants. As cobordisms, they are not equivalent. Each can be formed by sewing the appropriate $g = 0$ pieces with the appropriate number of $k$’s to either end of the formed surface with genus $g$.

Abrams [1] utilizes the left and right caps and pants as well as the cylinder as generators with the following relations to solidify the previous theorem in proposition 12. These generators and relations will allow the subsequent theorem to be proven almost effortlessly.
Theorem 7. Every two-dimensional TQFT \( Z : 2\text{COB}_{|C} \to \text{Complex Vector Spaces} \) is equivalent to a symmetric commutative Frobenius algebra \( \mathcal{A} \) which is finite-dimensional over \( \mathbb{C} \).

The spirit of the proof shows that Abrams’ generators and relations that \( 2\text{COB}_{|C} \) adheres to are paralleled in a Frobenius Algebra. A two-dimensional TQFT is a functor from \( 2\text{COB} \) to the category of Complex Vector Spaces. This theorem is being restricted to the subcategory \( 2\text{COB}_{|C} \). Pedantically, a TQFT over open/closed or just closed strings will be used to indicate the appropriate domain, \( 2\text{COB} \) or \( 2\text{COB}_{|C} \), respectively.

This proof is a sketch of the ideas of the equivalence of the categories. The actual proof utilizes Morse theory and will not be explored in this paper. Lauda and Pfeiffer [9] utilize Morse theory to describe the generators and relations that fully determine \( 2\text{COB} \).

**Proof. Part 1:** We show that a TQFT gives rise to a commutative symmetric Frobenius algebra \( \mathcal{A} \) in a natural way.

It suffices to simply consider a cap and pair of pants as these are the fundamental manifolds which when sewn together can form any relevant \((\Sigma, g, k)\) manifold. [1]

Since \( \mathbb{S}^1 \) is the only closed connected oriented 1 manifold, define the algebra \( \mathcal{A} \) as a vector space where \( Z(\mathbb{S}^1) = \mathcal{A} \).

A right-cap \( R \) is a cobordism from \( \mathbb{S}^1 \) to \( \emptyset \), so associate the cobordism \( R \) with the trace map \( \theta : \mathcal{A} \to \mathbb{C} \), so \( Z(R) = \theta : \mathcal{A} \to \mathbb{C} \).}

\[\mathcal{A} \xrightarrow{\theta} \mathbb{C}\]

A left-cap \( L \) is a cobordism from \( \emptyset \) to \( \mathbb{S}^1 \), thus the unit map \( \mathbb{C} \to \mathcal{A} \) is associated to the element \( Z(L) \in \text{Hom} (\mathbb{C}, \mathcal{A}) \) such that \( 1 \mapsto 1_{\mathcal{A}} \). To prove this we observe that \( \emptyset \sqcup \mathbb{S}^1 \cong \mathbb{S}^1 \), therefore the following objects are equivalent in \( d\text{COB} \):
If \( \phi(1) \) is the image of 1 of the left-cap and \( \vec{v} \in \mathbb{Z}(S^1) \), then
\[
1 \mapsto \phi(1) \otimes \vec{v} \mapsto \phi(1) \cdot \vec{v}
\]

These maps are identical to the identity map
\[
\vec{v} \mapsto \vec{v}.
\]

So we can conclude \( \phi(1) = 1_\mathcal{A} \).

A left-pair of pants considered as a cobordism from \( S^1 \) to \((S^1 \amalg S^1)\) is the product \( \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \).

A right-pair of pants considered as a cobordism from \( S^1 \) to \((S^1 \amalg S^1)\) is associated to comultiplication \( \alpha : \vec{v} \mapsto \vec{v} \otimes \vec{v} \) which can be defined as above.

Commutativity can be verified when considering the topology of a cobordism

\[
S^1 \amalg S^1 \rightarrow S^1 = S^1 \amalg S^1 \rightarrow S^1.
\]

Without regard to the ambient space, there exists a homeomorphism from a pair of pants onto itself which gives the necessary relations.

The Frobenius property can again be verified by considering that the topology of the manifold does not change. The base case is seen below where we have the cobordism

\[
(S^1 \amalg S^1) \amalg S^1 \rightarrow S^1 = S^1 \amalg (S^1 \amalg S^1) \rightarrow S^1.
\]
Part 2: A symmetric Frobenius algebra gives rise to a TQFT: As part 1 is bijective, the axioms and properties of a symmetric Frobenius algebra manifest themselves in a TQFT. Using left and right-pairs of pants, left and right-caps and cylinders one can generate each element in 2COB in the spirit of theorem 6. The relations given above by Abrams, complete the sketch of the proof showing that two diffeomorphic manifolds are the manifestation of isomorphic symmetric Frobenius algebras.

\[ \Box \]

5.2. Properties of 2-dimensional TQFT’s Over Closed Strings.

Property 5.2.1. Let \( T \) be a manifold defined by a torus with one outgoing boundary circle. Let \( \alpha \in \mathcal{A} \) be an element such that \( Z(T) = \alpha \). Then If \( K \) is a \((K, g, 0)\) surface, \( Z(K) = \theta(\alpha^g) \).

**Proof.** This can be done by induction.

Let \( g = 1 \) then \( K \) is composed of \( T \) sewn to a left-cap \( X \). We can now follow the formulations given to associate a symmetric Frobenius algebra with a 2-dimensional space. Therefore

\[
Z(K) \in \text{Hom} (Z(\emptyset), Z(S^1), Z(S^1)^*, Z(\emptyset)) = \theta(Z(T)) = \theta(\alpha).
\]

Let \( g \in \mathbb{N} \) then \( K \) is composed of \( (K_{g-1} \amalg T) \) sewn to a left-pair of pants \( X \), sewn to a left-cap \( X \). Then

\[
Z(K) = \text{Hom} (Z(K_{g-1} \amalg T), Z(X_1), Z(X_2))
\]

\[
= \text{Hom} \left( (Z(K_{g-1}) \otimes Z(T)), Z(X_1), Z(X_2) \right)
\]

\[
= \theta(\alpha^{g-1} \times \alpha)
\]

\[
= \theta(\alpha^g)
\]

**Property 5.2.2.** If \( \{e_i\} \) is a vector space basis for \( \mathcal{A} \), and \( \{e^*_i\} \) is the dual basis, then \( \alpha(1) = \sum e_i \lambda^{-1}(e^*_i) \) where \( \lambda: \mathcal{A} \to \mathcal{A}^* \) and \( \lambda^{-1}: \mathcal{A}^* \to \mathcal{A} \).
**Proof.** We begin by decomposing $T$ into a left-cap sewn to a right-pair of pants sewn to a left-pair of pants.

Following the cobordisms and the associated TQFT, we get

$$Z(\emptyset) \to Z(S^1) \to Z(S^1 \coprod S^1) \to Z(S^1)$$

which is associated algebraically to

$$\mathbb{C} \to \mathcal{A}^* \to (\mathcal{A} \otimes \mathcal{A}) \to \mathcal{A}.$$ 

Sewing the left cap to the right-pair of pants, we are left with the following maps:

$$\mathbb{C} \to (\mathcal{A} \otimes \mathcal{A}) \to \mathcal{A}.$$ 

If we follow where the element 1 maps to:

$$\alpha(1) = (\lambda^{-1} \otimes \lambda^{-1}) \circ \beta^* \circ \lambda(1)$$

$$= (\lambda^{-1} \otimes \lambda^{-1}) \theta$$

$$= (\lambda^{-1} \otimes \lambda^{-1}) \circ \Sigma(e_i^* \otimes \lambda(e_i))$$

$$= \sum \lambda^{-1}(e_i^*) \otimes e_i$$

Our remaining map

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

$$\sum e_i \otimes \lambda^{-1}(e_i^*) \mapsto \sum e_i \lambda^{-1}(e_i^*)$$

completes the proof. □

**Property 5.2.3.** For $\pi: \mathcal{A} \to \text{End } (\mathcal{A})$, the left regular representation, $\theta(aa) = \text{tr}(\pi(a))$ for $a \in \mathcal{A}$, tr the trace map.

**Proof.** The following set of maps

$$\mathcal{A} \xrightarrow{\pi} \text{End } (\mathcal{A}) \cong \mathcal{A} \otimes \mathcal{A}, \xrightarrow{\lambda \otimes \lambda^{-1}} \mathcal{A} \otimes \mathcal{A}$$

defines a coproduct map $\Psi$ of $\mathcal{A}$. The composition $\text{tr} \circ \pi$ is equal to

$$\mathcal{A} \xrightarrow{\Psi} \mathcal{A} \otimes \mathcal{A} \xrightarrow{(\cdot,\cdot)} \mathbb{C}.$$ 

Since

$$\langle (\cdot,\cdot) \rangle = \theta \circ \mu, \text{tr}(\pi(a)) = \theta \circ \mu \Psi(a),$$

we can rewrite this sequence as

$$(\theta \circ \mu \circ \alpha(\otimes id_{\mathcal{A}}))(a) = \theta(aa),$$

thus proving the property. □
Property 5.2.4. \( \mathcal{A} \) is semi-simple, i.e., a sum of copies of \( \mathbb{C} \), if and only if \( \alpha \) is invertible.

Proof. Since \( \theta(a, b) \) is non-degenerate for \( a, b \in \mathcal{A} \), then the trace function \( (a, b) \mapsto \text{tr}(a)\text{tr}(b) \) is non-degenerate if and only if \( \alpha \) is invertible since for non-zero \( a, b \in \mathcal{A} \) we have
\[
0 \neq \theta(a, b) = \theta(ab\alpha^{-1}) = \text{tr}(ab\alpha^{-1}),
\]
which implies that the finite-dimensional algebra is semi-simple by non-degeneracy.

\[\square\]

Property 5.2.5. If the characteristic polynomial of \( \alpha \) is \( \chi(t) = \prod(t - \lambda_i) \) then \( \Psi_g = \sum \lambda_i^{g-1} \) for \( g \geq 1 \).

Proof.
\[
\Psi_g = \theta(\alpha^g) = \theta(\alpha^g\alpha) = \text{tr}(\alpha^{g-1}) = \sum \lambda_i^{g-1}
\]
since \( \alpha \) can be written as a diagonal matrix of its eigenvalues.

\[\square\]

Property 5.2.6. If \( g = 1 \) then \( \Psi_1 = \theta(\alpha) = \theta(\sum e_i \lambda_i^{-1}(e_i^*)^g) = \dim(\mathcal{A}). \)

This property can be visualized as the identification of the ends of a cylinder to form a torus. Once formed, the torus can be decomposed into a left-cap, right-pair of pants, two cylinders, a left-pair of pants and a right-cap.

5.3. Two-Dimensional TQFTs with Open Strings. In a parallel development, we expand the results of studying 2-dimensional TQFTs defined on \( 2\text{COB}|_{lc} \) to all of \( 2\text{COB} \). We begin with a simpler construction and then introduce boundary conditions on the boundaries of the open strings. If boundary conditions are not considered (i.e. all boundary conditions are trivial), then certain properties that will be illustrated will be straightforward. If the reader prefers, they may ignore the boundary conditions of the open strings when they are introduced. The absences of boundary conditions provide a simpler model for the open string case.

We begin by expanding the definition of a cobordism.

Definition 14. A cobordism is a 4-tuple \( (\Sigma, X, Y, \partial_{exp} \Sigma) \) such that the boundary of \( \Sigma \) is the disjoint union of \( X \) and \( Y \), where \( \overline{X} \) indicates the orientation reversal of \( X \). \( \partial_{exp} \Sigma \) is the constrained boundary, a cobordism from \( \partial X \) to \( \partial Y \). In the case where \( \partial X = \partial Y = \emptyset \) as before, we will only refer to the 3-tuple \( (\Sigma, X, Y) \).

Similar to a \( (\Sigma, X, Y) \) cobordism, in a \( (\Sigma, X, Y, \partial_{exp} \Sigma) \), the direction in which the cobordism propagates will be determined by the orientation of \( \Sigma \).
Example 13. \((\Sigma, 1 1 \Pi 1 1, 1 1, 1 \Pi 1)\) is a cobordism from two line segments to one. So the reader does not confuse the lack of boundary conditions with a cobordism whose boundaries are closed strings, the identity element 1 is considered the boundary. Again, it is necessary to emphasize the order in which the disjoint union of 1-dimensional manifolds appear.

The introduction to Moore and Segal’s paper [13] is concerned with a generalization of an open string theory, where the constrained parts of the boundaries of morphisms are equipped with boundary conditions. The set of boundary conditions is itself a linear category, where the composition corresponds to

We now introduce boundary conditions, where \(B_0\) is a fixed linear category.

Definition 15. a. A \(B_0\) decorated string is an oriented compact 1-manifold (with or without boundary) with a labelling of the boundary components by elements of \(B_0\).
b. A \(B_0\) decorated morphism is a cobordism as in definition 14 with the labeling of the connected components of the constrained boundary by elements of \(B_0\).
Definition 16. $B_0$-decorated 2COB [2COB with boundary conditions] is the category whose objects consist of $B_0$-decorated strings (up to isomorphism), and the morphisms consist of equivalence classes of $B_0$-decorated morphisms, where the labelling of the morphism is consistent with the labelling of the unconstrained part of the boundary (up to isomorphism).

Example 14. $(\Sigma, ab \sqcup bc, ac, a \sqcup c)$ is a cobordism from the line segments $ab \sqcup bc$ to the line segment $ac$. Observe that $a$ and $c$ are the constrained boundaries - cobordisms from $\partial(ab \sqcup bc)$ to $\partial(ac)$. $a$, $b$ and $c$ will also be referred to as boundary conditions. It is particularly important to heed the orientation of the line segments. A line segment $ab$ is a cobordism from $b$ to $a$, whereas a line $ba$ is a cobordism from $a$ to $b$. Moreover, it is necessary to emphasize the order in which the disjoint union of 1-dimensional manifolds appear.

Henceforth, consider surfaces whose boundaries are both open and closed strings, i.e. objects and morphisms in the category 2COB. Since this section succeeds the results 2-dimensional TQFT’s with closed strings, many of the results will be straightforward or will echo previous developments.

Theorem 8. Expanding on theorem 5, let $\Sigma$ be a two-dimensional compact orientable manifold whose boundary is a disjoint union of closed and open strings, then $\Sigma$ can be categorized according to its genus and the number of closed and open strings in the boundary. Thus we say that $\Sigma$ is a $(\Sigma, g, k, l)$ manifold (with genus $g$, number of closed strings $k$ and number of open strings $l$).

Again, in the spirit of remark 5, two different cobordisms can be formed from a $(\Sigma, g, k, l)$ manifold.

Proof. Following theorem 5, form a $(\Sigma, g, k + l)$ manifold. Then glue $l$ of the following pieces to $l$ closed strings.
Definition 17. A $B_0$-decorated topological quantum field theory is a functor from $B_0$-decorated $2$COB to the category of linear algebra, where disjoint unions correspond to tensor products.

In the two-dimensional case, by property 4.0.1, a cobordism with open strings in the boundary gives a linear map:

\[
O_{ab} \otimes O_{bc} \rightarrow O_{ac}
\]

Observe that $ab$ is a cobordism from $b$ to $a$. Thus, we consider the vector space $Z(ab) = \text{Hom}(b, a) = O_{ab}$ to be the set of morphisms from $b$ to $a$. We will always assume that the theories are reduced as in Remark 3.

Theorem 9. In the undecorated category $2$COB, every two-dimensional $\text{TQFT } Z : 2$COB $\rightarrow$ Complex Vector Spaces is equivalent to the following algebraic structure:

- A finite-dimensional, symmetric and commutative Frobenius algebra over $\mathbb{C}$ with a non-degenerate trace map $\theta_A : \mathcal{A} \rightarrow \mathbb{C}$.
- A finite-dimensional, symmetric, not necessarily commutative Frobenius algebra over $\mathbb{C}$ with a non-degenerate trace map $\theta_{Z(I)} : Z(I) \rightarrow \mathbb{C}$.
- A homomorphism $\iota : \mathcal{A} \rightarrow Z(I)$ such that $\iota(1_{\mathcal{A}}) = 1_{Z(I)}$ and the image of $\mathcal{A}$ is in the center of $Z(I)$.

The algebraic consequences and properties of the theorem are outlined in section 5.4. The sketch of the proof is provided below. As there is no generalization of Morse Theory to manifolds with corners, Lauda and Pfeiffer’s proof [9] of the equivalence of these categories is illuminating and should be consulted for a greater understanding of the subject. Their account of the necessity and sufficiency of the generators and relations in the open / closed case nicely generalizes the closed string case. The reader should note that the conditions of the algebraic structure are defined by Lauda and Pfeiffer to be a knowledgeable Frobenius Algebra. The term knowledgeable comes from the mapping of $\mathcal{A}$ into the center of $Z(I)$ indicating that $\mathcal{A}$ gives some information on the structure of $Z(I)$.

Proof. Part 1: Again, we show that a $\text{TQFT}$ gives rise to a noncommutative symmetric Frobenius algebra in a natural way. We consider the open string
manifolds analogous to the closed string cap and pants. The only argument that is not analogous is the noncommutativity of the symmetric Frobenius algebra.

Consider the “flat right-cap” as a cobordism from $aa$ to $\emptyset$, associate to it the nondegenerate trace $\theta_a : O_{aa} \to \mathbb{C}$ on $O_{aa}$.

\[
O_{aa} \longrightarrow \mathbb{C}
\]

Noncommutativity is clear when one realizes that due to the boundary conditions, there is no homeomorphism from a manifold $M$ to itself that will give the necessary conditions. An attempt to imitate the closed string condition does not yield the desired result. If a TQFT over open strings were commutative, then for $\psi \in O_{ba}$ and $\phi \in O_{ab}$,

\[
\theta_b(\psi\phi) = \theta_a(\phi\psi)
\]

a priori, there is no reason for the above statement to hold true in all cases, and thus the Frobenius algebra is said to be not necessarily commutative.

**Part 2**: Due to the complicated nature of the objects and morphisms in 2COB, it is difficult to confirm that a list of generators and relations is complete. Lauda and Pfeiffer accomplish this goal by describing a normal form for a cobordism in 2COB (similar to the categorization of theorem 8). Next they construct moves that change a given cobordism to its normal form. Thus the relations found below along with those from the closed string case generate all objects in 2COB.
5.4. Properties of 2-dimensional TQFT’s Over Open Strings. Defining a TQFT for decorated open strings is trickier. Considerations need to be made since the objects in 2COB are also cobordisms. In a manner of speaking, a TQFT for decorated strings needs to be defined inductively to ensure that for an object \( ba \in 2\text{COB} \), which is a one-dimensional cobordism, the properties of a TQFT are also taken into account. Although we can now specify the category \( B \). Objects are the class of boundary conditions. The set of morphisms from \( b \) to \( a \) is the vector space \( O_{ab} \). Composition of morphisms is given by the relation above in equation (1). For the following properties, we will generalize to decorated open strings so the reader can appreciate where the boundaries need to be compatible. The relations given earlier are given alongside the algebraic information.
Property 5.4.1. To limit ourselves to a reduced TQFT in the spirit of Remark 3, we see that comparable to the cylinder for closed strings, a cobordism \((\Sigma, O_{ba}, O_{ba}, a \amalg b)\) acts as an identity element on \(O_{ba}\).

Open strings and closed strings are necessary to define a TQFT over an open string. Thus we relate them using the following properties.

Property 5.4.2. There exists a linear map

\[
\iota^a : O_{aa} \rightarrow \mathcal{A}
\]

Property 5.4.3. There also exists a linear map

\[
i_a : \mathcal{A} \rightarrow O_{aa}
\]

\(i_a\) has the following properties:

Property 5.4.4. \(i_a\) is an algebra homomorphism since for \(\phi_i \in \mathcal{A}\),

\[
i_a(\phi_1 \phi_2) = i_a(\phi_1)i_a(\phi_2).
\]

Property 5.4.5. \(i_a\) preserves the identity in the sense that \(i_a(1_\mathcal{A}) = 1_a\).

Property 5.4.6. \(i_a\) is central in that it maps into the center of \(O_{aa}\). \(i_a(\phi)\psi = \psi i_a(\phi)\) for all \(\phi \in \mathcal{A}, \psi \in O_{aa}\).
Property 5.4.7. \( t_a \) and \( t^a \) are adjoints \( \theta(t^a(\psi)\phi) = \theta_a(\psi t_a(\phi)) \) for all \( \psi \in O_{aa}, \phi \in \mathcal{A} \).

Property 5.4.8. The Cardy Condition: For a space \( O_{ab} \) with basis \( \psi_\mu \), then \( O_{ba} \) is its dual space with basis \( \psi^\nu \). Let \( \pi^a_b : O_{aa} \to O_{bb} \) be defined in the following manner.

\[
\pi^a_b(\psi) = \sum_p \psi_\mu \psi^\nu.
\]

This implies the Cardy condition

\[
\pi^a_b = t_b \circ t^a.
\]

We can understand the Cardy condition with the use of the following diagram:

The associated algebra is as follows:

\[
O_{aa} \rightarrow O_{ab} \otimes O_{ba} \rightarrow O_{ba} \otimes O_{ab} \rightarrow O_{bb}
\]

so that an element \( \psi \) gets mapped as follows:

\[
\psi \rightarrow c_{\mu\nu} \psi_\mu \otimes \psi^\nu \rightarrow c_{\mu\nu} \psi^\nu \otimes \psi_\mu \rightarrow c_{\mu\nu} \psi^\nu \psi_\mu
\]

To answer the question of what \( c_{\mu\nu} \) is, we follow the following diagram:
The diagram on the left is a map from $O_{ab} \otimes O_{ba} \otimes O_{aa}$ to $\mathbb{C}$. Apply the map to basis elements $\psi^0 \otimes \psi^{\mu_0} \otimes \psi$ for some fixed basis elements $\psi^0$, $\psi_{\mu_0}$ and $\psi \in O_{aa}$. Since the diagram on the right is equivalent to the diagram on the left, the triple $(\phi, \psi^0, \psi_{\mu_0})$ maps to $\theta_b(\psi_{\mu_0} \psi^{\mu_0})$. We can therefore conclude that $c_{\mu_0 \nu_0} = \theta_b(\psi_{\mu_0} \psi^{\mu_0})$.

We can validate this idea by checking that $\theta_b(\psi_{\mu_0} \psi^{\mu_0}) \psi_v$ equals $\psi_\mu \psi_v$. Begin by setting $\psi_\mu \psi_v = \psi \in O_{ba}$. Then we have $\theta_b(\phi \psi^\nu) \psi_v = \phi$ but we know that $\phi = \sum \phi^v \psi_v$ so $\phi^v = \theta_b(\phi \psi^\nu)$.

It is clear that one can consider a TQFT over closed strings as a restriction of a TQFT over open strings. What is unclear is whether the converse is true. Moore and Segal explore this question and use K-theory to illustrate results related to this question in [13].

6. Example: The Dijkgraaf-Witten Toy Model

There is an example of an $n$-dimensional TQFT associated to a finite group $G$, the Dijkgraaf-Witten Toy model.

**Definition 18.** Let $M$ be a topological space, $\mathbb{K}$ a field. Then an $\mathbb{K}$ vector bundle of rank $k$ over $M$ is a topological space $E$ with a surjective continuous map $\pi: E \to M$ such that

- for each $x \in M$, the set $E_x = \pi^{-1}(x) \subset E$, the fiber of $E$ over $x$, is endowed with the structure of a $k$-dimensional $\mathbb{K}$ vector space so that $E = \bigcup_{x \in M} E_x$.
- for each $x \in M$, there exist a neighborhood $U$ of $x$ in $M$ and a homeomorphism $\Phi: \pi^{-1}(U) \to U \times \mathbb{K}^k$ (called a local trivialization of $E$ over $U$) such that the following diagram commutes:

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\phi} & U \times \mathbb{K}^k \\
\downarrow & & \downarrow \pi_1 \\
E & \xrightarrow{\pi} & U
\end{array}
\]

where $\pi_1$ is projection on the first factor such that for each $q \in U$ the restriction of $\Phi$ to $E_q$ to $\{q\} \times \mathbb{K}^k \cong \mathbb{K}^k$ [11].
Definition 19. Let $G$ be a fixed finite group and $P$ a fiber bundle over a manifold $M$. If $G$ acts on $P$ from the right, then $P$ is a principal $G$ bundle if the following properties hold:

- Any fiber $\pi^{-1}(U) \cong G$
- $P \times G \to P$
- $(p, g) \mapsto pg$
- $p(g_1g_2) = (pg_1)g_2$
- If $pg_1 = pg_2$ for some $p \in P$ then $g_1 = g_2$

In the toy model, for $\Sigma$ a closed manifold, $\Psi_\Sigma$ is the sum of isomorphism classes of principal $G$ bundles over $\Sigma$ with weight $1/|\text{Aut } P|$, where $\text{Aut } P$ is the group of automorphisms of $P$ covering $\text{id}_\Sigma$. Define

$$\Psi_\Sigma := \sum_{[P]} \frac{1}{|\text{Aut } P|}$$

where $[P]$ are the isomorphism classes of principal $G$ bundles over $\Sigma$.

Reducing to the case where $\Sigma$ is connected, there exists the following correspondence.

Lemma 2. There is a bijection between the isomorphism classes of principal $G$ bundles $[P]$ over $\Sigma$ and homomorphisms $\rho \in \text{Hom} (\pi_1(\Sigma), G)$ modulo conjugacy.

Proof. Let $\hat{\Sigma}$ be the universal cover of $\Sigma$, $\rho : \pi_1(\Sigma) \to G$ a homomorphism, and $p : P \to \Sigma$ a surjective, continuous map. As $\hat{\Sigma}$ is the universal cover, $\pi_1(\Sigma)$ acts on $\hat{\Sigma}$. Consider the space $\hat{\Sigma} \times_\rho G \cong \hat{\Sigma} \times G/\sim_\rho$ where if $\alpha \in \pi_1(\Sigma)$, $(n, g) \sim (an, \rho(\alpha)g)$. Define $P \cong \hat{\Sigma} \times_\rho G$.

It remains to show that $P$ is a principal bundle. $G$ acts on $P$ from the right since for $h \in G$,

$$[(an, \rho(\alpha)g)] \cdot h \sim [(n, g)] \cdot h = [(n, gh)] \sim [(an, \rho(\alpha)gh)].$$

Finally, we show that $P/G \cong \Sigma$ by first considering elements in $\hat{\Sigma} \times G$ modulo $G$. We have that $[(n, g)] \sim [(n, 1)]$. Taking $\rho$ into account, $[(n, 1)] \sim [(an, \rho(\alpha))] \sim [(an, 1)]$. This shows that if $[(m, 1)] \sim [(n, 1)]$ then $m$ and $n$ differ by a multiple of an element of $\pi_1(\Sigma)$. Thus we see that $P/G \cong \hat{\Sigma}/\pi_1(\Sigma) \cong \Sigma$.

Thus, $P$ is a principal bundle associated to $\rho$. □

Lemma 3. The group $\text{Aut } (P)$ is identical to $C_G(\rho)$, the subgroup of elements of $G$ which commute with the image of $\rho$. 
Proof. First we choose an element \( g_\rho \in G \) that commutes with the image of \( \rho \) and define the map

\[
g_\rho : \tilde{\Sigma} \times_\rho G \to \tilde{\Sigma} \times_\rho G
\]

\[
[(n, h)] \mapsto [(n, g_\rho h)]
\]

\[
[(an, \rho(\alpha)h)] \mapsto [(an, g_\rho \rho(\alpha)h)] = [(an, \rho(\alpha)g_\rho h)]
\]

Since \([(n, h)] \sim [(an, \rho(\alpha)h)] \text{ and } [(n, g_\rho h)] \sim [(an, \rho(\alpha)g_\rho h)]\), we have a well defined action on equivalence classes. Therefore, \( g_\rho \) is an element of \( \text{Aut}(P) \).

Since \( \tilde{\Sigma} \times G \) is free over \( G \), choose \( \psi \in \text{Aut} P \). The \( \psi \)'s are the same as \( \pi_1(\Sigma) \) equivariant maps \( \tilde{\Sigma} \to \tilde{\Sigma} \times G \). Then we get \( (n, 1) \mapsto (n', \psi) \) for \( \psi \in G \) where \( \pi(n) = \pi(n') \). Since \( n \) and \( n' \) are in the same fiber, they are related by a deck transformation, so we can rewrite this as a map \( (n, 1) \mapsto (n, \psi(n)) \) for some \( \psi(n) \in G \).

The equivariance of \( \psi \) means that \( \psi \) commutes with \( \rho(\alpha) \), for all \( \alpha \in \pi_1(\Sigma) \), then the set of all such \( \psi \) lie in the centralizer of \( \rho(\pi_1(\Sigma)) \). \( \square \)

Consequently, the isomorphism class of \( P \) is isomorphic to \( G/C_G(\rho(\pi_1(\Sigma))) \). Therefore

\[
\sum_{[P]} \frac{|G|}{|C_G(\rho(\pi_1(\Sigma)))|} = |\text{Hom}(\pi_1(\Sigma), G)|
\]

\[
|G| \sum_{[P]} \frac{1}{|\text{Aut}(P)|} = |\text{Hom}(\pi_1, G)|
\]

Therefore

\[
\psi_Y = \frac{|\text{Hom}(\pi_1(Y), G)|}{|G|}
\]

We now describe the vector spaces associated to an \( (d - 1) \)-dimensional manifold \( X \) and the maps associated to cobordisms.

**Definition 20.** Two principal \( G \)-bundles \( \pi_1 : P_1 \to M, \pi_2 : P_2 \to M \) are equivalent if there exists a homeomorphism \( H : P_1 \to P_2 \) such that

\[
P_1 \xrightarrow{H} P_2
\]

\[
\xymatrix{ P_1 \ar[rr]^H \ar[dr]_{\pi_1} & & P_2 \ar[dl]^{\pi_2} \\
& M &}
\]

commutes and \( H(p \cdot g) = H(p) \cdot g \) for all \( g \in G \).
Let $P_X$ be the set of isomorphism classes of principal $G$ bundles on $X$ and $Z(X) := \mathbb{C}^{P_X}$. Let $\Sigma$ be a manifold where $\partial \Sigma = X$. And take $\psi_{\Sigma}(1) = Z(X) = \mathbb{C}^{P_X}$ to be determined as follows:

Let $P \to X$ be a $G$ bundle over $X$. Then

$$\psi_{\Sigma}(1)(P) = \sum_{[Q]} \frac{1}{|\text{Aut}(Q)|}$$

where $[Q]$ ranges over isomorphism classes of $G$ bundles over $\Sigma$ such that $Q|_{\partial \Sigma} = P$ and where the isomorphisms fix $Q|_{\partial \Sigma}$.

6.1. **The Two-Dimensional Case.** We use Dijkgraaf-Witten to calculate $Z(S^1)$ and $Z(O_{ba})$.

To show that the toy model manifested in two dimensions in the commutative symmetric Frobenius algebra $(\mathcal{A}, \theta_{S^1})$ where $\mathcal{A}$ is the center of the group ring $\mathbb{C}[G]$ and we define

$$\theta\left(\sum \lambda_g g\right) := \frac{1}{|G|} \lambda_1.$$

6.1.1. $Z(S^1) \cong Z(\mathbb{C}[G])$, the center of $\mathbb{C}[G]$. Fix a basepoint $x \in S^1$ and a finite group $G$. Let $P \cong \mathbb{R} \times \rho$ be a principal $G$-bundle on $S^1$. A $G$-bundle is determined up to isomorphism by the map $\rho: \mathbb{Z} \to G$. By mapping $1 \mapsto g$ for any $g \in G$, $|\text{Hom}(\mathbb{Z}, G)| = |G|$. Hence, we can associate any map $\rho$ with an element $g \in G$. To determine isomorphism classes of $G$ bundles on $S^1$, we simply consider what happens to the basepoint $p$ for two equivalent $G$-bundles. For a given $g$ associated with $p$, $g$ changes a basepoint $p \mapsto p \cdot g$. This change in basepoint induces a change in the map $\rho$. Thus two $G$-bundles are equivalent if two group elements acting on the space are conjugate.

Thus $Z(S^1)$ is the space of complex valued functions on isomorphism classes of $G$-bundles of $S^1$. Namely, since two $G$-bundles are isomorphic if they are induced by conjugate elements in $G$, $Z(S^1)$ is the center of the group ring $\mathbb{C}[G]$.

6.1.2. $Z(ba) \cong \mathbb{C}[G]$. Again, we follow the algorithm to determine $Z(ba)$. Fix a basepoint $x \in ba$. We consider the principal $G$-bundle on $ba$. There is only one principal bundle on $ba$ since $\pi_1(ba) = e$, the identity element. Thus the only principal bundle is defined by the identity map. Hence, there are no non-trivial isomorphism classes of $G$-bundles to be concerned with and $Z(ba) = \mathbb{C}[G]$. 
As we have boundary conditions, the category $B$ is the category of finite-dimensional complex representations $T$ of $G$. As $O_{aa}$ is the set of morphisms $\psi: a \to a$ for some boundary condition $a$, the trace $\theta_a: O_{aa} \to \mathbb{C}$ takes $\psi \in \text{End}(T) \mapsto 1/|G| \times \text{trace}(\psi)$.

**Theorem 10.** Specific to the Dijkgraaf-Witten model, we have the following properties.

**Property 6.1.1.** The group ring $\mathbb{C}[G] = \bigoplus \text{End}(V)$ where $V$ runs through the irreducible representations of $G$. Consider the set

$$e_\chi = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) \cdot g$$

as a basis for $\mathbb{C}[G]$:

Since $e_\chi$ is a function on $G$ with complex coefficients constant over the conjugacy classes of $G$, $e_\chi \in \mathbb{C}[G]$. Second $e_\chi \cdot e_\chi = \delta_{ij}$.

**Property 6.1.2.** $\theta(1_V) = \lambda_V^{-1}$ where $1_V$ is the identity element in $\text{End}(V)$ and $\lambda_V = \frac{|G|^2}{\dim(V)^2}$.

**Proof.**

$$\theta(1_V) = \theta(\chi_{V_0}) = \frac{1}{|G|^2} \chi_{V_0}(1) = \frac{(\dim V_0)^2}{|G|^2}$$

\[ \square \]

**Property 6.1.3.** $\alpha = \sum \lambda_V 1_V$.

**Proof.** We know that $\alpha = \sum e_i \Phi^*(e_i^*)$ and for basis elements $e_i$ and $e_i^*$, $\theta(e_i \Phi^*(e_i^*)) = \delta_{ij}$.

Consider $1_V$ to be a basis for $\text{End}(V)$, we show that $\Phi^*(e_i^*) = \lambda_V 1_V$.

Then we see that

$$\theta(1_V, \lambda_V 1_V) = \lambda_V \theta(1_V, 1_V) = \delta_{ij}$$

by the previous property. \[ \square \]

**Property 6.1.4.** $\Psi_\Sigma = |G|^{2g-2} \sum \frac{1}{(\dim V)^{2g-2}}$ for $\Sigma$ a closed manifold of genus $g$. 
Proof.

\[ \Psi_\Sigma = \theta(\alpha)^g \]
\[ = \theta((\sum V_1^g)^g) \]
\[ = \sum V_1^g \theta(1_V)^g \]
\[ = |G|^{2g-2} \sum \frac{1}{(\dim V)^{g-2}} \]

\[ \Box \]

7. Appendix: 2-Categories

Definition 21. [7] A 2-category \( C \) is composed of the following:

- A class of objects \( |C| \)
- For each pair of \( X, Y \in |C| \), a category \( \text{Hom}(X, Y) \).
  - Objects in \( \text{Hom}(X, Y) \) are 1-morphisms, i.e. morphisms between objects in \( C \)
  - Morphisms in \( \text{Hom}(X, Y) \) are 2-morphisms, i.e. morphisms between the morphisms between objects in \( C \).
- Horizontal composition of 1-morphisms and 2-morphisms: For each triple \( X, Y, Z \in |C| \) a functor
  \[ \mu_{X,Y,Z} : \text{Hom}(X, Y) \times \text{Hom}(Y, Z) \to \text{Hom}(X, Z) \]
  with the following conditions:
  - (Identity 1-morphism): For each object \( X \in |C| \), there exists an object \( \text{id}_X \in \text{Hom}(X, X) \) such that
    \[ \mu_{X,X,Y}(\text{id}_X, \cdot) = \mu_{X,Y,Y}(\cdot, \text{id}_Y) = I_{\text{Hom}(X,Y)}, \]
    where \( I_{\text{Hom}(X,Y)} \) is the identity functor in \( \text{Hom}(X, Y) \).
  - (Associativity of horizontal compositions): For each quadruple \( X, Y, Z, T \in \text{ob}C \),
    \[ \mu_{X,Z,T} \circ (\mu_{X,Y,Z} \times I_{\text{Hom}(Z,T)}) = \mu_{X,Y,T} \circ (I_{\text{Hom}(X,Y)} \times \mu_{Y,Z,T}) \]
This definition can be visualized with the use of diagrams.

Fix a 2-category \( C \) with objects \( X, Y, Z \) and \( T \). An object \( f \) in the category \( \text{Hom}(X, Y) \) is a 1-morphism of \( C \):

\[ X \xrightarrow{f} Y. \]

A morphism \( \alpha \) of the category \( \text{Hom}(X, Y) \) is a 2-morphism of \( C \):

\[ \begin{array}{c}
X \\
\alpha
\end{array} \quad \xrightarrow{f} \quad 
\begin{array}{c}
Y
\end{array} \]
Vertical composition of 2-morphisms is associative:
\[ γ \circ (β \circ α) = (γ \circ β) \circ α. \]

Given the diagram

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \downarrow \downarrow g \\
\downarrow \downarrow \downarrow h \\
Z
\end{array}
\]

there exists

\[
\begin{array}{c}
X \xrightarrow{\beta \circ \alpha} Y \\
\downarrow \downarrow \downarrow g \\
\downarrow \downarrow \downarrow h \\
Z
\end{array}
\]

Horizontal compositions of 2-morphism is associative:
\[ ((γ \ast β) \circ α = γ \ast (β \ast α)). \]

Given the diagram

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \downarrow \downarrow f' \\
\downarrow \downarrow \downarrow f'' \\
Z
\end{array}
\]

there exists

\[
\begin{array}{c}
X \xrightarrow{\beta \ast \alpha} Y \\
\downarrow \downarrow \downarrow g' \\
\downarrow \downarrow \downarrow g'' \\
Z
\end{array}
\]

There exists an identity for 2-morphisms. For \( X, Y, Z \in \text{obj} C \), \( f, g : X \to Y \) and \( h : Y \to Z \) all 1-morphisms. For each 1-morphism \( f \), there exists a 2-morphism \( I_f \) such that \( α \circ I_g = I_f \circ α = α \) for some \( α : g \Rightarrow f \). Composition also respects the identity in that \( I_h \ast I_f = I_{hof} \).

Compatibility of horizontal and vertical compositions of 2-morphisms. For a given diagram

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \downarrow \downarrow f' \\
\downarrow \downarrow \downarrow f'' \\
Z
\end{array}
\]

then \( (β' \circ β) \ast (α' \circ α) = (β' \ast α') \circ (β \ast α) \).

References