A MARKOV CHAIN APPROACH TO ANALYSIS OF COOPERATION IN MULTI-AGENT SEARCH MISSIONS

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We consider the effects of cueing in a cooperative search mission that involves several autonomous agents. Two scenarios are discussed: one in which the search is conducted by a number of identical search-and-engage vehicles, and one where these vehicles are assisted by a search-only (reconnaissance) asset. The cooperation between the autonomous agents is facilitated via cueing, i.e. the information transmitted to the agents by a searcher that has just detected a target. The effect of cueing on the target detection probability is derived from first principles using a Markov chain analysis. Exact solutions to Kolmogorov-type differential equations are presented, and existence of an upper bound on the benefit of cueing is demonstrated.

1. Introduction

In any system-of-systems analysis, consideration of dependencies between systems is imperative. In this paper, we consider a particular type of system interaction, called cueing. The interaction could be between similar systems, such as two or more wide area search munitions, or between dissimilar systems, such as a reconnaissance asset and a munition. In this paper, we consider two scenarios: one in which the search is conducted by a number of identical search-and-engage vehicles, and one where these vehicles are assisted by a search-only vehicle. The autonomous agents forming the system cooperatively interact via cueing.

In Shakespeare's day, the word "cue" meant a signal (a word, phrase, or bit of stage business) to a performer to begin a specific speech or action.[6] The word is now used more generally for anything serving a comparable purpose. In this paper, we mean any information that provides focus to a search; e.g., information that limits the search area or provides a search heading.

Search theory is one of the oldest areas of operations research,[8] with a solid foundation in mathematics, probability and experimental physics. Yet, search theory is clearly of more than academic interest. At times, a search can become an international priority, as in the 1966 search for the hydrogen bomb lost in the Mediterranean near Palomares, Spain.

That search was an immense operation involving 34 ships, 2,200 sailors, 130 frogmen and four mini-subs. The search took 75 days, but might have concluded much earlier if cueing had been utilized from the start. A Spanish fisherman had come forward quickly to say he'd seen something fall that looked like a bomb, but experts ignored him.

Instead, they focused on four possible trajectories calculated by a computer, but for weeks found only airplane pieces. Finally, the fisherman, Francisco Simo, was summoned back. He sent searchers in the right direction, and a two-man sub, the Alvin, located the 10-foot-long bomb under 2,162 feet of water.[11]

Cueing is a current topic in vision research. For example, Arrington et al.[2] study the role of objects in guiding spatial attention through a cluttered visual environment. Magnetic resonance imaging is used to measure brain activity during cued discrimination tasks requiring subjects to orient attention either to a region bounded by an object or to an unbounded region of space in anticipation of an upcoming target. Comparison between the two tasks revealed greater brain activity when an object cues the subjects attention.

Bernard Koopman[8] pioneered the application of mathematical process to military search problems during World War II. Koopman[4] discusses the case in which a searcher inadvertently provides information to the target, perhaps allowing the target to employ evasive action. The use of receivers on German U-boats to detect search radar signals in World War II is a classic example. Koopman referred to this type of cueing as target alerting.

This paper uses a detection rate approach to examine the effect of cueing on probability of target detection. Koopman[5] used a similar approach in his discussion of target detection. In Koopman's terminology, a quantity γ was called the "instantaneous probability of detection." From this starting point, Koopman derived the probability of detection as a function of time. It is very clear that Koopman's instantaneous probability of detection is precisely the individual searcher detection rate used here. The main difference is that Koopman considered a single searcher, while we consider the case of multiple interdependent searchers.

Wasburn[10] examines the case of a single searcher attempting to detect a randomly moving target at a discrete time. Given an effort distribution, bounded at each discrete time t, Washburn establishes an upper bound on the probability of target detection. It is noteworthy that Washburn mentions that the detection rate approach to computation of detection probabilities has proved to be more robust than approaches relying on geometric models.

Alpern and Gal[1] discuss the problem of searching for a submarine with a known initial location. Thomas and Washburn[9] considered dynamic search games in which the hider starts moving at time zero from a location known to both a searcher and a hider, while the searcher starts with a time delay known to both players; for example, a helicopter attempts to detect a submarine that reveals its position by torpedoing a ship.

In this paper, we use a Markov chain analysis to examine cueing as a coupling mechanism among several searchers. A Markov chain approach to target detection can be found in Stone,[8] which deals with the optimal allocation of effort to detect a target. A prior distribution of the target's location is assumed known to the searcher. Stone uses a Markov chain analysis to deal with the search for targets whose motion is Markovian. In Stone's formulation, the states correspond to cells that contain a target at a discrete time with a specified probability. In this research, the states correspond to detection states for individual search vehicles.

The rest of the paper is organized as follows. The next section discusses the effect of cueing on the performance of a cooperative system of several identical search agents. Section 3 presents analysis of a search system that involves a search-only vehicle that provides cues to a number of search-and-engage vehicles.

2. Cooperative search

Consider a system of N agents engaged in a cooperative search mission, where the objective of every agent is to find (*detect*) an object of interest (a *target*). The search capabilities of any agent are characterized by the detection rate θ, i.e. the probability of detecting a target within time interval Δt:

$$\mathsf{P}[\text{agent } i \text{ detects a target during time } \Delta t] = \theta \Delta t + o(\Delta t). \quad (1)$$

Upon detecting a target, an agent discontinues its search by *engaging* the detected target; we also say that such an agent becomes *inactive*. For example, in a search-and-rescue mission for passengers of a sinking ship the searchers will try to rescue the passenger(s) they find, instead of continuing the search; on the battlefield, an autonomous wide-area search munition will attack the detected target, etc. Moreover, it is assumed that upon engaging a target, the searcher immediately cues the remaining *active* agents, thereby potentially increasing their detection capabilities. Within the presented framework the informational content of the cueing signal is not important; instead, we are interested in the degree by which cueing impacts the search capabilities of individual agents in a cooperative system. In accordance to this, at any time $t \geq 0$ the detection rate of a searcher may change values as

$$\theta_0 \rightarrow \theta_1 \rightarrow \ldots \rightarrow \theta_{N-1}, \quad (2)$$

where θ_k is the detection rate common to $N-k$ active searchers. Clearly, θ_0 is equal to the initial "uncued" detection rate: $\theta_0 = \theta$. Also, it is natural to assume that cueing generally leads to improvement of search capabilities, whence $\theta_k \geq \theta, \ k = 1, \ldots, N-1$. The search mission terminates when there are no active searchers left, i.e., each searcher has found a target.

The outlined model, complemented by the usual assumptions of independence of target detections on non-overlapping time intervals etc., can be formulated as a continuous-time discrete-state Markov chain, or, furthermore, as a *pure death process*.[3,7] The states of the system are identified by the number k of inactive searchers, $k = 0, 1, \ldots, N$. The presented formulation, however, differs slightly from the traditional models for birth-death processes in that it defines the transition rate between states by the search rates of individual agents (2), versus the rates defined with respect to the entire population.[3] Let $\mathcal{S}_{N-k,k}$ be *a state* of the system in which there are k inactive searchers, and, correspondingly, $N-k$ active searchers; note that there are $\binom{N}{k}$ different states $\mathcal{S}_{N-k,k}$. Defining probabilities $P_{N-k,k}(t)$ as

$$P_{N-k,k}(t) = \mathsf{P}[\text{system is in one of the states } \mathcal{S}_{N-k,k} \text{ at time } t],$$

one can write the system of Chapman-Kolmogorov ODEs governing these probabilities as

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{N-k,\,k}(t) = -\bar{\delta}_{kN}\,(N-k)\,\theta_k\,P_{N-k,\,k}(t) + \bar{\delta}_{k0}\,k\theta_{k-1}\,P_{N-k+1,\,k-1}(t), \qquad k = 0, \ldots, N, \quad (3)$$

where $\bar{\delta}_{ij}$ is the negation of the Kroneker symbol δ_{ij}

$$\bar{\delta}_{ij} = 1 - \delta_{ij} = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{if } i \neq j. \end{cases} \quad (4)$$

Equations (3) admit a simple interpretation: since in a state $\mathcal{S}_{N-k,k}$ there are $N-k$ active searchers with search rate θ_k, the probability of the system being in this state decreases at rate $(N-k)\theta_k P_{N-k,k}(t)$ as each of $N-k$ searchers may detect a target and turn the system into a state $\mathcal{S}_{N-k-1,k+1}$. On the other hand, probability $P_{N-k,k}(t)$ increases at rate $k\theta_{k-1}P_{N-k+1,k-1}(t)$ as there are exactly $\binom{k}{1} = k$ states $\mathcal{S}_{N-k+1,k-1}$ that may lead to a (given) state $\mathcal{S}_{N-k,k}$. Indeed, let $A_k = \{i_1, \ldots, i_k\}$ be any set containing k inactive searchers, $|A_k| = k$. Trivially, A_k can be represented in k different ways as $A_k = A^j_{k-1} \cup \{i_j\}$, where $A^j_{k-1} = A_k \backslash \{i_j\} \subset A_k$, $|A^j_{k-1}| = k-1$, $j = 1, \ldots, k$. Thus, a state $\mathcal{S}_{N-k,k}$ with k inactive searchers can only be obtained from exactly k states $\mathcal{S}_{N-k+1,k-1}$ with $k-1$ inactive searchers. Factors $\bar{\delta}_{N,k}$ and $\bar{\delta}_{k,0}$ in (3) have the obvious function of handling the extreme cases of $k = 0$ and $k = N$. Denoting $P_{N-k,k}(t) = \hat{P}_k(t)$, $k = 0, \ldots, N$, equations (3) can be rewritten in the matrix form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{P}} = M\hat{\mathbf{P}}, \quad (5)$$

where $\hat{\mathbf{P}} = (\hat{P}_0, \ldots, \hat{P}_N)^T$, and matrix $M = \{m_{ij}\} \in \mathbb{R}^{(N+1)\times(N+1)}$ has the following non-zero elements:

$$\begin{aligned} m_{ii} &= -(N-i)\theta_i, \qquad && i = 0, \ldots, N, \\ m_{i,i-1} &= i\theta_{i-1}, && i = 1, \ldots, N. \end{aligned} \quad (6)$$

Explicitly, the matrix M in (5) can be written as

$$M = \begin{pmatrix} -N\theta_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \theta_0 & -(N-1)\theta_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2\theta_1 & -(N-2)\theta_2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ & & & \cdots & -2\theta_{N-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & (N-1)\theta_{N-2} & -\theta_{N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & N\theta_{N-1} & 0 \end{pmatrix}. \quad (7)$$

The initial conditions for equations (5) reflect the fact that at $t = 0$ the system is in the state $\mathcal{S}_{N,0}$ with probability 1:

$$\hat{P}_0(0) = 1, \hat{P}_1(0) = \hat{P}_2(0) = \ldots = \hat{P}_N(0) = 0. \tag{8}$$

Using (5) and (8) it is straightforward to verify that the probabilities $\hat{P}_k(t)$ satisfy the identity

$$\sum_{k=0}^{N} \binom{N}{k} \hat{P}_k(t) = 1, \quad t \geq 0. \tag{9}$$

Clearly, matrix (7) is lower-triangular, hence system (5) has the characteristic equation of the form

$$\lambda \prod_{j=0}^{N-1} \big(\lambda + (N-j)\theta_j\big) = 0, \tag{10}$$

Assuming that all eigenvalues of M are distinct,

$$(N-i)\theta_i \neq (N-j)\theta_j, \quad 0 \leq i < j \leq N-1, \tag{11}$$

the solution of the Cauchy problem (5), (8) can be written in a simple closed form

$$\hat{P}_i(t) = \sum_{j=0}^{i} a_{ji} e^{m_{jj} t}, \quad i = 0, \ldots, N, \tag{12}$$

with coefficients a_{ji} defined recursively as

$$a_{ji} = \frac{m_{i,i-1}\, a_{j,i-1}}{m_{jj} - m_{ii}}, \quad 0 \leq j < i \leq N, \tag{13}$$

$$a_{ii} = -\sum_{s=0}^{i-1} a_{si}, \quad i = 1, \ldots, N, \qquad a_{00} = \hat{P}_0(0) = 1. \tag{14}$$

One of the possible measures of performance (MOE) for a cooperative search system is the time required for all N searchers to find targets. In the scope of the presented approach, this characteristic is embodied by the probability $P_{0,N}(t) = \hat{P}_N(t)$ of the system being in the state $\mathcal{S}_{0,N}$ at time t. Noting that dependencies (14)–(13) can be reexpressed for all $0 \leq j < i \leq N-1$ as

$$a_{ji} = \beta_{ji} a_{ii}, \quad \text{where} \quad \beta_{ji} = \frac{\theta_j}{\theta_i} \frac{\binom{N-1}{j}}{\binom{N-1}{i}} \prod_{s=j+1}^{i} \left[1 - \frac{(N-j)\theta_j}{(N-i)\theta_i}\right]^{-1}, \tag{15}$$

the probability $\hat{P}_N(t)$ can be represented from equality (12) in the form

$$\hat{P}_N(t) = 1 - \sum_{j=0}^{N-1} \binom{N}{j} \prod_{s=j+1}^{N-1} \left[1 - \frac{(N-j)\theta_j}{(N-s)\theta_s}\right]^{-1} a_{jj}\, e^{-(N-j)\theta_j\, t}, \qquad (16)$$

where a_{jj} are calculated due to (14) and (15) as

$$a_{jj} = -\sum_{r=0}^{j-1} \beta_{rj} a_{rr}, \quad j = 1, \ldots, N-1, \qquad (17)$$

and the usual convention $\prod_{i=i_1}^{i_2} (\cdot)_i = 1$ for $i_1 > i_2$ is adopted.

Expression (16) can now be used for the analysis of the effect of cueing on the cooperation among the searchers. For example, it is easy to see that when cueing has no impact on the detection rates of the searchers, then $\hat{P}_N(t)$ is equal to the probability of all N agents detecting targets independently:

Proposition 2.1. *If $\theta_0 = \theta_1 = \ldots = \theta_{N-1} = \theta$, i.e. the cueing has no effect on the detection capabilities of the searchers, then*

$$\hat{P}_N(t) = \left(1 - e^{-\theta t}\right)^N. \qquad (18)$$

Proof. If all the cued detection rates (2) are equal to the uncued rate θ, then the product term in expressions (15) and (16) for β_{ji} and $\hat{P}_N(t)$ reduces to

$$\prod_{s=j+1}^{i} \left[1 - \frac{(N-j)\theta_j}{(N-s)\theta_s}\right]^{-1} = (-1)^{i-j},$$

which immediately yields

$$\beta_{ji} = (-1)^{i-j} \binom{i}{j}, \quad 0 \le j < i \le N-1, \qquad (19)$$

and

$$\hat{P}_N(t) = 1 + \sum_{j=0}^{N-1} a_{jj} \binom{N}{j} (-1)^{N-j} e^{-(N-j)\theta t}.$$

The last expression verifies the statement (18) of the proposition provided that $a_{jj} = 1,\ j = 0, \ldots, N-1$. Using the induction argument, we have that $a_{00} = \hat{P}_0(0) = 1$ from (14), and, assuming that $a_{11} = \ldots = a_{jj} = 1$ for some j, by means of (17) and (19) we obtain

$$a_{j+1,j+1} = -\sum_{s=0}^{j} a_{s,j+1} = -\sum_{s=0}^{j} (-1)^{j+1-s} \binom{j+1}{s} a_{ss} = 1. \qquad (20)$$

The last equality in (20) follows from the Newton binom formula $(1-1)^{j+1} = \sum_{j=0}^{j}(-1)^{j+1-s}\binom{j+1}{s} + 1$. $\square$

Obviously, cueing has the purpose of improving the system's effectiveness by increasing the cued detection rates $\theta_1, \ldots, \theta_{N-1}$. Indeed, it can be verified that higher values of $\theta_1, \ldots, \theta_{N-1}$ increase the probability of detection $\hat{P}_N(t)$ for a given t. Below we demonstrate that under quite general conditions the effect of cueing on the system's performance is bounded, i.e., there exists an upper bound for the state probability $\hat{P}_N(t)$ when the cued detection rates $\theta_1, \ldots, \theta_{N-1}$ increase indefinitely.

Proposition 2.2. *Let the cued detection rates approach infinity,* $\theta_i \to \infty$, $i = 1, \ldots, N-1$, *in such a way that*

$$\lim_{\theta_i,\, \theta_j \to \infty} \frac{(N-i)\theta_i}{(N-j)\theta_j} \neq 1, \quad i \neq j.$$

Then the probability $\hat{P}_N(t)$ *of all agents having detected a target at time* t *has the limit*

$$\lim_{\theta_1, \ldots, \theta_{N-1} \to \infty} \hat{P}_N(t) = 1 - e^{-N\theta t}. \tag{21}$$

Proof. To establish the statement of the proposition, it suffices to rewrite expression (16) for the probability $\hat{P}_N(t)$ in the form

$$\begin{aligned} \hat{P}_N(t) = 1 - \prod_{s=1}^{N-1} \left[1 - \frac{N\theta_0}{(N-s)\theta_s}\right]^{-1} a_{00}\, e^{-N\theta_0\, t} \\ - \sum_{j=1}^{N-1} \binom{N}{j} \prod_{s=j+1}^{N-1} \left[1 - \frac{(N-j)\theta_j}{(N-s)\theta_s}\right]^{-1} a_{jj}\, e^{-(N-j)\theta_j\, t}. \end{aligned} \tag{22}$$

Taking into account that $a_{00} = 1$, and that under the conditions of the proposition $\theta_0/\theta_s \to 0$ for all $s = 1, \ldots, N-1$, and $R(\theta_j, \theta_s)\, e^{-(N-j)\theta_j t} \to 0$ for all rational functions $R(\cdot)$ and all $1 \leq s, j \leq N-1$, we observe that the second term in the right-hand side of (22) is approaching $\left(-e^{-N\theta t}\right)$, while the third term vanishes. This yields expression (21). $\square$

A numerical illustration of the performance of the cooperative search system as defined by (16) is presented in Figures 1, 2, and 3. In these examples, the cueing rates dynamics is assumed to follow

$$\theta_i = \theta\kappa^{\frac{N-i}{N-1}}, \quad i = 1, \ldots, N-1, \quad \text{for some } \kappa > 1. \tag{23}$$

Then in the case of 5 searchers ($N = 5$) the probability of detection $\hat{P}_5(t)$ for various values of κ is displayed in Figure 1. The black line, marked by $\kappa \gg 1$ corresponds to the upper bound (21) for the detection probability $\hat{P}_5(t)$.

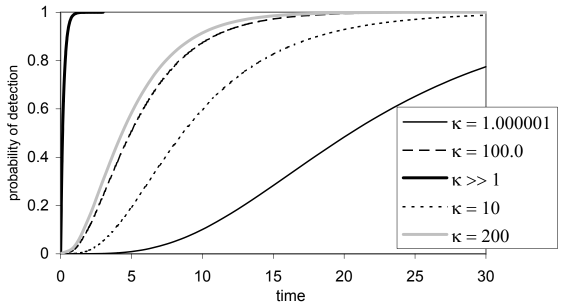

Figure 1. Probability of detection $\hat{P}_N(t)$ for 5 searchers ($N = 5$).

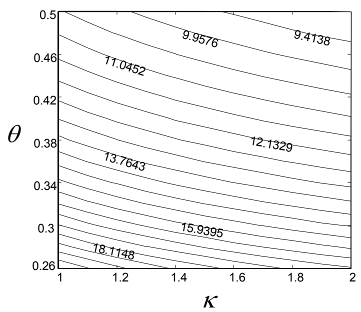

Figure 2. Isolines of the *time to engage,* $T_{0.95}$, for 2 searchers.

Another measure of effectiveness of the considered cooperative search system may be considered to be the *time to engage*, i.e., such a time $t = T_{0.95}$ that the probability that all N agents have detected a target by this time is 95 percent:

$$\mathsf{P}[\text{all searchers have detected targets by the time } T_{0.95}] = \hat{P}_N(T_{0.95}) = 0.95.$$

Figures 2 and 3 display the curves $T_{0.95} = \text{const}$ for systems comprising 2 and 10 searchers. From these charts it is evident, for example, that in large

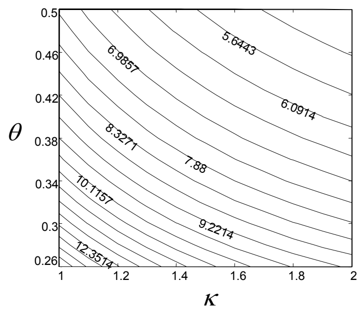

Figure 3. Isolines of the *time to engage,* $T_{0.95}$, for 10 searchers.

systems, where the cueing dynamics obeys relation (23), a more efficient time to engage is achieved by increasing the initial detection rate θ.

3. Bi-level cooperative search model

In this section we consider a cooperative search mission that involves a dedicated search-only vehicle and N search-and-engage vehicles. As follows from the adopted nomenclature, the search-and-engage vehicles are capable of conducting stand-alone search for a target, and they engage the target upon detecting one. Generally, the search capabilities of the search-only vehicle are assumed to be superior to those of search-and-engage vehicles, and its role is to provide cues to them, thereby increasing their detection rates. In contrast to the model considered above, now it is assumed that search-and-engage vehicles do *not* cue each other. The model for the described scenario builds upon the Markov chain approach presented in the previous section.

For example, the search-only vehicle has two possible states, as shown in the left portion of Figure 4. Note that the search-only vehicle transitions from the "search" state to the "detect and cue" state at rate λ, and that the transition back to the "search" state is instantaneous, denoted by an infinite transition rate. That is, once the search vehicle cues a search-and-engage vehicle, the search vehicle immediately resumes its search for additional targets.

The states and transition rates for the search-and-engage vehicles are shown in the right portion of Figure 4. Search-and-engage vehicles have three possible states, "search uncued," "search cued," and "detect and engage," denoted as $\mathcal{U}$, $\mathcal{C}$, and $\mathcal{D}$, respectively. The transition rate from the uncued to the cued state depends on the detection rate λ of the search-only vehicle. We assume that the search-only vehicle will cue only one search-and-engage vehicle for each target detected, and that the cues are equally distributed to the uncued search-and-engage vehicles, i.e., the transition rate from "search uncued" to "search cued" is λ/i if there are i search-and-engage vehicles in the state $\mathcal{U}$. This assumption implies that the search-only vehicle is aware of the current state of all the search-and-engage vehicles, and the search-only vehicle can transmit information to a single search-and-engage vehicle. Even if a transmission is broadcast on a common frequency, we assume that the transmitted data can be "tagged" for use only by an individual search-and-engage vehicle. After a search-and-engage vehicle receives a cue, its detection rate changes to θ_1 (recall that search-and-engage vehicles have the ability to search independently, so that a vehicle could make a direct transition from the "search uncued" to the "detect and engage" state at rate θ_0). In general, it is assumed that $\theta_1 \geq \theta_0$.

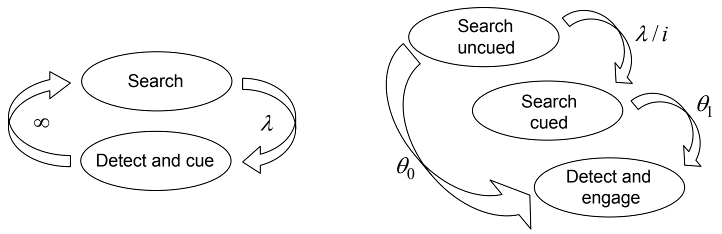

Figure 4. State diagrams for the search-only vehicle (left) and search-and-engage vehicle (right)

Similarly to the above, let $\mathcal{S}_{ijk}$ be *a state* in which there are i search-and-engage vehicles (also called "searchers") in the state $\mathcal{U}$, j searchers in the state $\mathcal{C}$, and k searchers in the state $\mathcal{D}$, where $i+j+k=N$. It is easy to see that for given i, j, and k there are $\binom{N}{i\ \ j\ \ k} = \frac{N!}{i!\,j!\,k!}$ different states $\mathcal{S}_{ijk}$. Further, it is important to note that there are $\binom{N+3-1}{N} = \frac{(N+2)(N+1)}{2}$ different triplets (i,j,k) such that $i+j+k=N$. By defining the probability of the cooperative system occupying a state $\mathcal{S}_{ijk}$ at time t as $P_{ijk}(t)$, one can describe the corresponding Markov model with a finite number of states via the following system of Kolmogorov equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{ijk}(t) = -\,\bar{\delta}_{kN}\left[i\theta_0 + j\theta_1 + \bar{\delta}_{i0}\lambda\right]P_{ijk}(t) + \bar{\delta}_{iN}\bar{\delta}_{j0}\left[\frac{j\lambda}{i+1}\right]P_{i+1,\,j-1,\,k}(t) + \bar{\delta}_{iN}\bar{\delta}_{k0}\left[k\theta_0\right]P_{i+1,\,j,\,k-1}(t) + \bar{\delta}_{jN}\bar{\delta}_{k0}\left[k\theta_1\right]P_{i,\,j+1,\,k-1}(t), \quad i+j+k=N. \tag{24}$$

Indeed, in the most general case a state $\mathcal{S}_{i,j,k}$ with i uncued searchers, j cued searchers, and k inactive searchers can be obtained

- from a state $\mathcal{S}_{i+1,j-1,k}$ due to a transition $\mathcal{U} \to \mathcal{C}$, i.e. when one of the $i+1$ uncued searchers receives a cueing signal from the search-only vehicle. Since each of the $i+1$ uncued searchers is being cued at rate $\frac{\lambda}{i+1}$, and there are $j = \binom{j}{1}$ states $\mathcal{S}_{i+1,j-1,k}$ that can result in the given state $\mathcal{S}_{ijk}$, transitions $\mathcal{U} \to \mathcal{C}$ increase the probability $P_{ijk}(t)$ at the rate $\frac{j\lambda}{i+1} P_{i+1,j-1,k}(t)$. This amounts to the second term in equation (24).
- from a state $\mathcal{S}_{i+1,j,k-1}$ due to a transition $\mathcal{U} \to \mathcal{D}$, i.e., when one of the $i+1$ uncued searchers detects a target before receiving a cue by the search-only vehicle. The search rate of an uncued agent is θ_0, and there are $k = \binom{k}{1}$ different states $\mathcal{S}_{i+1,j,k-1}$ that can lead to the given state $\mathcal{S}_{ijk}$. Thus, due to transitions $\mathcal{U} \to \mathcal{D}$ the probability $P_{ijk}(t)$ increases at the rate $k\theta_0 P_{i+1,j,k-1}(t)$, which amounts to the third term in (24).
- from a state $\mathcal{S}_{i,j+1,k-1}$ due to a transition $\mathcal{C} \to \mathcal{D}$, when one of the $j+1$ cued searchers detects a target. The search rate of a cued agent is θ_1, and there are $k = \binom{k}{1}$ different states $\mathcal{S}_{i,j+1,k-1}$ that can lead to the given state $\mathcal{S}_{ijk}$. Thus, due to transitions $\mathcal{C} \to \mathcal{D}$ the probability $P_{ijk}(t)$ increases at the rate $k\theta_1 P_{i,j+1,k-1}(t)$, which amounts to the fourth term in (24).
- finally, the first term in the right-hand side of (24) accounts for the possibility of transition from the given state $\mathcal{S}_{ijk}$ to states $\mathcal{S}_{i-1,j,k+1}$, $\mathcal{S}_{i,j-1,k+1}$, and $\mathcal{S}_{i-1,j+1,k}$ correspondingly.

Analogously to (9), probabilities $P_{ijk}(t)$ satisfy

$$\sum_{\substack{j+j+k=N \\ i,\, j,\, k \geq 0}} \begin{pmatrix} & N & \\ i & j & k \end{pmatrix} P_{ijk}(t) = 1, \quad t \geq 0.$$

Solution of the system of equations (24) is facilitated via representing (24) in a matrix form, with a lower-triangular matrix. A lower-triangular form of equations (24) is obtained by introducing the notation $P_{ijk}(t) = \tilde{P}_\ell(t)$, where the index ℓ runs from 0 to $L = \frac{(N+2)(N+1)}{2} - 1 = \frac{N(N+3)}{2}$ and is determined by the indices i, j, and k as

$$\ell = \sum_{r=0}^{j+k-1} (r+1) + k = \frac{(j+k)(j+k+1)}{2} + k \quad \text{for all } \ 0 \le j+k \le N. \tag{25}$$

Explicitly, the introduced relation between $P_{ijk}(t)$ and $\tilde{P}_\ell(t)$ enumerates as

$$\begin{aligned}
\tilde{P}_0(t) &= P_{N00}(t),\\
\tilde{P}_1(t) &= P_{N-1,1,0}(t),\\
\tilde{P}_2(t) &= P_{N-1,0,1}(t),\\
\tilde{P}_3(t) &= P_{N-2,2,0}(t),\\
\tilde{P}_4(t) &= P_{N-2,1,1}(t),\\
\tilde{P}_5(t) &= P_{N-2,0,2}(t),\\
&\ \vdots\\
\tilde{P}_{L-1}(t) &= P_{0,1,N-1}(t),\\
\tilde{P}_L(t) &= P_{00N}(t).
\end{aligned}$$

It is easy to see that such a correspondence between $P_{ijk}(t)$ and $\tilde{P}_\ell(t)$ allows one to represent equations (24) in a matrix form

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{P}} = \tilde{M}\tilde{\mathbf{P}}, \tag{26}$$

where the matrix $\tilde{M} \in \mathbb{R}^{(L+1)\times(L+1)}$ is lower-triangular. The initial conditions for the above system are formulated similarly to (8):

$$\tilde{P}_0(0) = 1, \quad \tilde{P}_\ell(0) = 0, \quad 1 \le \ell \le L. \tag{27}$$

Since $\tilde{M}$ is generally not diagonal, the solution to the Cauchy problem (26)–(27) has the form analogous to (12),

$$\tilde{P}_\ell(t) = \sum_{i=0}^{\ell} a_{i\ell} e^{\tilde{m}_{ii} t}, \tag{28}$$

but the expressions for coefficients $a_{i\ell}$ are more complicated comparing to (13):

$$a_{i\ell} = \sum_{j=i}^{\ell-1} \frac{\tilde{m}_{\ell j}\, a_{ij}}{\tilde{m}_{ii} - \tilde{m}_{\ell\ell}}, \quad i < \ell, \tag{29}$$

$$a_{ii} = -\sum_{j=0}^{i-1} a_{ji}, \quad a_{00} = \tilde{P}_0(0) = 1. \tag{30}$$

Above, it is assumed that the eigenvalues of matrix $\tilde{M}$ are all different:

$$i\theta_0 + j\theta_1 + \bar{\delta}_{i0}\lambda \neq i'\theta_0 + j'\theta_1 + \bar{\delta}_{i'0}\lambda \text{ for all } 0 \leq i+j \leq N \text{ and } 0 \leq i'+j' \leq N.$$

The developed solution to equations (24) is illustrated on a system comprised by one search-only vehicle and five search-and-engage vehicles ($N = 5$). The value of information transmitted in the cues is determined by parameter κ,

$$\theta_1 = \kappa\theta_0.$$

As before, the system's effectiveness is measured by the probability $\tilde{P}_L(t) = P_{00N}(t)$ that all search-and-engage vehicles have detected targets by time t. Figures 5 and 6 contain graphs of the probability $\tilde{P}_L(t)$ for the case when the initial detection rate θ_0 of the search-and-engage vehicles is equal to 0.1, and the cueing effectiveness κ and the search rate λ of the search-only vehicle vary. In particular, the presented graphs imply that increments in κ have more pronounced effect on increasing the probability $\tilde{P}_L(t)$ than increments in λ. In other words, precise cueing is more valuable than high target detection rate of the search-only vehicle (in the considered case).

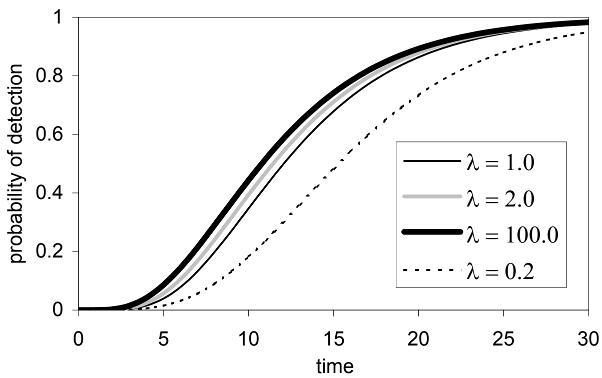

Figure 5. Probability of detection, $\tilde{P}_L(t)$, for a system of one search-only vehicle and 5 search-and-engage vehicles, where $\theta = 0.1$ and $\kappa = 1.9$.

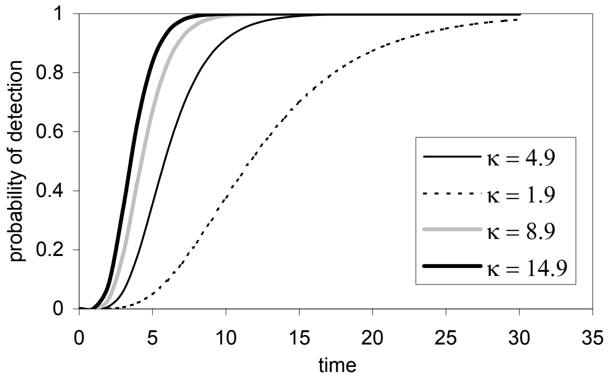

Figure 6. Probability of detection, $\tilde{P}_L(t)$, for a system of one search-only vehicle and 5 search-and-engage vehicles where $\theta = 0.1$ and $\lambda = 1.5$.

Conclusions

We have proposed a Markov chain approach to quantification of the effect of cueing in cooperative search systems. It has been shown that cueing can dramatically affect the probability of detection over a fixed time interval. We have also shown that there is an upper bound on the benefit of cueing, at least for the problem defined.

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