# On valid inequalities for mixed integer $p$-order cone programming

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**Abstract**

We discuss two families of valid inequalities for linear mixed integer programming problems with cone constraints of arbitrary order, which arise in the context of stochastic optimization with downside risk measures. In particular, we extend the results of Atamtürk and Narayanan (Math. Program., 2010, 2011), who developed mixed integer rounding cuts and lifted cuts for mixed integer programming problems with second order cone constraints. Numerical experiments conducted on randomly generated problems and portfolio optimization problems with historical data demonstrate the effectiveness of the proposed methods.

**Keywords:** valid inequalities; nonlinear cuts; mixed integer $p$-order cone programming; stochastic optimization; risk measures

## 1 Introduction

In this work we consider mixed integer programming problems with linear objective and $p$-order cone constraints, which represent an extension of mixed integer second order cone programming (MISOCP) problems and subsequently are referred to as mixed integer $p$-order cone programmig (MIpOCP) problems. Specifically, we focus on a class of MIpOCP instances that arise in stochastic optimization problems with risk-based objective functions or constraints.

There exists a substantial literature on solution approaches for mixed integer conic programming problems. In many cases, the proposed methods attempt to extend some of the techniques developed for mixed integer linear programming. One of such research directions concerns construction of branch-and-bound schemes based on outer polyhedral approximations of cones. This potentially allows for computational savings in traversing the branch-and-bound tree due to the "warm start" capabilities of linear programming solvers. In particular, Vielma et al. [21] proposed a branch-and-bound method for MISOCP that employed lifted polyhedral approximations of second order cones due to Ben-Tal and Nemirovski [5]. Vinel and Krokhmal [22] discuss further development of this approach in the case of MIpOCP. Drewes [10] presented subgradient-based linear outer approximations for the second order cone constraints in mixed integer programs. With respect to mixed integer nonlinear programming, a similar idea has been exploited by Bonami et al. [6] and Tawarmalani and Sahinidis [20].

Two approaches to generation of valid inequalities for MISOCP problems have been proposed by Atamtürk and Narayanan [2, 3]. In the first paper the authors introduced a reformulation of a second order cone constraint using a set of two-dimensional second order cones and then derived valid inequalities for the resulting mixed integer sets. The obtained cuts were termed by the authors conic mixed integer rounding cuts. In [3], a general lifting procedure for deriving nonlinear conic valid inequalities was proposed and applied to 0-1 MISOCP problems.

In a recent work of Belotti et al. [4], disjunctive conic cuts for MISOCP problems are introduced. For the case of general convex sets, the authors are able to describe the convex hull of the intersection of a convex set and a linear disjunction. And in the particular case of the feasible set of the continuous relaxation of a

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MISOCP problem they derive a closed-form expression for such a convex hull, thus obtaining a new nonlinear conic cut.

Among other approaches to solving mixed integer cone programming problems one can mention the split closure of a strictly convex body [9], lift-and-project algorithm [19], Chvátal-Gomory and disjunctive cuts for 0-1 conic programming [8].

It is worth noting that the vast majority of the existing literature on mixed integer cone programming problems addresses the case of self-dual cones, and particularly second order cones, with relatively little attention paid to problems involving cones that are not self-dual, as in the case of MIpOCP with $p \in ]1,2[ \cup ]2,\infty[$. In this work, we consider derivation of valid inequalities for mixed integer problems with $p$-order cone constraints following the techniques [2, 3] proposed for MISOCP. We derive closed form expressions for two families of valid inequalities for MIpOCP problems: mixed integer rounding conic cuts and lifted conic cuts. We also propose to use outer polyhedral approximations as a practical way of employing nonlinear lifted cuts within branch-and-cut framework. With such an approach, we are able to obtain promising computational results on a number of portfolio optimization problems with real-life data.

The paper is organized as follows. In Section 2 we present mixed integer rounding cuts for $p$-cone constrained mixed integer sets. Section 3 discusses (nonlinear) lifted cuts for 0-1 and mixed integer $p$-order cone programming problems. Computational studies of the developed techniques on randomly generated MIpOCP problems as well as portfolio optimization problems with real-life data are discussed in Section 4, followed by concluding remarks in Section 5.

## 2 Conic Mixed Integer Rounding Cuts for *p*-Order Cones

In this section we present a class of mixed integer rounding cuts for MIpOCP problems arising in the context of risk-averse stochastic optimization. A mixed integer $p$-order cone programming problem has the form

$$\begin{aligned}
\min \quad & \mathbf{c}_x^\top \mathbf{x} + \mathbf{c}_y^\top \mathbf{y} \\
\text{s. t.} \quad & \mathbf{D}_x \mathbf{x} + \mathbf{D}_y \mathbf{y} \leq \mathbf{d} \\
& \|\mathbf{A}_j \mathbf{x} + \mathbf{G}_j \mathbf{y} - \mathbf{b}_j\|_{p_j} \leq \mathbf{e}_j^\top \mathbf{x} + \mathbf{f}_j^\top \mathbf{y} - h_j, \quad j = 1, \dots, k \\
& \mathbf{x} \in \mathbb{Z}_+^n, \ \mathbf{y} \in \mathbb{R}_+^q,
\end{aligned} \tag{1}$$

where $p_j \in (1, \infty)$, and $\|\cdot\|_p$ is the usual $p$-norm in the Euclidean space of an appropriate dimension: $\|\mathbf{r}\|_p = (|r_1|^p + \ldots + |r_N|^p)^{1/p}$.

MIpOCP problems (1) can be obtained from stochastic programming models that involve specific families of risk measures in objectives or constraints. Namely, given a probability space $(\Omega, \mathcal{F}, \mu)$, let the cost or loss function $Y$ be an element of the linear space $\mathcal{L}_p(\Omega, \mathcal{F}, \mu)$ of $\mathcal{F}$-measurable functions $Y : \Omega \mapsto \mathbb{R}$, where $p \geq 1$. Then, the *higher-moment coherent risk measures* $\mathsf{HMCR}_{p,\alpha}(Y)$ are defined as the optimal values of the following convex stochastic optimization problem [12]

$$\mathsf{HMCR}_{p,\alpha}(Y) = \min_{\eta \in \mathbb{R}} \eta + (1-\alpha)^{-1} \big\| [Y - \eta]_+ \big\|_p, \quad \alpha \in (0,1), \quad p \geq 1, \tag{2}$$

where $[Y]_+ = \max\{0, Y\}$ and $\|Y\|_p = (\mathsf{E}|Y|^p)^{1/p}$. A related family of *semi-moment coherent risk measures,* or risk measures of semi-$\mathcal{L}_p$ type [11], is given as

$$\mathsf{SMCR}_{p,\beta}(Y) = \mathsf{E}Y + \beta \big\| [Y - \mathsf{E}Y]_+ \big\|_p, \quad \beta \in [0,1], \quad p \geq 1. \tag{3}$$

In the case when the set $\Omega$ is finite, $\Omega = \{\omega_1, \dots, \omega_m\}$, and the cost function $Y = Y(\mathbf{u}, \omega)$ is a piecewise linear convex function of the decision vector $\mathbf{u}$, terms with HMCR or SMCR measures in the objective function and/or constraints can be implemented via linear inequalities involving $Y(\mathbf{u}, \omega_i)$ and $p$-order cone constraints $t \geq \|(w_1, \ldots, w_m)\|_p$, thus leading to MIpOCP problem of the form

$$
\begin{aligned}
\min \quad & \mathbf{c}_x^\top \mathbf{x} + \mathbf{c}_y^\top \mathbf{y} \\
\text{s. t.} \quad & \mathbf{D}_x \mathbf{x} + \mathbf{D}_y \mathbf{y} \leq \mathbf{d} \\
& \|[\mathbf{A}_j \mathbf{x} + \mathbf{G}_j \mathbf{y} - \mathbf{b}_j]_+\|_{p_j} \leq \mathbf{e}_j^\top \mathbf{x} + \mathbf{f}_j^\top \mathbf{y} - h_j, \quad j = 1, \ldots, k \\
& \mathbf{x} \in \mathbb{Z}_+^n, \ \mathbf{y} \in \mathbb{R}_+^q,
\end{aligned}
\tag{4}
$$

Formulation (4) differs from (1) by the presence of operator $[\,\cdot\,]_+$, which explicitly accounts for the problem structure induced by downside risk measures such as (2)–(3). For simplicity, we consider the case of a single $p$-cone constraint in (4), $k = 1$. Following the approach of [2] for constructing mixed integer rounding cuts for problems of type (1) with $p = 2$, we rewrite the $p$-cone constraint in (4) as

$$
\begin{aligned}
& t_0 \leq \mathbf{e}^\top \mathbf{x} + \mathbf{f}^\top \mathbf{y} - h \\
& t_i \geq [\mathbf{a}_i^\top \mathbf{x} + \mathbf{g}_i^\top \mathbf{y} - b_i]_+, \quad i = 1, \ldots, m \\
& t_0 \geq \|(t_1, \ldots, t_m)\|_p,
\end{aligned}
$$

where $\mathbf{a}_i$ and $\mathbf{g}_i$ denote the $i$-th rows of matrices $\mathbf{A}$ and $\mathbf{G}$, respectively. Then, the task of deriving valid inequalities for the original $p$-cone mixed integer set in (4) can be reduced to obtaining valid inequalities for the polyhedral mixed integer set

$$
T = \left\{\mathbf{x} \in \mathbb{Z}_+^n, \ \mathbf{y} \in \mathbb{R}_+^p, \ t \in \mathbb{R} \ : \ [\mathbf{a}^\top \mathbf{x} + \mathbf{g}^\top \mathbf{y} - b]_+ \leq t\right\},
$$

or, without loss of generality, the set

$$
\widetilde{T} = \left\{(y^+, y^-, t, \mathbf{x}) \in \mathbb{R}_+^3 \times \mathbb{Z}_+^n \ : \ [\mathbf{a}^\top \mathbf{x} + y^+ - y^- - b]_+ \leq t\right\}. \tag{5}
$$

The following two propositions provide an expression for a family of such inequalities.

**Proposition 1** *For $\alpha \neq 0$, the inequality*

$$
\sum_{j=1}^{n} \phi_{f_{|\alpha|}}\left(\frac{a_j}{|\alpha|}\right) x_j - \phi_{f_{|\alpha|}}\left(\frac{b}{|\alpha|}\right) \leq \frac{t + y^-}{|\alpha|}, \tag{6}
$$

*where* $f_\alpha = \frac{b}{|\alpha|} - \left\lfloor \frac{b}{|\alpha|} \right\rfloor$ *and*

$$
\phi_f(a) = \begin{cases} (1 - f)n, & n \leq a < n + f \\ (1 - f)n + (a - n) - f, & n + f \leq a < n + 1 \end{cases}
$$

*is valid for* $\widetilde{T}$.

**Proposition 2** *Inequalities* (6) *with $\alpha = a_j$, $j = 1, \ldots, n$, are sufficient to cut off all fractional extreme points of the relaxation of* $\widetilde{T}$.

Proofs of Propositions 1 and 2 are furnished in the Appendix. It is worth noting, however, that since (5) is a polyhedral mixed integer set, the derived valid inequalities can also be obtained using the general theory of mixed integer rounding (MIR) inequalities; see, for example, [15]. An advantage of the direct derivation is that it provides a natural way of dealing with continuous variables $y^+, y^-, t$. Propositions 1 and 2 justify the usage of inequalities of type (6) as cuts in a branch-and-cut procedure; following [2], we refer to these inequalities as conic MIR cuts. The results of numerical experiments on utilization of conic MIR cuts (6) in MIpOCP problems are presented in Section 4.

# 3 Lifted Conic Cuts for $p$-Order Cones

## 3.1 General Framework

Lifting for conic mixed integer programming was studied in [3], where a general approach for constructing valid nonlinear conic inequalities for mixed inter conic programming problems was proposed. Namely, consider a general mixed integer conic set

$$S^n(\mathbf{b}) = \left\{ (\mathbf{x}^0, \ldots, \mathbf{x}^n) \in X^0 \times \cdots \times X^n \ : \ \mathbf{b} - \sum_{i=0}^{n} \mathbf{A}^i \mathbf{x}^i \in \mathcal{C} \right\}, \tag{7}$$

where $\mathbf{A}^i \in \mathbb{R}^{m \times n_i}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathcal{C}$ is a proper cone (a closed, convex, pointed cone with a nonempty interior), and each $X^i \subset \mathbb{R}^{n_i}$ is a mixed integer set. Similarly, $S^0(\mathbf{b}), \ldots, S^{n-1}(\mathbf{b})$ are restrictions of the set $S^n(\mathbf{b})$. Further, it is assumed that the following conic inequality

$$\mathbf{h} - \mathbf{F}^0 \mathbf{x}^0 \in \mathcal{K},$$

where $\mathcal{K}$ is a proper cone, is known to be valid for the restriction $S^0(\mathbf{b})$. The approach proposed in [3] is to iteratively find a sequence $\mathbf{F}^1, \ldots, \mathbf{F}^n$, such that

$$\mathbf{h} - \sum_{j=0}^{i} \mathbf{F}^j \mathbf{x}^j \in \mathcal{K} \tag{8}$$

is valid for the respective restriction $S^i(\mathbf{b})$ for all $i$. Such a procedure is called *lifting* and the resulting inequality that is valid for the initial mixed integer set $S^n(\mathbf{b})$ is called *lifted inequality*. In order to determine the values of $\mathbf{F}^1, \ldots, \mathbf{F}^n$, the *lifting set* is introduced for $\mathbf{v} \in \mathbb{R}^m$ as

$$\Phi_i(\mathbf{v}) = \left\{ \mathbf{d} \in \mathbb{R}^s \ : \ \mathbf{h} - \sum_{j=0}^{i} \mathbf{F}^j \mathbf{x}^j - \mathbf{d} \in \mathcal{K} \ \text{ for all } \ (\mathbf{x}^0, \ldots, \mathbf{x}^i)^\top \in S^i(\mathbf{b} - \mathbf{v}) \right\}.$$

Then, a necessary and sufficient condition for (8) to be valid can be formulated, which essentially provides a description of the set of valid inequalities.

**Proposition 3 ([3])** *Inequality* (8) *is valid for* $S^i(\mathbf{b})$ *if and only if* $\mathbf{F}^i \mathbf{t} \in \Phi_i(\mathbf{A}^i \mathbf{t})$ *for all* $\mathbf{t} \in X^i$ *and* $i = 0, \ldots, n$.

The condition established by Proposition 3 is still too general to be used for derivation of conic cuts. For example, it can be seen that in this way the resulting inequalities are *sequence-dependent*, i.e., a change in the order in which variables $\mathbf{x}^i$ are introduced will change the sets $\Phi_i(\mathbf{v})$. The following theorem provides a "sequence-independent" approach to construction of lifting procedure.

**Theorem 1 ([3])** *If* $\Upsilon(\mathbf{v}) \subseteq \Phi_0(\mathbf{v})$ *for all* $\mathbf{v} \in \mathbb{R}^m$ *and* $\Upsilon$ *is superadditive, then* (8) *is a lifted valid inequality for* $S^n(\mathbf{b})$ *whenever* $\mathbf{F}^i \mathbf{t} \in \Upsilon(\mathbf{A}^i \mathbf{t})$ *for all* $\mathbf{t} \in X^i$ *and* $i = 0, \ldots, n$.

Then, the following procedure can be formulated for derivation of lifted conic inequalities:

***Step 1.*** Compute $\Phi_0(\mathbf{v})$.

***Step 2.*** If $\Phi_0(\mathbf{v})$ is not superadditive, find a superadditive $\Upsilon(\mathbf{v}) \subset \Phi_0(\mathbf{v})$.

***Step 3.*** For each $i$ find $\mathbf{F}^i$ such that $\mathbf{F}^i \mathbf{t} \in \Upsilon(\mathbf{A}^i \mathbf{t})$ is satisfied for all $\mathbf{t} \in X^i$.

In [3] this process was employed to obtain nonlinear lifted conic cuts for 0-1 MISOCP problems; however, no computational results were reported. Below we apply this procedure to derive nonlinear lifted conic cuts for 0-1 and mixed integer $p$-order cone programming problems with risk-based constraints, and also discuss polyhedral approximations of these cuts that are used in numerical implementation.

## 3.2 Lifting Procedure for 0-1 $p$-Order Cone Programming Problems

In the case of 0-1 $p$-order cone programming problem, consider the following conic set

$$S_p^n(b) = \left\{ (\mathbf{x}, \eta_+, \eta_-, y, t) \in \{0,1\}^n \times \mathbb{R}_+^4 \;:\; \Big[ \sum_{i=1}^n a_i x_i + \eta_+ - \eta_- - b \Big]_+^p + y^p \le t^p \right\},$$

where $p \in (1, \infty)$. The set $S_p^n(b)$ represents a relaxation of a high dimensional 0-1 mixed integer $p$-order conic set: all but one dimensions of the $p$-cone are aggregated into the term $y^p$. By complementing the binary variables, if necessary, we can assume that all $a_i \ge 0$. The restriction $S_p^0$ of this set can be taken as

$$S_p^0(b) = \big\{ (x, y, t) \in \{0,1\} \times \mathbb{R}_+^2 \;:\; [x - b]_+^p + y^p \le t^p \big\}.$$

Notice that $S_p^0(b)$ has one extreme point $(b, 0, 0)$, which is fractional when $b \in (0, 1)$. Thus, in the only interesting case we have $\lfloor b \rfloor = 0$. Using the results of the previous section, the initial valid inequality can be selected as $\big|(1 - f)(x - \lfloor b \rfloor)\big|^p + y^p \le t^p$, where $f = b - \lfloor b \rfloor$ (the fact that this inequality is valid can be verified directly by examining the possible values of $x, y, t$). Now, by definition, in order to compute $\Phi_0(v)$ we need to find such $d$ that inequality

$$\big|(1 - f)(x - \lfloor b \rfloor) + d\big|^p + y^p \le t^p \tag{9}$$

is satisfied for all $x, y, t$ such that $[x - b + v]_+^p + y^p \le t^p$.

Recalling that $\lfloor b \rfloor = 0$ and, therefore, $f = b$, we obtain that (9) can be rewritten as $|(1-b)x + d|^p + y^p \le t^p$ for all $x, y, t$ such that $[x - b + v]_+^p + y^p \le t^p$. Given that $x \in \{0, 1\}$, for $x = 0$ we have $|d| \le [v - b]_+$, and for $x = 1$ we have $|1 - b + d| \le [1 - b + v]_+$. Thus, if $v \ge b$ then $|d| \le v - b$, and if $v < b$ then $d = 0$, meaning that $|d| \le [v - b]_+$, whereby $\Phi_0(v) = \{d \;:\; |d| \le [v - b]_+\}$, which is superaddive. Finally, the following proposition holds.

**Proposition 4** *Conic inequality*

$$\left|(1 - f)(x - \lfloor b \rfloor) + \sum_{i=1}^n \alpha_i x_i\right|^p + y^p \le t^p \tag{10}$$

*with* $\alpha_i = [a_i - b]_+$ *is valid for the set* $S_p^n(b)$.

**Proof:** Since $\Phi_0(v)$ is superadditive, by Theorem 1 we only need to verify that the chosen values of $\alpha_i$ satisfy $\alpha_i x \in \Phi_0(a_i x)$ for $x \in \{0, 1\}$, which follows readily from the expression for $\Phi_0(v)$. □

## 3.3 Lifting Procedure for MIpOCP Problems

Similarly, in the case of MIpOCP problem we consider the set

$$\hat{S}_p^n(b) = \left\{ (\mathbf{x}, \eta_+, \eta_-, y, t) \in \mathbb{Z}_+^n \times \mathbb{R}_+^4 \;:\; \Big[ \sum_{i=1}^n a_i x_i + \eta_+ - \eta_- - b \Big]_+^p + y^p \le t^p \right\},$$

where $p \in (1, \infty)$. Once again, the set $\hat{S}_p^n(b)$ represents a relaxation of a high dimensional mixed integer $p$-order cone constraint. Let us also assume that values $x_i$ are bounded, e.g., $x_i \in \{0, \ldots, M\}$ for all $i$. Again, let us assume without loss of generality that $a_i > 0$. The restriction of $\hat{S}_p^n(b)$ can be selected as

$$\hat{S}_p^0(b) = \big\{ (x, y, t) \in \mathbb{Z}_+ \times \mathbb{R}_+^2 \;:\; [x - b]_+^p + y^p \le t^p \big\}, \tag{11}$$

but in this case let us choose a weaker initial valid inequality, $\big[(1 - f)(x - \lfloor b \rfloor)\big]_+^p + y^p \le t^p$. The problem of computing $\Phi_0(v)$ is then reduced to the problem of finding values of $d$ such that

$$\big[(1 - f)x - \lfloor b \rfloor(1 - f) + d\big]_+ \le \big[x - b + v\big]_+. \tag{12}$$

Recall that we are only interested in a superadditive subset $\Upsilon(v)$ of such set. One of the possible choices is $\Upsilon(v) = \{d \geq 0 \ : \ d \leq [v - b + \lfloor b \rfloor(1-f)]_+\}$. Indeed, $0 \in \Upsilon(v)$ by definition, and (12) is a consequence of inequality $(1-f)x - \lfloor b \rfloor(1-f) + d \leq x - b + v$, which yields the above expression for $\Upsilon(v)$. Lastly, the following proposition holds.

**Proposition 5** *Conic inequality*

$$\left[(1-f)(x - \lfloor b \rfloor) + \sum_{i=1}^{n} \alpha_i x_i\right]_+^p + y^p \leq t^p \tag{13}$$

*with* $\alpha_i = \left[\dfrac{a_i - b + \lfloor b \rfloor(1-f)}{M}\right]_+$ *is valid for* $\hat{S}_p^n(b)$.

**Proof:** Indeed, in accordance to Section 3.1 it suffices to show that for such a choice of $\alpha_i$ we have $\alpha_i x \in \Upsilon(a_i x)$ for all $x$. For $x \neq 0$ we have

$$\Upsilon(a_i x) = \{d \geq 0 \ : \ d \leq [a_i x - b + \lfloor b \rfloor(1-f)]_+\},$$

and

$$\alpha_i x = \left[\frac{a_i - b + \lfloor b \rfloor(1-f)}{M}\right]_+ x \leq [a_i - b + \lfloor b \rfloor(1-f)]_+ \leq [a_i x - b + \lfloor b \rfloor(1-f)]_+.$$

On the other hand, for $x = 0$ it is clear that $0 \in \Upsilon(0)$. □

## 3.4 Polyhedral Approximations of $p$-Order Cones

Observe that lifted cuts (10) and (13) for, respectively, 0-1 and mixed integer $p$-order cone programming problems have the form of $p$-order cones themselves. Thus, one may expect that while addition of such cuts can reduce the number of nodes explored in the branch-and-bound tree, the computational cost of solving the relaxed problem with extra $p$-cone constraints at the nodes may increase. In view of this, we propose to replace the nonlinear $p$-order cone cuts (10) and (13) with their polyhedral approximations during the branch-and-cut procedure. A detailed discussion of polyhedral approximations of $p$-order cones can be found in [22].

Since in our case the lifted cuts have the form of 3-dimensional $p$-cones, we use a simple gradient polyhedral approximation. Particularly, a gradient polyhedral approximation for the conic set $\mathcal{K}_p^{(3)} = \{\boldsymbol{\xi} \in \mathbb{R}_+^3 \ : \ \xi_3 \geq \|(\xi_1, \xi_2)\|_p\}$, $p \in (1, \infty)$, can be constructed as

$$\mathcal{H}_{p,\ell}^{(3)} = \{\boldsymbol{\xi} \in \mathbb{R}_+^3 \ : \ \xi_3 \geq \alpha_i^{(p)}\xi_1 + \beta_i^{(p)}\xi_2, \ \ i = 0, \ldots, \ell\}, \tag{14}$$

where

$$\begin{bmatrix} \alpha_i^{(p)} \\ \beta_i^{(p)} \end{bmatrix} = (\cos^p \theta_i + \sin^p \theta_i)^{\frac{1-p}{p}} \begin{bmatrix} \cos^{p-1} \theta_i \\ \sin^{p-1} \theta_i \end{bmatrix}, \quad \theta_i = \frac{\pi i}{2\ell}, \quad i = 0, \ldots, \ell.$$

Here $\mathcal{H}_{p,\ell}^{(3)}$ is an approximation of $\mathcal{K}_p^{(3)}$ in the sense that $\boldsymbol{\xi} \in \mathcal{K}_p^{(3)}$ implies $\boldsymbol{\xi} \in \mathcal{H}_{p,\ell}^{(3)}$, and $\boldsymbol{\xi} \in \mathcal{H}_{p,\ell}^{(3)}$ implies $(1+\varepsilon)\xi_3 \geq \|(\xi_1, \xi_2)\|_p$, where $\varepsilon = \varepsilon(\ell)$ is the accuracy of approximation. In the case of polyhedral approximation (14), the latter can be estimated as [13]

$$\varepsilon(\ell) \approx \begin{cases} \frac{1}{p}\left(1 - \frac{1}{p}\right)^p \left(\frac{\pi}{2\ell}\right)^p, & p \in (1, 2), \\ \frac{1}{8}(p-1)\left(\frac{\pi}{2\ell}\right)^2, & p \in [2, \infty). \end{cases}$$

For example, for $p = 4.0$ it suffices to have $\ell = 25$ facets in the approximation to ensure an accuracy of $10^{-3}$.

# 4 Computational Results

In this section we report the results of numerical experiments on applying the derived MIR and lifted conic cuts to MIpOCP problem instances. In our case study, three types of problem instances were considered: the first type represents the "generic" MIpOCP instances with randomly generated data, and the second and third types of instances represent portfolio optimization problems with cardinality constraints and lot-buying constraints, respectively. Historical financial data were used for both types of portfolio optimization problems. A detailed description of each problem type is given below.

Computations were ran on a 3GHz PC with 4GB RAM, and CPLEX 12.2 solver was used. Since CPLEX cannot natively handle $p$-cone constraints with $p \neq 2$, a second-order cone reformulation [1, 14, 16] was applied to $p$-order cone constraints with rational $p > 2$. The derived cuts were added at the root node of the branch-and-bound tree using CPLEX callback routines. In addition, each instance was solved using the default mixed integer CPLEX solver with built-in cuts. In both cases, default solver configuration was used, with the exceptions that the number of threads was limited to one and QCP relaxations of the model were used at each node.

## 4.1 Problem Formulations

**Randomly generated MIpOCP problems** The first set of problem instances consisted of randomly generated mixed integer $p$-order cone programming problems of the general form. Specifically, the following formulation was used:

$$\begin{aligned} \min \quad & \mathbf{c}^\top \mathbf{x} + y^+ + y^- \\ \text{s. t.} \quad & \left\| [\mathbf{A}\mathbf{x} + y^+\mathbf{1} - y^-\mathbf{1} - \mathbf{b}]_+ \right\|_p \le \mathbf{e}^\top \mathbf{x} + f y^+ - g y^- - h \\ & \mathbf{x} \in \mathbb{Z}^n_+, \ \ y^+, y^- \in \mathbb{R}_+, \end{aligned} \tag{15}$$

where $\mathbf{A} \in \mathbb{R}^{n\times m}$, $\mathbf{c}, \mathbf{b}, \mathbf{e} \in \mathbb{R}^n$, $f, g, h \in \mathbb{R}$, and $\mathbf{1} = (1, \ldots, 1)^\top$. Each of the parameters $\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{e}, f, g, h$ in (15) was selected from the uniform $U(1, 1000)$ distribution.

**Portfolio optimization with cardinality constraints.** The second set of problem instances consisted of portfolio optimization problems with cardinality constraints. Specifically, portfolio risk as given by HMCR measure was minimized while requiring that the portfolio's expected return was not below some prescribed level $r_0$. No short sales were allowed, and the cardinality constraint ensured that the portfolio was comprised of no more than $K$ assets:

$$\min_{\mathbf{y}\in\mathbb{R}^n_+,\ \mathbf{x}\in\{0,1\}^n} \Big\{ \mathsf{HMCR}_{\alpha,p}(-\mathbf{r}^\top\mathbf{y}) \ : \ \mathsf{E}(\mathbf{r}^\top\mathbf{y}) \ge r_0, \ \ \mathbf{1}^\top\mathbf{y} \le 1, \ \ \mathbf{y} \le \mathbf{x}, \ \ \mathbf{1}^\top\mathbf{x} \le K \Big\}, \tag{16}$$

where vectors $\mathbf{y}$ and $\mathbf{r} = \mathbf{r}(\omega)$ represented the weights of assets in the portfolio and the assets' uncertain returns, respectively. Using definition (2) of HMCR measures and assuming that the stochastic vector $\mathbf{r}(\omega)$ is discretely distributed with $m$ scenarios $\mathbf{r}(\omega_i)$, $i = 1, \ldots, m$, the portfolio optimization problem (16) can be formulated as a 0-1 MIpOCP problem with $(m + 1)$-dimensional $p$-cone constraint. In our computations we set $K = 5$ and $\alpha = 0.9$ in (16).

**Portfolio optimization with lot-buying constraints.** The last type of problems considered in this case study represents portfolio optimization problems with lot-buying constraints. The lot-buying constraints reflect the real-life trading policies of many financial markets (see, e.g., [7, 17, 18] and references therein), where the investors are allowed to buy or sell shares of financial instruments only in *lots* of standard size $L$, e.g., in multiples of $L = 1{,}000$ shares. Following the same setup as above, a risk-minimizing portfolio allocation problem with lot-buying constraints is formulated as

$$\min_{\mathbf{y}\in\mathbb{R}^n_+,\ \mathbf{x}\in\mathbb{Z}^n_+} \Big\{ \mathsf{HMCR}_{\alpha,p}(-\mathbf{r}^\top\mathbf{y}) \ : \ \mathsf{E}(\mathbf{r}^\top\mathbf{y}) \ge r_0, \ \ \mathbf{1}^\top\mathbf{y} \le 1, \ \ \mathbf{y} = \frac{L}{C}\,\mathrm{Diag}(\mathbf{p})\,\mathbf{x} \Big\}. \tag{17}$$

Here $L \in \mathbb{N}$ is the given lot size, $C > 0$ is the available capital (in dollars), vector $\mathbf{p} \in \mathbb{R}^n_+$ represents the current (observable) asset prices per share, and Diag($\mathbf{a}$) denotes a matrix whose diagonal elements are equal to the corresponding elements of vector $\mathbf{a}$, and off-diagonal elements are zero. Similarly to the above, portfolio problem (17) reduces to a MIpOCP problem with $(m+1)$-dimensional $p$-cone constraint, where $m$ is the number of scenarios in stochastic representation of the vector of assets' returns $\mathbf{r}$. The values of parameters $L$ and $C$ in our experiments were set at $L = 1{,}000$ and $C = \$100{,}000$.

For portfolio optimization problems, we used historical data for $n$ stocks chosen at random from the S&P500 index, and returns over $m$ consequent 10-day periods starting at a (common) randomized date were used to construct the set of $m$ scenarios for the stochastic vector $\mathbf{r}$ in (16), (17).

## 4.2 Discussion of Results: Conic MIR Cuts

**Randomly generated MIpOCP problems** For each pair of parameters $(n, m)$ that determine the number of integer variables and the dimensionality of $p$-cone, 50 randomly generated instances of problem (15) were solved. The results are summarized in Table 1, where the average computational time (in seconds), the average number of nodes explored in the search tree, and the average number of cuts added during the solution procedure are reported. In addition, we report the percentage of cases in which addition of conic MIR cuts improves the computational time and the number of nodes explored, respectively, as compared to the default CPLEX routines. It has also been noted that randomly generated problems are relatively easy to solve; in fact, many instances were solved at the root node. Therefore, in addition to the results averaged over all instances of a given problem size $(n, m)$, Table 1 presents the results averaged over "difficult" instances, i.e., instances that could not be solved at the root node by CPLEX solver with default parameter settings. As one can see, in most cases utilization of conic MIR cuts reduces the average solution time and the number of nodes explored in the solution tree, with the improvement being more noticeable for "difficult" instances and larger sizes of the problem. It is also worth noting that while solution times vary for different values of the parameter $p$, the observed improvement due to implementation of conic MIR cuts stays approximately the same.

**Portfolio optimization with cardinality constraints.** For each problem size we generated 30 problem instances. The obtained results are summarized in Table 2. We can again conclude that for the majority of the instances, introduction of conic MIR cuts leads to an improved performance in comparison to the default CPLEX solution procedures, although the improvement is considerably smaller comparing to that observed on randomly generated problems. Note also that a significantly smaller number of cuts were generated in problem instances of this type; moreover, in many cases the default CPLEX optimizer did not add any cuts to the problem.

**Portfolio optimization with lot-buying constraints.** The results averaged over 30 instances for each problem size are summarized in Table 3. Note that in many instances of problems of this type, no user cuts of the proposed structure have been found. It can also be noted that regardless of the number of cuts found, solution times are rather comparable to those of the default CPLEX optimizer, which may indicate that conic MIR cuts do not make a significant difference in problems of this type.

## 4.3 Discussion of Results: Lifted Conic Cuts

**Portfolio Optimization.** For evaluation of the performance of lifted cuts derived in Section 3, we used both types of portfolio optimization problems, with parameters set up as described above. As it has been already noted, each lifted nonlinear cut was replaced by its outer gradient polyhedral approximation. Specifically, the approximation accuracy was set at $10^{-3}$. Since in this case each cut results in multiple additional linear constraints, we restricted the number of lifted cuts to be added at the root node to two. The results obtained for portfolio optimization problems with cardinality constraints (16) and lot-buying constraints (17), each averaged over 30 problem instances, are summarized in Tables 2 and 3, respectively. We observed similar improvements in computational time for both types of problems. Also, it has been observed that utilization

| $p = 2.0$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | all instances | | | "difficult" instances | | |
| $n$ | $m$ | | default CPLEX | conic MIR | % better | default CPLEX | conic MIR | % better |
| 500 | 200 | time | 26.88 | 22.88 | 29.41% | 58.22 | 43.77 | 61.11% |
| | | nodes | 2.0 | 0.75 | 100.00% | 5.67 | 2.11 | 100.0% |
| | | cuts | 16.74 | 48.65 | – | 16.06 | 50.94 | – |
| | 600 | time | 218.0 | 224.72 | 52.83% | 356.27 | 369.85 | 67.86% |
| | | nodes | 3.34 | 3.17 | 92.45% | 6.32 | 6.0 | 85.71% |
| | | cuts | 73.45 | 53.90 | – | 19.08 | 55.82 | – |
| | 1000 | time | 1117.45 | 856.59 | 45.61% | 2045.46 | 1418.66 | 65.22% |
| | | nodes | 1.68 | 0.60 | 96.49% | 4.17 | 1.48 | 91.30% |
| | | cuts | 102.54 | 63.40 | – | 76.00 | 50.87 | – |

| $p = 3.0$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | all instances | | | "difficult" instances | | |
| $n$ | $m$ | | default CPLEX | conic MIR | % better | default CPLEX | conic MIR | % better |
| 500 | 200 | time | 12.60 | 11.10 | 37.25% | 24.11 | 20.68 | 76.92% |
| | | nodes | 0.88 | 0.31 | 100.00% | 1.23 | 3.46 | 100.0% |
| | | cuts | 11.71 | 49.65 | – | 11.38 | 50.94 | – |
| | 600 | time | 189.76 | 71.90 | 51.92% | 421.64 | 133.0 | 87.50% |
| | | nodes | 6.92 | 2.13 | 100.00% | 22.94 | 7.06 | 100.00% |
| | | cuts | 18.92 | 54.58 | – | 15.37 | 48.26 | – |
| | 1000 | time | 910.04 | 560.12 | 66.67% | 1741.93 | 974.53 | 61.90% |
| | | nodes | 1.53 | 0.35 | 98.25% | 4.14 | 0.95 | 95.24% |
| | | cuts | 32.81 | 63.40 | – | 22.0 | 50.87 | – |

| $p = 4.0$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | all instances | | | "difficult" instances | | |
| $n$ | $m$ | | default CPLEX | conic MIR | % better | default CPLEX | conic MIR | % better |
| 500 | 200 | time | 31.92 | 26.54 | 35.29% | 62.04 | 48.06 | 52.17% |
| | | nodes | 2.29 | 0.98 | 98.04% | 5.09 | 2.17 | 95.65% |
| | | cuts | 26.16 | 48.65 | – | 29.17 | 63.83 | – |
| | 600 | time | 582.88 | 324.86 | 43.40% | 875.88 | 471.92 | 55.88% |
| | | nodes | 9.25 | 8.0 | 88.84% | 14.41 | 12.47 | 82.36% |
| | | cuts | 76.75 | 53.91 | – | 37.87 | 60.01 | – |

**Table 1:** Performance of conic MIR cuts for randomly generated MIpOCP problems. The "% better" column represents the percentage of problem instances for which conic MIR cuts approach outperformed CPLEX with default parameters in terms of solution time and number of nodes, respectively. "Difficult" instances represent problem instances that cannot be solved at the root node.

of lifted cuts in portfolio optimization with lot-buying constraints does not generally lead to a reduction in the number of nodes explored in the solution tree. Thus, based on this observation and results of the experiments of the previous section, we can suggest that the observed improvement is probably partially due to considerably less time spent while looking for cuts. In contrast, in portfolio problems with cardinality constraints we observe reductions in both the number of nodes and solution times due to utilization of lifted cuts.

# 5 Concluding Remarks

The recent progress in solving mixed integer programming problems can partially be attributed to the advances in utilization of valid inequalities for integer and mixed integer sets. Mixed integer cuts allow for tightening of the bounds given by the continuous relaxation of the problem during the branch-and-cut procedure and, as a result, can lead to reductions in the number of nodes explored in the branch-and-bound tree and in the overall computational time. Typically, valid inequalities exploit specific structure of the feasible set of the problem.

This paper presents two families of valid inequalities for mixed integer $p$-order programming problems that arise in risk-averse stochastic optimization with downside risk measures. Particularly, we developed mixed integer rounding cuts and nonlinear lifted cuts for mixed integer $p$-order conic sets, extending the corresponding results for mixed integer second order programming problems [2, 3]. Computational studies on randomly generated problems as well as discrete portfolio optimization problems with historical data demonstrate that both conic MIR cuts and lifted conic cuts lead to improved solution times.

In general, nonlinear cuts are not yet as prevalent as linear ones, partly due to the fact that additional

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $p = 2.0$ | | | | | |
| | | default CPLEX | | | conic MIR cuts | | | lifted conic cuts | | |
| $n$ | $m$ | time | nodes | cuts | time | nodes | cuts | time | nodes | cuts |
| 100 | 600 | 360.97 | 31.31 | 0.10 | 315.98 | 31.90 | 3.00 | **281.34** | 30.59 | 2.00 |
| | 1000 | 787.16 | 31.15 | 0.00 | 772.44 | 77.90 | 3.00 | **595.66** | 30.77 | 2.00 |
| | 1400 | 916.18 | 37.58 | 0.00 | 766.14 | 55.50 | 3.00 | **664.73** | 25.8 | 2.00 |
| 150 | 600 | 446.11 | 41.80 | 0.00 | 400.02 | 41.20 | 3.00 | **377.87** | 40.20 | 2.00 |
| | 1000 | 1566.79 | 53.44 | 0.00 | 1436.57 | 53.20 | 3.00 | **1326.74** | 52.33 | 2.00 |
| | 1400 | 2601.84 | 40.69 | 0.00 | 2343.03 | 38.83 | 3.00 | **2196.61** | 39.92 | 2.00 |
| | | | | | $p = 3.0$ | | | | | |
| | | default CPLEX | | | conic MIR cuts | | | lifted conic cuts | | |
| $n$ | $m$ | time | nodes | cuts | time | nodes | cuts | time | nodes | cuts |
| 100 | 600 | 813.62 | 47.93 | 0.00 | **537.14** | 45.63 | 3.00 | 610.98 | 45.35 | 2.00 |
| | 1000 | 1449.75 | 49.78 | 0.00 | 1216.24 | 49.90 | 3.00 | **1213.02** | 49.67 | 2.00 |
| | 1400 | 1671.64 | 36.38 | 0.00 | 1518.44 | 59.87 | 3.00 | **1428.81** | 40.2 | 2.00 |
| 150 | 600 | 488.07 | 41.40 | 0.20 | 415.92 | 40.67 | 3.00 | **354.40** | 39.80 | 2.00 |
| | 1000 | 2877.30 | 80.81 | 0.05 | 2661.90 | 83.87 | 3.00 | **2514.82** | 86.71 | 2.00 |
| | 1400 | 4307.80 | 70.72 | 0.11 | 4006.54 | 70.43 | 3.00 | **3739.91** | 69.89 | 2.00 |
| | | | | | $p = 4.0$ | | | | | |
| | | default CPLEX | | | conic MIR cuts | | | lifted conic cuts | | |
| $n$ | $m$ | time | nodes | cuts | time | nodes | cuts | time | nodes | cuts |
| 100 | 600 | 1234.58 | 47.08 | 0.10 | 1186.99 | 45.83 | 3.00 | **1062.46** | 45.58 | 2.00 |
| | 1000 | 2368.82 | 45.05 | 0.00 | 2204.83 | 48.20 | 3.00 | **2062.06** | 47.87 | 2.00 |
| | 1400 | 3243.04 | 33.49 | 0.00 | 2630.18 | 34.40 | 3.00 | **2552.70** | 31.48 | 2.00 |
| 150 | 600 | 435.52 | 34.50 | 0.17 | 371.95 | 58.65 | 3.00 | **340.62** | 33.33 | 2.00 |
| | 1000 | 5913.61 | 94.71 | 0.00 | 5451.90 | 47.95 | 3.00 | **5168.28** | 97.57 | 2.00 |
| | 1400 | 6442.82 | 62.50 | 0.05 | 6087.91 | 31.30 | 3.00 | **5286.47** | 62.85 | 2.00 |

**Table 2:** Performance of conic MIR and lifted cuts in cardinality constrained portfolio optimization problems. Entries in bold correspond to the minimum solution time for each row. Results are averaged over 30 instances for each problem size.

nonlinear inequalities in the bounding (relaxed) problem tend to have deteriorating effect on the computational time of branch-and-bound procedure. In order to improve the computational tractability of the derived nonlinear lifted cuts within the branch-and-cut framework, we proposed replacing them with their polyhedral approximations; since the nonlinear lifted cuts constitute low-dimensional $p$-cones, the corresponding polyhedral approximations are relatively inexpensive. In this respect, our computational results are among the first successful applications of nonlinear cuts in nonlinear mixed integer programming problems.

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| | | default CPLEX | | | conic MIR cuts | | | lifted conic cuts | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $p = 2.0$ | | | | | |
| $n$ | $m$ | time | nodes | cuts | time | nodes | cuts | time | nodes | cuts |
| 10 | 200 | 9.09 | 4.13 | 1.50 | 9.59 | 5.10 | 0.00 | **8.03** | 5.31 | 2.00 |
| | 600 | 45.53 | 4.67 | 2.61 | 40.08 | 5.57 | 0.13 | **32.98** | 6.17 | 2.00 |
| | 1000 | 117.78 | 11.47 | 2.37 | 111.44 | 13.97 | 0.33 | **102.81** | 14.74 | 2.00 |
| 20 | 200 | 42.49 | 20.79 | 3.64 | 37.17 | 23.13 | 0.40 | **32.00** | 25.36 | 2.00 |
| | 600 | 103.28 | 12.80 | 5.00 | 101.67 | 16.93 | 0.13 | **94.96** | 20.16 | 2.00 |
| | 1000 | 188.04 | 13.63 | 3.19 | 177.53 | 13.83 | 1.10 | **168.88** | 13.63 | 2.00 |
| 50 | 200 | 54.50 | 42.94 | 4.38 | 51.21 | 45.40 | 0.50 | **46.55** | 47.44 | 2.00 |
| | 600 | 307.66 | 33.19 | 6.19 | 286.28 | 41.27 | 1.50 | **268.13** | 46.75 | 2.00 |
| | 1000 | 640.82 | 49.71 | 3.71 | **635.54** | 62.03 | 0.00 | 664.29 | 69.35 | 2.00 |
| | | | | | $p = 3.0$ | | | | | |
| | | default CPLEX | | | conic MIR cuts | | | lifted conic cuts | | |
| $n$ | $m$ | time | nodes | cuts | time | nodes | cuts | time | nodes | cuts |
| 10 | 200 | 18.56 | 4.79 | 3.57 | 17.33 | 7.73 | 0.03 | **15.50** | 9.50 | 2.00 |
| | 600 | 49.60 | 8.33 | 2.22 | 42.32 | 9.73 | 0.03 | **34.46** | 10.39 | 2.00 |
| | 1000 | 96.15 | 10.19 | 2.38 | 94.93 | 12.97 | 0.03 | **90.25** | 15.38 | 2.00 |
| 20 | 200 | 34.05 | 9.06 | 3.11 | 27.11 | 10.97 | 1.10 | **21.23** | 12.00 | 2.00 |
| | 600 | 96.98 | 9.51 | 4.22 | 79.78 | 12.00 | 1.10 | **66.74** | 13.84 | 2.00 |
| | 1000 | **130.59** | 4.53 | 4.35 | 134.93 | 4.67 | 1.23 | 141.49 | 4.53 | 2.00 |
| 50 | 200 | 78.29 | 30.55 | 5.10 | 70.07 | 35.93 | 0.03 | **57.25** | 39.95 | 2.00 |
| | 600 | 316.89 | 37.39 | 5.33 | 275.04 | 38.17 | 0.03 | **210.81** | 37.67 | 2.00 |
| | 1000 | 540.25 | 22.58 | 5.37 | 500.46 | 36.87 | 1.00 | **459.55** | 47.74 | 2.00 |
| | | | | | $p = 4.0$ | | | | | |
| | | default CPLEX | | | conic MIR cuts | | | lifted conic cuts | | |
| $n$ | $m$ | time | nodes | cuts | time | nodes | cuts | time | nodes | cuts |
| 10 | 200 | 23.29 | 6.29 | 2.29 | 17.93 | 6.13 | 2.00 | **13.58** | 5.71 | 2.00 |
| | 600 | 44.50 | 3.57 | 2.21 | 41.56 | 3.93 | 7.03 | **37.73** | 4.21 | 2.00 |
| | 1000 | **122.08** | 8.00 | 2.29 | 123.10 | 10.13 | 25.03 | 125.04 | 12.71 | 2.00 |
| 20 | 200 | 49.11 | 7.93 | 4.07 | 43.88 | 16.07 | 0.13 | **40.19** | 20.40 | 2.00 |
| | 600 | 110.42 | 16.47 | 3.31 | 101.32 | 18.00 | 12.50 | **89.95** | 18.24 | 2.00 |
| | 1000 | 315.87 | 10.89 | 4.94 | 279.44 | 11.10 | 34.23 | **256.45** | 10.89 | 2.00 |
| 50 | 200 | 127.20 | 43.78 | 5.17 | 118.54 | 46.67 | 0.46 | **112.06** | 48.06 | 2.00 |
| | 600 | 416.48 | 36.76 | 4.68 | 344.87 | 33.93 | 21.40 | **294.47** | 29.32 | 2.00 |
| | 1000 | 993.53 | 44.50 | 5.71 | 825.43 | 46.20 | 33.17 | **682.21** | 56.59 | 2.00 |

**Table 3:** Performance of conic MIR and lifted cuts in portfolio optimization problems with lot-buying constraints. Entries in bold correspond to the minimum solution time for each row. Results are averaged over 30 instances for each problem size.


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# A A Direct Derivation of Conic Mixed Integer Rounding Cuts for Mixed Integer $p$-Order Cone Programming Problems

Following [2], let us first consider a simple case of the following set

$$T = \big\{(y, w, t, x) \in \mathbb{R}^3_+ \times \mathbb{Z} : [x + y - w - b]_+ \le t\big\}.$$

Let us denote by $\operatorname{relax}(T)$ the continuous relaxation of $T$ and by $\operatorname{conv}(T)$ its convex hull. It can be seen that the extreme rays of $\operatorname{relax}(T)$ are as follows: $(1, 0, 0, 1)$, $(-1, 0, 0, 0)$, $(1, 0, 1, 0)$, $(-1, 1, 0, 0)$, and its only extreme point is $(b, 0, 0, 0)$. Let us also denote $f = b - \lfloor b \rfloor$. Clearly, the case of $f = 0$ is not interesting, hence it can be assumed that $f > 0$, whereby $\operatorname{conv}(T)$ has four extreme points: $(\lfloor b \rfloor, 0, 0, 0)$, $(\lfloor b \rfloor, f, 0, 0)$, $(\lceil b \rceil, 0, 1 - f, 0)$, $(\lceil b \rceil, 0, 0, 1 - f)$. With these observations in mind we can formulate the following proposition.

**Proposition 6** *Inequality*

$$(1 - f)(x - \lfloor b \rfloor) \le t + w \tag{18}$$

*is valid for $T$ and cuts off all points in* $\operatorname{relax}(T) \setminus \operatorname{conv}(T)$.

**Proof:** First, let us show the validity of (18). The base inequality for $T$ is

$$[x + y - w - b]_+ \le t. \tag{19}$$

Now, let $x = \lfloor b \rfloor - \alpha$ and $\alpha \ge 0$. In this case, (19) turns into $t \ge [y - w - f - \alpha]_+$ and (18) becomes $t \ge -(1-f)\alpha - w$. Observing that $[y-w-f-\alpha]_+ - (-(1-f)\alpha - w) = \max\{y - f - \alpha f, (1-f)\alpha + w\} \ge 0$, one obtains that (19) implies (18) for $x \le \lfloor b \rfloor$.

On the other hand, if $x = \lceil b \rceil + \alpha$ with $\alpha \ge 0$, then (19) becomes $t \ge [y - w + (1 - f) + \alpha]_+$ and (18) turns into $t \ge (1 - f)(1 + \alpha) - w$. Similarly to above,

$$\begin{aligned}
&[y - w + (1 - f) + \alpha]_+ - ((1 - f)(1 + \alpha) - w)\\
&= \max\{y - w + (1 - f) + \alpha - (1 - f) - \alpha(1 - f) + w, w - (1 - f)(1 + \alpha)\}\\
&= \max\{y + \alpha f, w - (1 - f)(1 + \alpha)\} \ge 0,
\end{aligned}$$

which means that (19) implies (18) for $x \ge \lceil b \rceil$. Hence, (18) is valid for $T$.

To prove the remaining part of the proposition, consider the polyhedron $\acute{T}$ defined by the inequalities

$$x + y - w - b \leq t, \tag{20}$$
$$0 \leq t, \tag{21}$$
$$0 \leq y, \tag{22}$$
$$0 \leq w, \tag{23}$$
$$(1 - f)(x - \lfloor b \rfloor) \leq t + w. \tag{24}$$

Since $\acute{T}$ has four variables, the basic solutions of $\acute{T}$ are defined by four of these inegualities at equality. They are:

- Inequalities (20), (21), (22), (23): $(x, y, w, t) = (b, 0, 0, 0)$ is infeasible if $f \neq 0$.
- Inequalities (20), (21), (22), (24): $(x, y, w, t) = (\lceil b \rceil, 0, 1 - f, 0)$.
- Inequalities (20), (21), (23), (24): $(x, y, w, t) = (\lfloor b \rfloor, f, 0, 0)$.
- Inequalities (20), (22), (22), (24): $(x, y, w, t) = (\lceil b \rceil, 0, 0, 1 - f)$.
- Inequalities (21), (23), (22), (24): $(x, y, w, t) = (\lfloor b \rfloor, 0, 0, 0)$.

Hence, $\mathrm{conv}(T)$ has exactly the same extreme points as $\acute{T}$, which completes the proof. □

In the general case, let

$$\widehat{T} = \left\{(y^+, y^-, t, \mathbf{x}) \in \mathbb{R}^3_+ \times \mathbb{Z}^n_+ \ : \ [\mathbf{a}^\top \mathbf{x} + y^+ - y^- - b]_+ \leq t\right\}, \tag{25}$$

and consider the following function

$$\phi_f(a) = \begin{cases} (1 - f)n, & n \leq a < n + f \\ (1 - f)n + (a - n) - f, & n + f \leq a < n + 1. \end{cases}$$

**Proposition 7** *For $\alpha \neq 0$ the following inequality*

$$\sum_{j=1}^{n} \phi_{f_{|\alpha|}}\Big(\frac{a_j}{|\alpha|}\Big) x_j - \phi_{f_{|\alpha|}}\Big(\frac{b}{|\alpha|}\Big) \leq \frac{t + y^-}{|\alpha|}, \tag{26}$$

*where $f_{|\alpha|} = \frac{b}{|\alpha|} - \lfloor \frac{b}{|\alpha|} \rfloor$, is valid for $\widehat{T}$.*

**Proof:** First consider the case $\alpha = 1$. We can rewrite the base inequality for (25) as

$$\Big[\Big(\sum_{f_j \leq f} \lfloor a_j \rfloor x_j + \sum_{f_j > f} \lceil a_j \rceil x_j\Big) + \Big(\sum_{f_j \leq f} f_j x_j + y^+\Big) - \Big(\sum_{f_j > f} (1 - f_j) x_j + y^-\Big) - b\Big]_+ \leq t,$$

where $f_j = a_j - \lfloor a_j \rfloor$. Observe that

$$\acute{x} = \sum_{f_j \leq f} \lfloor a_j \rfloor x_j + \sum_{f_j > f} \lceil a_j \rceil x_j \in \mathbb{Z}, \quad \acute{y} = \sum_{f_j \leq f} f_j x_j + y^+ \geq 0, \quad \acute{w} = \sum_{f_j > f} (1 - f_j) x_j + y^- \geq 0.$$

Hence, we can apply simple conic MIR inequality (18) with variables $(\acute{x}, \acute{y}, \acute{w}, t)$:

$$(1 - f)\Big(\sum_{f_j \leq f} \lfloor a_j \rfloor x_j + \sum_{f_j > f} \lceil a_j \rceil x_j - \lfloor b \rfloor\Big) \leq t + \sum_{f_j > f} (1 - f_j) x_j + y^-.$$

Rewriting it with the help of function $\phi_f(a)$, we obtain that

$$\sum_{j=1}^{n} \phi_f(a_j)x_j - \phi_f(b) \leq t + y^-.$$

So, by Proposition 6 inequality (26) is valid for $\alpha = 1$. In order to see that the result holds for all $\alpha \neq 0$ we only need to scale the base inequality:

$$\left[\frac{1}{|\alpha|}(\mathbf{a}^\top \mathbf{x} + y^+ - y^- - b)\right]_+ \leq \frac{t}{|\alpha|}.$$

$\square$

**Proposition 8** *Inequalities* (26) *with* $\alpha = a_j,\ j = 1, \ldots, n$ *are sufficient to cut off all fractional extreme points of* $\mathrm{relax}(\widehat{T})$.

**Proof:** The set $\mathrm{relax}(\widehat{T})$ is defined by $n + 3$ variables and $n + 4$ constraints. Therefore, if $x_j > 0$ in an extreme point, then the remaining $n + 3$ constraints must be active. Thus, the continuous relaxation has at most $n$ fractional extreme points $(x^j, 0, 0, 0)$ of the form $x^j_j = \frac{b}{a_j} > 0$, and $x^j_i = 0$, for $i \neq j$. Such points are infeasible if $\frac{b}{a_j} \notin \mathbb{Z}$. Now, let $a_j > 0$. For such a fractional extreme point inequality (26) reduces to

$$\phi_{f_{a_j}}(1)x_j - \phi_{f_{a_j}}\left(\frac{b}{a_j}\right) \leq \frac{t + y^-}{a_j}, \quad \text{or} \quad (1 - f_{a_j})x_j - (1 - f_{a_j})\left\lfloor \frac{b}{a_j} \right\rfloor \leq \frac{t + y^-}{a_j},$$

which by Proposition 6 cuts off fractional extreme point with $x^j_j = \frac{b}{a_j}$.

Now, let us consider $a_j < 0$. In this case we observe that the inequality (26) reduces to

$$\phi_{f_{|a_j|}}(-1)x_j - \phi_{f_{|a_j|}}\left(\frac{b}{|a_j|}\right) \leq \frac{t + y^-}{|a_j|}, \quad \text{or} \quad -(1 - f_{|a_j|})x_j - (1 - f_{|a_j|})\left\lfloor \frac{b}{|a_j|} \right\rfloor \leq \frac{t + y^-}{|a_j|},$$

which again, cuts off fractional extreme point with $x^j_j = \frac{b}{a_j}$. $\square$