Effects of Cueing in Cooperative Search*

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Abstract

We consider the effects of cueing in a cooperative search mission that involves several autonomous agents. Two scenarios are discussed: one in which the search is conducted by a number of identical search-and-engage vehicles, and one where these vehicles are assisted by a search-only (reconnaissance) asset. The cooperation between the autonomous agents is facilitated via cueing, i.e. the information transmitted to the agents by a searcher that has just detected a target. The effect of cueing on the target detection probability is derived from first principles using a Markov chain analysis. In particular, it is demonstrated that the benefit of cueing on the system's effectiveness is bounded.

Key Words: Cooperative search; target detection; cueing; Markov chains

1 Introduction

In any system-of-systems analysis, consideration of dependencies between systems is imperative. In this paper, we consider a particular type of system interaction, called *cueing*, in application to a *search task* that involves several cooperating subsystems. The interaction could be between similar systems, such as two or more wide area search munitions, or between dissimilar systems, such as a reconnaissance asset and a munition. We discuss two scenarios: one in which the search is conducted by a number of identical search-and-engage vehicles, and one where these vehicles are assisted by a search-only vehicle. The autonomous agents forming the cooperative system interact via cueing.

In Shakespeare's day, the word "cue" meant a signal (a word, phrase, or bit of stage business) to a performer to begin a specific speech or action [6]. The word is now used more generally for anything serving a comparable purpose. In this paper, by cue we mean any

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information that provides focus to a search; e.g., information that limits the search area or provides a search heading.

Search theory is one of the oldest areas of operations research [10] with a solid foundation in mathematics, probability and experimental physics. Yet, search theory is clearly of more than academic interest. At times, a search can become an international priority, as in the 1966 search for the hydrogen bomb lost in the Mediterranean near Palomares, Spain.

That search was an immense operation involving 34 ships, 2,200 sailors, 130 frogmen and four mini-subs. The search took 75 days, but might have concluded much earlier if cueing had been utilized from the start. A Spanish fisherman had come forward quickly to say he'd seen something fall that looked like a bomb, but experts ignored him.

Instead, they focused on four possible trajectories calculated by a computer, but for weeks found only airplane pieces. Finally, the fisherman, Francisco Simo, was summoned back. He sent searchers in the right direction, and a two-man sub, the Alvin, located the 10-foot-long bomb under 2,162 feet of water [13].

It has to be emphasized that the present paper is not concerned with the traditional topics of search theory per se; instead, it is focused on the possible interaction among searchers in the form of cueing and the extent to which this interaction can improve the overall effectiveness of the search mission. Below we briefly mention some works in the search theory that are relevant to the present development.

Bernard Koopman pioneered the application of mathematical process to military search problems during World War II [10]. Koopman [4] discusses the case in which a searcher inadvertently provides information to the target, perhaps allowing the target to employ evasive action. The use of receivers on German U-boats to detect search radar signals in World War II is a classic example. Koopman referred to this type of cueing as target alerting. This paper uses a detection rate approach to examine the effect of cueing on probability of target detection, which is similar to Koopman's discussion of target detection [5].

Washburn [12] examines the case of a single searcher attempting to detect a randomly moving target at a discrete time. Given an effort distribution, bounded at each discrete time t, Washburn establishes an upper bound on the probability of target detection. It is noteworthy that Washburn mentions that the detection rate approach to computation of detection probabilities has proved to be more robust than approaches relying on geometric models.

Alpern and Gal [1] discuss the problem of searching for a submarine with a known initial location. Thomas and Washburn [11] consider dynamic search games in which the hider starts moving at time zero from a location known to both a searcher and a hider, while the searcher starts with a time delay known to both players; for example, a helicopter attempts to detect a submarine that reveals its position by torpedoing a ship. A Markov chain approach to target detection can be found in Stone [10], which deals with the optimal allocation of effort to detect a target whose motion is Markovian.

In this paper, we use a Markov chain framework to examine cueing as a coupling mechanism among several searchers. We show that the effect of cooperation (via cueing) on the overall system's effectiveness (e.g., probability of detection) is bounded. As an introductory illustration to the proposed approach, we first consider a model with several autonomous searchers that cue each other once a target is detected (Section 2). Then, in Section 3 we discuss a more complicated model of hierarchical search system, where cueing is an essential element that aids in tracking of time-critical targets.

The primary contribution of this paper consists in application of classical tools of Markov processes and queueing theory to the emerging field of cooperative control of autonomous systems.

2 A generic cooperative search model

Consider a cooperative search mission involving N *search-and-engage* agents, each having the objective of finding (*detecting*) an object of interest (a *target*). The search capabilities of any agent are characterized by the detection rate θ, i.e. the probability of detecting a target within time interval Δt:

$$\mathsf{P}[\text{vehicle } i \text{ detects a target during time } \Delta t] = \theta\,\Delta t + o(\Delta t), \quad i = 1, \ldots, N. \tag{1}$$

Upon detecting a target the agent *engages* it and stops searching any further; we also say that in such a case the searcher becomes *inactive*. For example, in a search-and-rescue mission for passengers of a sinking ship the searchers will try to rescue the passenger(s) they find, instead of continuing the search; on the battlefield, an autonomous wide-area search munition will attack the detected target, etc. Further, it is assumed that upon engaging a target, the searcher instantaneously cues the remaining *active* agents, thereby potentially increasing their detection capabilities. Within the presented framework the informational content of the cueing signal is not important; instead, we are interested in the degree by which cueing impacts the search capabilities of individual agents in a cooperative system. Namely, it is assumed that the detection rate of a searcher may change value in time as

$$\theta_0 \;\to\; \theta_1 \;\to\; \cdots \;\to\; \theta_{N-1}, \tag{2}$$

where θ_k is the detection rate common to $N-k$ active searchers, and θ_0 is the initial "uncued" detection rate of a searcher, $\theta_0 \equiv \theta$. It is natural to presume that cueing generally leads to improvement of search capabilities, i.e., $\theta_k \geq \theta_0,\ k = 1, \ldots, N-1$. The search mission terminates when there are no active searchers left, i.e., each searcher has engaged a target. Such a setup obviously presumes that the number of searchers does not exceed the number of targets; we make this assumption implicitly throughout the paper.

The outlined model, complemented by the usual assumptions of independence of target detections on non-overlapping time intervals etc., can be described by a continuous-time discrete-state Markov chain, or, furthermore, a *pure death process* (see, among others, [3, 8]). The states of the system are designated by the number k of inactive searchers, $k = 0, \ldots, N$. The formulation presented below is somewhat different from the traditional models for birth-death processes in that it defines the transition rate between states using the search rates of individual vehicles (2), versus the rates defined with respect to the entire population [3, 9].

Let $\mathcal{S}_k$ represent a state in which k searchers are inactive (and, correspondingly, $N-k$ searchers are active); note that there are $\binom{N}{k}$ states $\mathcal{S}_k$. Since the search capabilities of all (active) agents are identical, the states $\mathcal{S}_k$ are equiprobable for a given k, thus by defining the corresponding probabilities $P_k(t)$ as

$$P_k(t) = \mathsf{P}[\text{system is in state } \mathcal{S}_k \text{ at time } t],$$

one can write the system of Chapman-Kolmogorov ODEs governing these probabilities:

$$\frac{\mathrm{d}}{\mathrm{d}t} P_k(t) = -\big[(N-k)\,\theta_k\big] P_k(t) + \big[k\theta_{k-1}\big] P_{k-1}(t), \quad k = 0, \ldots, N. \tag{3}$$

Equations (3) have a simple interpretation: the probability of the system being in a state $\mathcal{S}_k$ decreases at the rate $(N-k)\theta_k P_k(t)$ as each of $N-k$ active searchers may detect a target and turn the system into a state $\mathcal{S}_{k+1}$. On the other hand, probability $P_k(t)$ increases at the rate $k\theta_{k-1} P_{k-1}(t)$ as there are exactly $\binom{k}{1} = k$ states $\mathcal{S}_{k-1}$ that may lead to a (given) state $\mathcal{S}_k$.[1] Initial conditions for equations (3) reflect the fact that at $t = 0$ the system is in the state $\mathcal{S}_0$ with probability 1:

$$P_0(0) = 1,\ P_1(0) = P_2(0) = \ldots = P_N(0) = 0. \tag{4}$$

Using (3) and (4) it is straightforward to verify that the probabilities $P_k(t)$ satisfy the identity

$$\sum_{k=0}^{N} \binom{N}{k} P_k(t) = 1, \quad t \geq 0. \tag{5}$$

A closed-form solution of equations (3) is easily obtained by means of the Laplace transform [9]. Equations (3) yield that the Laplace transforms

$$p_k(s) = \mathscr{L}[P_k(t)] = \int_0^\infty e^{-st} P_k(t)\,\mathrm{d}t, \quad \mathrm{Re}(s) \geq 0,$$

of the probabilities $P_k(t)$ have the form

$$p_k(s) = k!\,\theta_0 \cdots \theta_{k-1} \prod_{j=0}^{k} \big[s + (N-j)\theta_j\big]^{-1}, \quad k = 0, \ldots, N. \tag{6}$$

Assuming that

$$(N-i)\theta_i \neq (N-j)\theta_j, \quad 0 \leq i < j \leq N-1, \tag{7}$$

which corresponds to all eigenvalues of the matrix of system (3) being different, and applying the inverse Laplace transform to (6), we obtain closed-form expressions for probabilities $P_k(t)$

$$P_k(t) = \sum_{i=0}^{k} \frac{k!\,\theta_0 \cdots \theta_{k-1}}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{k} \big[(N-j)\theta_j - (N-i)\theta_i\big]}\, e^{-(N-i)\theta_i t}, \quad k = 0, \ldots, N. \tag{8}$$

[1] Indeed, let $A_k = \{i_1, \ldots, i_k\}$ be any set containing k inactive searchers. Trivially, A_k can be represented in k different ways as $A_k = A_{k-1}^j \cup \{i_j\}$, where A_{k-1}^j is a subset of $k-1$ inactive searchers: $A_{k-1}^j = A_k \backslash \{i_j\} \subset A_k$, $j = 1, \ldots, k$. Thus, a state $\mathcal{S}_k$ with k inactive searchers can only be obtained from exactly k states $\mathcal{S}_{k-1}$ with $k-1$ inactive searchers.

Evidently, assumptions (7) are by no means restrictive, and solution (8) can be readily generalized for the case when conditions (7) do not hold for some values i or j.

Expressions (8) allow for a detailed analysis of the considered model. In particular, the *probability of detection* $P_N(t)$ can be used to determine various measures of effectiveness (MOE) of the cooperative search system. For instance, one may be interested in the average time $\overline{T}_N$ needed to find N targets,

$$\overline{T}_N = \sum_{i=0}^{N-1} \frac{1}{(N-i)\theta_i} \prod_{\substack{j=0 \\ j \neq i}}^{N-1} \left[1 - \frac{(N-i)\theta_i}{(N-j)\theta_j} \right]^{-1}. \tag{9}$$

Another MOE is embodied by the *time to engage* T_α, or the time needed for all N searchers to detect and engage targets with probability $\alpha \in (0, 1)$: $T_\alpha = P_N^{-1}(\alpha)$.

Hence, the effect of cueing on the cooperation among searchers can be demonstrated on the probability of detection $P_N(t)$. For example, it is easy to see that when cueing has no impact on the detection rates of the searchers, then $P_N(t)$ is equal to the probability of all N agents detecting targets independently:

$$P_N(t) \equiv 1 + \sum_{i=0}^{N-1} \frac{N!}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{N} (i-j)} \, e^{-(N-i)\theta_i t} = \left(1 - e^{-\theta_0 t}\right)^N. \tag{10}$$

The first identity in (10) is a restatement of (8) with $k = N$, and the last equality is established by substituting $\theta_i = \theta_0$ and using Newton's binomial formula along with the identity

$$\prod_{\substack{j=0 \\ j \neq i}}^{N} (i-j) = (-1)^{N-i} i!(N-i)!.$$

Despite being rather evident, this fact manifests an important property of the model, and can be stated as

Proposition 1 *If $\theta_0 = \theta_1 = \ldots = \theta_{N-1}$, i.e., the cueing has no effect on the detection capabilities of the searchers, then*

$$P_N(t) = \left(1 - e^{-\theta_0 t}\right)^N.$$

Remark A constant common detection rate implicitly assumes that the number of targets in the search area is much larger than the number of searchers, in which case detection of one target leaves the number of undetected targets, and, correspondingly, the detection rate of any active searcher virtually unchanged. Note, however, that the general solution (8) is free of this assumption and applies to the case when the number of targets is finite.

Obviously, cueing has the purpose of improving the system's effectiveness by increasing the cued detection rates $\theta_1, \ldots, \theta_{N-1}$. Indeed, it can be verified that larger values of $\theta_1, \ldots, \theta_{N-1}$ increase the probability of detection $P_N(t)$ for a given t. Below we demonstrate that under quite general conditions the effect of cueing on the system's performance is bounded, i.e., there exists an upper bound for the state probability $P_N(t)$ when the cued detection rates $\theta_1, \ldots, \theta_{N-1}$ increase indefinitely.

Proposition 2 *Let the cued detection rates approach infinity, $\theta_i \to \infty,\ i = 1, \dots, N-1$. Then, the probability $P_N(t)$ of all agents having detected a target at time t and the expected duration $\overline{T}_N$ of the search mission have the limiting values*

$$\lim_{\theta_1,\dots,\theta_{N-1}\to\infty} P_N(t) = 1 - e^{-N\theta_0 t} \qquad \textit{and} \qquad \lim_{\theta_1,\dots,\theta_{N-1}\to\infty} \overline{T}_N = \frac{1}{N\theta_0}. \tag{11}$$

To establish the statement of the proposition, it suffices to rewrite expression (8) for the probability $P_N(t)$ in the form

$$\begin{aligned} P_N(t) = 1 - \prod_{j=1}^{N-1} \left[1 - \frac{N\theta_0}{(N-j)\theta_j}\right]^{-1} e^{-N\theta_0\, t} \\ - \sum_{i=1}^{N-1} \prod_{\substack{j=0 \\ j\neq i}}^{N-1} \left[1 - \frac{(N-i)\theta_i}{(N-j)\theta_j}\right]^{-1} e^{-(N-i)\theta_i\, t}. \end{aligned} \tag{12}$$

Then, it is evident that under the conditions of the proposition $\theta_0/\theta_j \to 0$ for all $j = 1, \dots, N-1$, and also $R(\theta_i, \theta_j)\, e^{-(N-i)\theta_i t} \to 0$ for all rational functions $R(\cdot,\cdot)$. Hence, the second term in the right-hand side of (12) approaches $\left(-e^{-N\theta_0 t}\right)$, while the third term vanishes. This yields expression (11) for P_N. The corresponding limiting value for $\overline{T}_N$ follows immediately.

As it will be shown below, the generic model delineated in this section can be used as a foundation or a building block for more sophisticated models of interaction in cooperative systems.

3 A hierarchical cooperative search model

In the current research on cooperative control of autonomous systems, of particular interest is the topic of coordination and cooperation in systems with hierarchical, or "layered" architecture. In such systems, groups of autonomous agents are structured into several functional layers with respect to specific capabilities of an agent. Hierarchical search systems may be used for detection and tracking of movable time-critical targets that may randomly pop-up or disappear in a given area. As an example, here we consider a *two-layer* search system where N search-and-engage vehicles are assisted by *search-only* vehicles. These search-only vehicles are assumed to possess superior search capabilities (greater field of view, improved target recognition, etc), but are unable to effectively engage targets (e.g., to achieve greater field of view, the search-only vehicles must fly at high altitudes). Their function is to provide information on targets to more agile search-and-engage vehicles, thereby increasing the overall effectiveness of the cooperative search system.

The principal difference of the model presented below with the one considered in Section 2 is that the mechanism of cueing (in the form of transmission of information from search-only vehicles to search-and-engage vehicles) is a *necessary* feature of the system's design. On the contrary, cueing in the model of Section 2 can be regarded as a *supplementary* mechanism that is called to improve the system's operation.

Without loss of generality, below we discuss a system of one search-only vehicle and N search-and-engage vehicles (henceforth also called "searchers"). In contrast to the situation considered in Section 2, now it is assumed that the search-and-engage vehicles do *not* cue each other. The model for the described scenario builds upon the Markov chain approach presented in the previous section.

For example, the search-only vehicle has two possible states, as shown in the left portion of Figure 1. Note that the search-only vehicle transitions from the "search" state to the "detect and cue" state at rate λ, and that the transition back to the "search" state is instantaneous, denoted by an infinite transition rate. That is, once the search vehicle cues a search-and-engage vehicle, the search vehicle immediately resumes its search for additional targets.

The states and transition rates for the search-and-engage vehicles (searchers) are shown in the right portion of Figure 1. Each searcher has three possible states, "search uncued," "search cued," and "detect and engage," denoted as $\mathcal{U}$, $\mathcal{C}$, and $\mathcal{D}$, respectively. The transition rate from the uncued to the cued state depends on the detection rate λ of the search-only vehicle. We assume that the search-only vehicle will cue only one search-and-engage vehicle for each target detected, and that the cues are equally distributed to the uncued searches, i.e., the transition rate from "search uncued" to "search cued" is λ/i if there are i searches in the state $\mathcal{U}$. This assumption implies that the search-only vehicle is aware of the current state of all the search-and-engage vehicles, and the search-only vehicle can transmit information to a single search-and-engage vehicle. Even if a transmission is broadcast on a common frequency, we assume that the transmitted data can be "tagged" for use only by an individual search-and-engage vehicle. After a searcher receives a cue, its detection rate changes from the initial "uncued" rate θ_u to the "cued" rate θ_c. Recall that search-and-engage vehicles also have the ability to search independently, so that a searcher could make a direct transition from the "search uncued" to the "detect and engage" state at rate θ_u. It is natural to presume that $\theta_c \geq \theta_u$; we also assume that the number of targets is much larger than the number of searchers, so that the detection rates λ, θ_u, and θ_u are not affected by the number of detected targets during the mission.

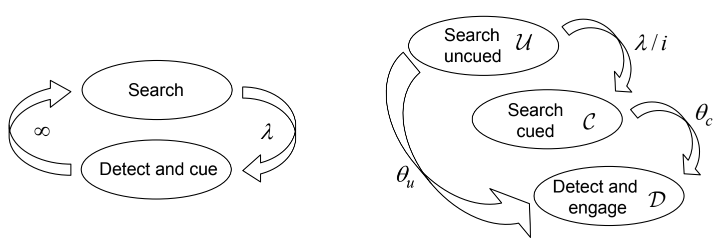

Figure 1: State diagrams for the search-only vehicle (left) and search-and-engage vehicle (right)

Similarly to the exposition of Section 2, let $\mathcal{S}_{ijk}$ represent a state in which there are i searchers in the state $\mathcal{U}$, j searchers in the state $\mathcal{C}$, and k searchers in the state $\mathcal{D}$, $i+j+k = N$. It is easy to see that for a given triple (i, j, k) there are $\binom{N}{i\ j\ k} = \frac{N!}{i!\,j!\,k!}$ states $\mathcal{S}_{ijk}$. Further, it is important to note that there are $\binom{N+3-1}{N} = \frac{(N+2)(N+1)}{2}$ different triplets (i, j, k) such that $i+j+k = N$. From symmetry considerations, the $\binom{N}{i\ j\ k}$ states $\mathcal{S}_{ijk}$ are equiprobable for fixed $i,\ j$, and k, therefore by defining the respective probabilities $P_{ijk}(t)$ as

$$P_{ijk}(t) = \mathsf{P}[\text{system is in state } \mathcal{S}_{ijk} \text{ at time } t],$$

one can describe the corresponding Markov model with a finite number of states via the following system of Chapman-Kolmogorov equations:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} P_{ijk}(t) = &- \bar{\delta}_{kN}\Big[i\theta_u + j\theta_c + \bar{\delta}_{i0}\lambda\Big] P_{ijk}(t) + \bar{\delta}_{iN}\left[\frac{j\lambda}{i+1}\right] P_{i+1,\, j-1,\, k}(t) \\ &+ \bar{\delta}_{iN}\big[k\theta_u\big]\, P_{i+1,\, j,\, k-1}(t) + \bar{\delta}_{jN}\big[k\theta_c\big]\, P_{i,\, j+1,\, k-1}(t), \quad i + j + k = N, \quad (13)\end{aligned}$$

where factors $\bar{\delta}_{ij}$ represent the negation of the Kronecker delta δ_{ij}

$$\bar{\delta}_{ij} = 1 - \delta_{ij} = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{if } i \neq j, \end{cases}$$

and have the obvious function of handling the extreme cases of $i,\ j$, or k being equal to 0 or N. Let us present the interpretation of equations (13). In the most general case a state $\mathcal{S}_{ijk}$ with i uncued searchers, j cued searchers, and k inactive searchers can be obtained

- from a state $\mathcal{S}_{i+1,j-1,k}$ due to a transition $\mathcal{U} \to \mathcal{C}$, i.e. when one of the $i+1$ uncued searchers receives a cueing signal from the search-only vehicle. Since each of the $i+1$ uncued searchers is being cued at rate $\frac{\lambda}{i+1}$, and there are $j = \binom{j}{1}$ states $\mathcal{S}_{i+1,j-1,k}$ that can result in a given state $\mathcal{S}_{ijk}$, transitions $\mathcal{U} \to \mathcal{C}$ increase the probability $P_{ijk}(t)$ at the rate $\frac{j\lambda}{i+1} P_{i+1,j-1,k}(t)$. This amounts to the second term in equation (13).
- from a state $\mathcal{S}_{i+1,j,k-1}$ due to a transition $\mathcal{U} \to \mathcal{D}$, i.e., when one of the $i+1$ uncued searchers detects a target before being cued by the search-only vehicle. The search rate of an uncued agent is θ_u, and there are $k = \binom{k}{1}$ different states $\mathcal{S}_{i+1,j,k-1}$ that can lead to a given state $\mathcal{S}_{ijk}$. Thus, due to transitions $\mathcal{U} \to \mathcal{D}$ the probability $P_{ijk}(t)$ increases at the rate $k\theta_u P_{i+1,j,k-1}(t)$, which amounts to the third term in (13).
- from a state $\mathcal{S}_{i,j+1,k-1}$ due to a transition $\mathcal{C} \to \mathcal{D}$, when one of the $j+1$ cued searchers detects a target. The search rate of a cued agent is θ_c, and there are $k = \binom{k}{1}$ different actual states $\mathcal{S}_{i,j+1,k-1}$ that can lead to a given state $\mathcal{S}_{ijk}$. Thus, due to transitions $\mathcal{C} \to \mathcal{D}$ the probability $P_{ijk}(t)$ increases at the rate $k\theta_c P_{i,j+1,k-1}(t)$, which amounts to the fourth term in (13).
- finally, the first term in the right-hand side of (13) accounts for the possibility of transition from the given state $\mathcal{S}_{ijk}$ to states $\mathcal{S}_{i-1,j,k+1}$, $\mathcal{S}_{i,j-1,k+1}$, and $\mathcal{S}_{i-1,j+1,k}$ correspondingly.

Similarly to (5), probabilities $P_{ijk}(t)$ satisfy the identity

$$\sum_{\substack{j+j+k=N \\ i,\, j,\, k \geq 0}} \binom{N}{i \quad j \quad k} P_{ijk}(t) = 1, \quad t \geq 0.$$

Application of the Laplace transform method to equations (13) leads to unwieldy expressions for probabilities $P_{ijk}(t)$. Thus, we employ an alternative procedure to derive the solution of equations (13) by representing them in a matrix form, with a lower-triangular matrix. A lower-triangular form of equations (13) is obtained by introducing the notation $P_{ijk}(t) = \tilde{P}_\ell(t)$, where the index ℓ runs from 0 to $L = \frac{(N+2)(N+1)}{2} - 1 = \frac{N(N+3)}{2}$ and is determined by the indices i, j, and k as

$$\ell = \sum_{r=0}^{j+k-1} (r+1) + k = \frac{(j+k)(j+k+1)}{2} + k \quad \text{for all } 0 \le j+k \le N. \tag{14}$$

The above relation between $P_{ijk}(t)$ and $\tilde{P}_\ell(t)$ can be explicitly enumerated as

$$\begin{aligned} \tilde{P}_0(t) &= P_{N00}(t), & \tilde{P}_5(t) \quad &= P_{N-2,0,2}(t), \\ \tilde{P}_1(t) &= P_{N-1,1,0}(t), & &\cdots \\ \tilde{P}_2(t) &= P_{N-1,0,1}(t), & &\cdots \\ \tilde{P}_3(t) &= P_{N-2,2,0}(t), & \tilde{P}_{L-1}(t) &= P_{0,1,N-1}(t), \\ \tilde{P}_4(t) &= P_{N-2,1,1}(t), & \tilde{P}_L(t) \quad &= P_{0,0,N}(t). \end{aligned}$$

It is easy to see that such a correspondence between $P_{ijk}(t)$ and $\tilde{P}_\ell(t)$ allows one to represent equations (13) in the matrix form

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{P}} = \mathbf{M}\,\tilde{\mathbf{P}}, \tag{15}$$

where the matrix $\mathbf{M} \in \mathbb{R}^{(L+1)\times(L+1)}$ is lower-triangular. Similarly to (4), the initial conditions for the above system are formulated as

$$\tilde{P}_0(0) = 1, \quad \tilde{P}_\ell(0) = 0, \quad 1 \le \ell \le L. \tag{16}$$

Then, the solution to the Cauchy problem[2] (15)–(16) has the form

$$\tilde{P}_\ell(t) = \sum_{i=0}^{\ell} a_{i\ell} e^{m_{ii} t}, \quad \ell = 0, \dots, L, \tag{17}$$

where the coefficients $a_{i\ell}$ are determined in a recursive manner

$$a_{i\ell} = \sum_{j=i}^{\ell-1} \frac{m_{\ell j}\, a_{ij}}{m_{ii} - m_{\ell\ell}}, \quad i < \ell, \tag{18a}$$

$$a_{ii} = -\sum_{j=0}^{i-1} a_{ji}, \quad a_{00} = \tilde{P}_0(0) = 1. \tag{18b}$$

Throughout this section it is assumed that the eigenvalues of the matrix $\mathbf{M}$ are all different:

$$i\theta_u + j\theta_c + \bar{\delta}_{i0}\lambda \ne i'\theta_u + j'\theta_c + \bar{\delta}_{i'0}\lambda \quad \text{for all } 0 \le i+j \le N \text{ and } 0 \le i'+j' \le N.$$

[2]See, e.g., text by Ince [2].

The exact solution (17)–(18) of the governing equations (13) allows for a detailed analysis of the cooperative search system. For example, one can easily deduce the expected time $\overline{T}_N$ needed for all N searchers to engage targets, $\overline{T}_N = \sum_{i=0}^{L-1} a_{iN}/m_{ii}$, and so on.

In the context of the analysis of cooperation among the autonomous agents in the described search system it is of interest to determine the limiting effects of cueing on the probability of detection $\tilde{P}_L(t) = P_{00N}(t)$. In the considered model, cooperation/cueing is governed by two parameters, the search rate λ of the search-only vehicle, and the cued search rate θ_c of the search-and-engage vehicles. Below we discuss the limiting behavior of the cooperative system in two cases: (i) when the search rate λ of the search-only vehicle increases infinitely (i.e., the search-only asset can provide information on all the targets in the area instantly) and (ii) when the cued search rate θ_c of the search-and-engage vehicle is infinite (the search-only asset provides information to a search-and-engage agent that leads to immediate detection of a target).

Proposition 3 *If the search rate λ of the search-only vehicle increases infinitely, the probability $P_{00N}(t)$ that all N search-and-engage vehicles will detect targets by the time t approaches the limit*

$$\lim_{\lambda\to\infty} P_{00N}(t) = \left(1 - e^{-\theta_c t}\right)^N. \tag{19}$$

Proof: This result can be obtained more directly by rewriting equations (13) as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_0(t) &= -\big[N\theta_u + \lambda\big]\tilde{P}_0(t),\\
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_1(t) &= -\big[(N-1)\theta_u + \theta_c + \lambda\big]\tilde{P}_1(t) + \frac{\lambda}{N}\tilde{P}_0(t),\\
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_2(t) &= -\big[(N-1)\theta_u + \lambda\big]\tilde{P}_2(t) + \theta_u\tilde{P}_0(t) + \theta_c\tilde{P}_1(t),\\
&\ \ \vdots\\
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_{L-1}(t) &= -\theta_c\tilde{P}_{L-1}(t) + \lambda\tilde{P}_{L-N-1}(t) + (N-1)\theta_u\tilde{P}_{L-N-2}(t) + (N-1)\theta_c\tilde{P}_{L-2}(t),\\
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_L(t) &= N\theta_u\tilde{P}_{L-N-1}(t) + N\theta_c\tilde{P}_{L-1}(t).
\end{aligned} \tag{20}$$

Consider the first $L-N-1$ equations that involve derivatives of $\tilde{P}_0(t),\ \tilde{P}_1(t),\dots,\ \tilde{P}_{L-N-1}(t)$. These equations can be solved in a successive order, starting from the equation for $\tilde{P}_0(t)$, whose solution, trivially, is $\tilde{P}_0(t) = e^{-(N\theta_u+\lambda)t}$; observe that $\tilde{P}_0(t)$ vanishes exponentially when $\lambda\to\infty$. The remaining equations can be represented as non-homogeneous ODEs of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_\ell(t) = -\big(a_\ell\theta_u + b_\ell\theta_c + \lambda\big)\tilde{P}_\ell(t) + \Phi_\ell(t), \quad \ell = 1,\dots,L-N-1, \tag{21}$$

where $\Phi_\ell(t)$ is a known function involving expressions for $\tilde{P}_k(t)$, $k<\ell$. Given the initial conditions (16), the solution of the above equations reads as

$$\tilde{P}_\ell(t) = \exp\big\{-(a_\ell\theta_u + b_\ell\theta_c + \lambda)t\big\}\int_0^t \Phi_\ell(\tau)\,\exp\big\{(a_\ell\theta_u + b_\ell\theta_c + \lambda)\tau\big\}\,\mathrm{d}\tau. \tag{22}$$

From (17) and the structure of matrix $\mathbf{M}$ it is easy to see that function $\Phi_\ell(t)$ has the form $\phi_\ell(\lambda)e^{-\lambda t}$, where $\phi_\ell(\lambda)$ is a rational function of λ. Hence, for $0 \leq \ell \leq L - N - 1$ probabilities $\tilde{P}_\ell(t)$ vanish exponentially with $\lambda \to \infty$. The remaining $N + 1$ equations for $\tilde{P}_{N-L}(t),\ \tilde{P}_{N-L+1}(t), \dots, \tilde{P}_L(t)$ can be written similarly to (21)

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_\ell(t) = -\bar{\delta}_{\ell L}\,(L-\ell)\,\theta_c\tilde{P}_\ell(t) + \bar{\delta}_{\ell,L-N}\,(\ell - L + N)\theta_c\tilde{P}_{\ell-1}(t) + \Phi_\ell(t),$$
$$\ell = L-N, \dots, L, \tag{23}$$

where $\Phi_\ell(t)$ may contain only $\tilde{P}_0(t), \dots, \tilde{P}_{L-N-1}(t)$. In other words, $\Phi_\ell(t)$ in (23) all vanish exponentially as λ approaches infinity. Thus, $\tilde{P}_L(t)$ for $\lambda \to \infty$ is determined by the system of ODEs (23) that is a special case of system (3). By virtue of Proposition 1 we obtain that the limiting value of $\tilde{P}_L(t)$ is represented by (19). $\square$

The above result has a simple interpretation. If $\lambda = \infty$, at the beginning of the mission all N searchers receive a cue and transition immediately to state $\mathcal{C}$. Then, in accordance to Proposition 1, $P_{00N}(t)$ equals to the probability that N search-and-engage vehicles with a common search rate θ_c detect and engage targets independently.

Next we examine the limiting behavior of the cooperative system when the value of the cues provided by the search-only vehicle increases infinitely: $\theta_c \to \infty$. This corresponds to the situation when cues provided by the search-only vehicle lead to immediate target detection by a search-and-engage vehicle.

Proposition 4 *If $\theta_c \to \infty$, then the limiting value of the probability of detection $P_{00N}(t) = \tilde{P}_L(t)$ has the form*

$$\lim_{\theta_c \to \infty} P_{00N}(t) = 1 - \sum_{k=0}^{N-1} \prod_{\substack{s=0 \\ s \neq k}}^{N-1} \left[1 - \frac{(N-k)\theta_u + \lambda}{(N-s)\theta_u + \lambda}\right]^{-1} e^{-\left((N-k)\theta_u + \lambda\right) t}. \tag{24}$$

Proof: The statement of the proposition can be proven by demonstrating that when $\theta_c \to \infty$, the $L + 1 = (N+1)(N+2)/2$ equations (13) reduce to $N + 1$ equations with respect to the probabilities $P_{N-k,0,k}(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{N-k,0,k}(t) = -\,\bar{\delta}_{kN}\big[(N-k)\theta_u + \lambda\big]P_{N-k,0,k}(t) \tag{25}$$
$$+\,k\left[\theta_u + \frac{\lambda}{N-k+1}\right]P_{N-k+1,0,k-1}(t), \quad k = 0, \dots, N, \quad \theta_c \to \infty.$$

Note that the above system is a special case of equations (3) with $\theta_k = \theta_u + \frac{\lambda}{N-k}$, $k = 0, \dots, N-1$. The asymptotic relation between (13) and (25) can be proved by showing that the probabilities $P_{N-k,j,k-j}(t)$ vanish with $\theta_c \to \infty$ for all $j > 0$, whereas the probabilities $P_{N-k,0,k}(t)$ remain finite and are governed by asymptotical equations (25). Performing a successive asymptotic integration of equations (13), and employing the principle of mathematical induction, the following relations can be established for probabilities

$P_{N-k,k,0}(t),\ P_{N-k,k-1,1}(t), \dots, P_{N-k,1,k-1}(t)$ for each $k = 1, \dots, N$:

$$P_{N-k,k,0}(t) \sim \left(\frac{\lambda}{\theta_c}\right)^k \frac{(N-k)!}{N!} e^{-(N\theta_u+\lambda)t}, \qquad \theta_c \to \infty, \tag{26a}$$

$$P_{N-k,j,k-j}(t) \sim \frac{1}{\theta_c} \frac{\lambda}{N-k+1} P_{N-k+1,j-1,k-j}(t), \quad \theta_c \to \infty, \quad j = k-1, \dots, 1, \tag{26b}$$

which, upon substitution in (13), yield equations (25). To obtain relations (26) one can represent equations (13) in the non-homogeneous form (21) and then perform integration by parts in (22)

$$\tilde{P}_\ell(t) = \frac{\Phi_\ell(t)}{a_\ell\theta_u + b_\ell\theta_c + \lambda} - \frac{e^{-(a_\ell\theta_u+b_\ell\theta_c+\lambda)t}}{a_\ell\theta_u + b_\ell\theta_c + \lambda}\Phi_\ell(0) - \frac{e^{-(a_\ell\theta_u+b_\ell\theta_c+\lambda)t}}{a_\ell\theta_u + b_\ell\theta_c + \lambda}\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}\Phi_\ell(\tau) e^{(a_\ell\theta_u+b_\ell\theta_c+\lambda)\tau}\mathrm{d}\tau. \tag{27}$$

Estimating the last integral for large values of θ_c [7], we obtain that the solutions of (21) are asymptotically representable in the form

$$\tilde{P}_\ell(t) = \frac{1}{a_\ell\theta_u + b_\ell\theta_c + \lambda}\Phi_\ell(t) + \mathcal{O}\left(\theta_c^{-2}\right), \quad \theta_c \to \infty. \tag{28}$$

Then, expression (24) represents the solution $P_{N-k,0,k}(t)$ for $k = N$ of equations (25), which can be obtained from (8) by substitution $\theta_k = \theta_u + \frac{\lambda}{N-k}$. □

An intuitive interpretation of Proposition 4 can be given as follows. An infinite cued rate $\theta_c = \infty$ entails instantaneous target detection by a searcher, once a cue from search-only vehicle is received. Thus, the "effective" search rate of each of i uncued searchers can be represented as $\theta_u + \lambda/i$, or as a sum of the searcher's own uncued rate θ_u and the rate λ/i at which this searcher receives cues from the search-only vehicle.

It is interesting to note that unlike (11), bound (19) in the hierarchical search model is quite tight (see, for example, Fig. 2, where $P_{005}(t)$ is displayed for values $\theta_c = 0.1$ and $\theta_u = 0.19$). Hence, one can infer that an increase in the cued rate θ_c has a more pronounced effect on the probability of detection $P_{00N}(t)$ than an increase in the search rate λ of the search-only vehicle. In other words, to increase the probability of detection, it is more appropriate to improve quality of cues generated by the search-only vehicle instead of increasing its search rate λ.

Another interesting observation is that in both cases of $\theta_c \gg 1$ and $\lambda \gg 1$ the limiting behavior of the two-layer system considered herein can be described by the generic model of Section 2 with appropriately selected dynamics of cued search rates θ_k in (3).

Conclusions

We have proposed a Markov chain approach to quantification of the effect of interaction via cueing in cooperative search systems. Using the classical mathematical techniques of

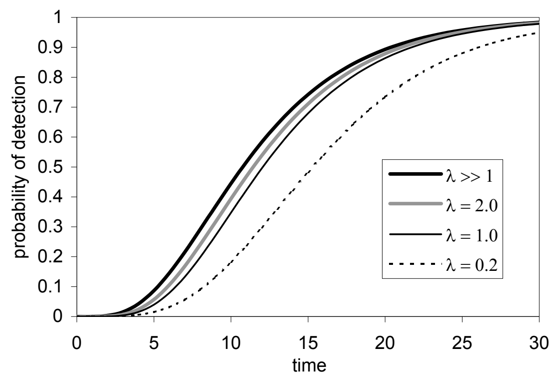

Figure 2: Probability of detection, $P_{005}(t)$, for a system of one search-only vehicle and 5 search-and-engage vehicles where $\theta_u = 0.1$ and $\theta_c = 0.19$.

Markov processes and queueing theory, we developed models of interaction among autonomous agents in search systems of different architectures. It has been shown that cueing can dramatically affect the probability of detection over a fixed time interval. In particular, it has been demonstrated that in the considered two-level hierarchical search system a better efficiency is achieved by improving the quality of information transmitted via cues, versus an increase in the volume of the transmitted detection data. We have also shown that there is an upper bound on the benefit of cueing, at least for the models defined.

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