

# On the Number of Local Minima for the Multidimensional Assignment Problem

Don A. Grundel\*    Pavlo A. Krokhmal<sup>†</sup>    Carlos A. S. Oliveira<sup>‡</sup>  
Panos M. Pardalos<sup>§</sup>

April 2006

## Abstract

The Multidimensional Assignment Problem (MAP) is an NP-hard combinatorial optimization problem occurring in many applications, such as data association, target tracking, and resource planning. As many solution approaches to this problem rely, at least partly, on local neighborhood search algorithms, the number of local minima affects solution difficulty for these algorithms. This paper investigates the expected number of local minima in randomly generated instances of the MAP. Lower and upper bounds are developed for the expected number of local minima,  $E[M]$ , in an MAP with iid standard normal coefficients. In a special case of the MAP, a closed-form expression for  $E[M]$  is obtained when costs are iid continuous random variables. These results imply that the expected number of local minima is exponential in the number of dimensions of the MAP. Our numerical experiments indicate that larger numbers of local minima have a statistically significant negative effect on the quality of solutions produced by several heuristic algorithms that involve local neighborhood search.

**Keywords:** Multidimensional Assignment Problem, Random Costs, Combinatorial Optimization, Local Minima, Neighborhood Search.

## 1 Introduction

The multidimensional assignment problem (MAP) is a higher dimensional version of the standard (two-dimensional, or linear) assignment problem. The objective of the MAP is to find tuples of elements from given sets, such that the total cost of the tuples is minimal. This

---

\*Air Force Research Lab/Munitions Directorate, Eglin AFB. E-mail: don.grundel@eglin.af.mil

<sup>†</sup>Department of Mechanical and Industrial Engineering, The University of Iowa. E-mail address: krokhmal@engineering.uiowa.edu (*corresponding author*). Partially supported by the NSF grant DMI-0457473.

<sup>‡</sup>School of Industrial Engineering and Management, Oklahoma State University. E-mail address: coliv@okstate.edu

<sup>§</sup>Center for Applied Optimization, Department of Industrial and Systems Engineering, University of Florida. E-mail address: pardalos@ufl.edu

problem has applications in numerous areas, such as data association (Andrijich and Caccetta, 2001), image recognition (Veenman et al., 1998), multisensor multitarget problems (Murphey et al., 1998), etc. More examples and an extensive discussion of the subject can be found in Pardalos and Pitsoulis (2000).

A well-known instance of the MAP is the three-dimensional assignment problem (3DAP). An example of the 3DAP is to minimize the total cost of assigning  $n_i$  items to  $n_j$  locations at  $n_k$  points in time. This is also referred to in the literature as the three-dimensional axial assignment problem, and may be extended to higher dimensions. The three-dimensional axial MAP can be formulated as

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n_i} \sum_{j=1}^{n_j} \sum_{k=1}^{n_k} c_{ijk} x_{ijk} \\
\text{s. t.} \quad & \sum_{j=1}^{n_j} \sum_{k=1}^{n_k} x_{ijk} = 1, \quad i = 1, \dots, n_i, \\
& \sum_{i=1}^{n_i} \sum_{k=1}^{n_k} x_{ijk} \leq 1, \quad j = 1, \dots, n_j, \\
& \sum_{i=1}^{n_i} \sum_{j=1}^{n_j} x_{ijk} \leq 1, \quad k = 1, \dots, n_k, \\
& x_{ijk} \in \{0, 1\}, \quad n_i \leq n_j \leq n_k,
\end{aligned} \tag{1}$$

where  $c_{ijk}$  is the cost of assigning item  $i$  to location  $j$  at time  $k$ . In this formulation, the variable  $x_{ijk}$  is equal to 1 if and only if the  $i$ -th item is assigned to the  $j$ -th location at time  $k$ .

Without loss of generality,  $d$ -dimensional axial MAP can be written in the form where each dimension has the same number  $n$  of elements, i.e.,

$$\min_{x \in \{0,1\}^{nd}} \left\{ \sum_{\substack{i_k \in \{1, \dots, n\} \\ k \in \{1, \dots, d\}}} c_{i_1 \dots i_d} x_{i_1 \dots i_d} \mid \sum_{\substack{i_k \in \{1, \dots, n\} \\ k \in \{1, \dots, d\} \setminus j}} x_{i_1 \dots i_d} = 1, \quad i_j = 1, \dots, n, \quad j = 1, \dots, d \right\}. \tag{2}$$

A feasible solution to (2) can be specified by enumerating the indices of non-zero binary variables  $x_{i_1 \dots i_d}$ . In such a way, the triple  $\{111, 222, 333\}$  represents a ‘‘diagonal’’ feasible solution to 3DAP (1) in which  $x_{ijk} = 1$  if  $i = j = k$ , and  $x_{ijk} = 0$  otherwise. Also, it is sometimes convenient to describe a feasible solution to MAP (2) by specifying its cost. Generally, the cost of a feasible solution to (2) has the form

$$z = c_{i_1^{(1)} \dots i_d^{(1)}} + c_{i_1^{(2)} \dots i_d^{(2)}} + \dots + c_{i_1^{(n)} \dots i_d^{(n)}}, \tag{3}$$

where  $\{i_k^{(1)}, i_k^{(2)}, \dots, i_k^{(n)}\}$  is a permutation of the set  $\{1, 2, \dots, n\}$  for every  $k = 1, \dots, d$ . This observation leads to an equivalent formulation for MAP in terms of permutations  $\pi_1, \dots, \pi_{d-1}$  of numbers 1 to  $n$ ,

$$\min_{\pi_1, \dots, \pi_{d-1} \in \Pi_n} \sum_{i=1}^n c_{i, \pi_1(i), \dots, \pi_{d-1}(i)}, \tag{4}$$

where  $\Pi_n$  is the set of all permutations of  $\{1, \dots, n\}$ .

In general, the MAP is known to be NP-hard, a fact which follows from a reduction of the three dimensional matching problem (Garey and Johnson, 1979). Solving even moderate sized instances of the MAP has been a difficult task, since a linear increase in the number of dimensions brings an exponential increase in the size of the problem. Despite its inherent difficulty, several exact and heuristic algorithms (Balas and Saltzman, 1991; Aiex et al., 2005; Clemons et al., 2003; Pasiliao, 2003) have been proposed to this problem. In most of these algorithms, some form of local neighborhood search is used.

In this paper we study the expected number of local minima in instances of MAP (2), where assignment costs  $c_{i_1 \dots i_d}$  are independent identically distributed (iid) random variables drawn from some continuous distribution.

A *local minimum* in a combinatorial problem can be defined with respect to a certain *local neighborhood* (see, e.g., Lin and Kernighan, 1973). Regarding the MAP (2), local neighborhoods are usually taken as  $p$ -exchange neighborhoods (see Section 2). In particular, the 2-exchange local neighborhood appears to be the most commonly used neighborhood in meta-heuristics such as GRASP that are applied to the MAP as evidenced in Aiex et al. (2005); Clemons et al. (2003); Murphey et al. (1998).

The motivation of this paper is to study the effects that the number of local minima in an MAP may have on heuristic solution algorithms that rely, at least partly, on repeated local searches in neighborhoods of feasible solutions Yong and Pardalos (1992). Intuitively, if the number of local minima is small then one may expect better performance from meta-heuristic algorithms that rely on local neighborhood searches. A solution landscape is considered to be *rugged* if the number of local minima is exponential with respect to the dimensions of the problem (Palmer, 1991). Evidence by Angel and Zissimopoulos (2001) showed that ruggedness of the solution landscape has a direct impact on the effectiveness of the simulated annealing heuristic in solving at least one other hard problem, the quadratic assignment problem. The present paper will further investigate the assumption that the number of local minima impacts the effectiveness of algorithms such as simulated annealing in solving the MAP.

The next section provides some additional background on the  $p$ -exchange local search neighborhoods. Section 3 describes the expected number of local minima for MAPs of size of  $n = 2$  and  $d \geq 3$  where the cost elements are independent identically distributed random variables from any probability distribution. Section 4 describes presents bounds for the expected number of local minima for all sizes of MAPs where assignment costs are independent standard normal random variables. Then in Section 5, we investigate how the expected number of local minima impacts the performance of various algorithms that rely on local searches. Some concluding remarks are given in the last section.

## 2 Local minima and $p$ -exchange neighborhoods in MAP

As it has been mentioned in the Introduction, we consider local minima of an MAP with respect to a local neighborhood, in the sense of Lin and Kernighan (1973). For any  $p = 2, \dots, n$ , we define the  $p$ -exchange local neighborhood  $\mathcal{N}_p(i)$  of the  $i$ -th feasible solution  $\{i_1^{(1)} \dots i_d^{(1)}, \dots, i_1^{(n)} \dots i_d^{(n)}\}$  of the MAP (2) as the set of solutions obtained from  $i$  by permut-

ing  $p$  or less elements in one of the dimensions  $1, \dots, d$ . More formally,  $\mathcal{N}_p(i)$  is the set of  $n$ -tuples  $\{j_1^{(1)} \cdots j_d^{(1)}, \dots, j_1^{(n)} \cdots j_d^{(n)}\}$  such that  $\{j_k^{(1)}, \dots, j_k^{(n)}\}$  is a permutation of  $\{1, \dots, n\}$  for all  $1 \leq k \leq d$ , and, furthermore, there exists only one  $k_0 \in \{1, \dots, d\}$  such that

$$2 \leq \sum_{r=1}^n \bar{\delta}_{i_{k_0}^{(r)} j_{k_0}^{(r)}} \leq p, \quad \text{while} \quad \sum_{r=1}^n \bar{\delta}_{i_k^{(r)} j_k^{(r)}} = 0 \quad \text{for all} \quad k \in \{1, \dots, d\} \setminus k_0, \quad (5)$$

where  $\bar{\delta}_{ij}$  is the negation of the Kronecker delta,  $\bar{\delta}_{ij} = 1 - \delta_{ij}$ . As an example, consider the following feasible solution to a  $d = 3, n = 3$  MAP (1):  $\{111, 222, 333\}$ . Then, one of its 2-exchange neighbors is  $\{111, 322, 233\}$ , another one is  $\{131, 222, 313\}$ ; a 3-exchange neighbor is given by  $\{311, 122, 233\}$ , etc. Evidently, one has  $\mathcal{N}_p \subset \mathcal{N}_{p+1}$  for  $p = 2, \dots, n - 1$ .

**Proposition 1** *For any  $p = 2, \dots, n$ , the size  $|\mathcal{N}_p|$  of the  $p$ -exchange local neighborhood of a feasible solution of a MAP (2) is equal to*

$$|\mathcal{N}_p| = d \sum_{k=2}^p D(k) \binom{n}{k}, \quad \text{where} \quad D(k) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j!. \quad (6)$$

**Proof:** Defining  $\mathcal{N}_p^*(\cdot)$  as the local neighborhood of a feasible solution obtained by permuting exactly  $p$  elements in any of the dimensions  $1, \dots, d$ , the set  $\mathcal{N}_p(\cdot)$  can be represented as

$$\mathcal{N}_p(\cdot) = \bigcup_{k=2}^p \mathcal{N}_k^*(\cdot), \quad \text{whence} \quad |\mathcal{N}_p| = \sum_{k=2}^p |\mathcal{N}_k^*|. \quad (7)$$

It is easy to see that  $|\mathcal{N}_k^*| = d \binom{n}{k} D(k)$ , where  $D(k)$  is the number of *derangements* of a  $k$ -element set Stanley (1986), i.e., the number of permutations  $\{1, 2, \dots, k\} \mapsto \{i^{(1)}, i^{(2)}, \dots, i^{(k)}\}$  such that  $i^{(1)} \neq 1, \dots, i^{(k)} \neq k$ . Then,  $D(k)$  satisfy the identity

$$k! - 1 = \sum_{j=1}^k \binom{k}{j} D(j) \quad k \geq 1,$$

which leads to expression (6). □

**Remark 1.1** Quantities  $D(k)$  can be easily calculated by means of the recurrent relation (see Stanley, 1986)

$$D(k) = kD(k-1) + (-1)^k, \quad D(1) = 0,$$

so that, for example,  $D(2) = 1, D(3) = 2, D(4) = 9$ , and so on. Then, according to Proposition 1, the size of a 2-exchange neighborhood is  $|\mathcal{N}_2| = d \binom{n}{2}$ , the size of a 3-exchange neighborhood is  $|\mathcal{N}_3| = d \left[ \binom{n}{2} + 2 \binom{n}{3} \right]$ , etc.

Note also that size of the  $p$ -exchange neighborhood is *linear* in the number of dimensions  $d$ . Depending on  $p$ ,  $|\mathcal{N}_p|$  is either *polynomial* or *exponential* in the number of elements  $n$  per dimension, as follows from the representation

$$D(n) = n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!} \right) \approx \frac{n!}{e}, \quad n \gg 1.$$

The definition of a local minimum with respect to the  $p$ -exchange neighborhood is then straightforward. The  $k$ -th feasible solution with cost  $z_k$  is a  $p$ -exchange local minimum iff  $z_k \leq z_j$  for all  $j \in \mathcal{N}_p(k)$ . Continuing the example above, the solution  $\{111, 222, 333\}$  is a 2-exchange local minimum iff its cost  $z_1 = c_{111} + c_{222} + c_{333}$  is less than or equal to costs of all of its 2-exchange neighbors.

The number  $M_p$  of local minima of the MAP is obtained by counting the feasible solutions that are local minima with respect to neighborhoods  $\mathcal{N}_p$ . In a random MAP, where the assignment costs are random variables,  $M_p$  becomes a random quantity itself. In this paper we are interested in determining the expected number  $\mathbb{E}[M_p]$  of local minima in random MAPs that have iid assignment costs with continuous distribution.

### 3 Expected number of local minima in MAP with $n = 2$

In a special case of an MAP where  $n = 2$ ,  $d \geq 3$ , and cost elements are independent identically distributed random variables from some continuous distribution with cumulative distribution function (c.d.f.)  $F$ , the feasible set possesses a special structure that allows for a closed-form expression for the expected number of local minima  $\mathbb{E}[M]$ . In doing so, we will need the following well-known fact, which we state as

**Proposition 2** *Given  $k$  iid continuous random variables, the probability that the  $r$ -th,  $r = 1, \dots, k$ , variable is the minimum value is  $1/k$ .*

In other words, in a sample of iid continuous random variables the position of the minimum value is uniformly located within the sequence regardless of the distribution. We are now in the position to prove the main result of this section.

**Theorem 1** *In a  $n = 2$ ,  $d \geq 3$  MAP with cost coefficients that are iid continuous random variables, the expected number of local minima is given by*

$$\mathbb{E}[M] = \frac{2^{d-1}}{d+1}. \quad (8)$$

**Proof:** In an  $n = 2$  MAP the largest local neighborhood is  $\mathcal{N}_2$ , hence the total number of local minima is equal to the number of 2-exchange local minima. Next, observe that in an instance of the MAP with  $n = 2$  and cost coefficients that are iid random variables with continuous distribution  $F$ , the costs of distinct feasible solutions are iid with distribution  $F_2$  that is the convolution of the distribution  $F$  with itself:  $F_2 = F * F$  (see Grundel et al., 2005).

Denoting  $N = 2^{d-1}$  as the number of feasible solutions to the  $n = 2$  MAP, and introducing indicator variables  $Y_k$  such that  $Y_k = 1$  if the  $k$ -th solution,  $k = 1, \dots, N$ , is a local minimum, and  $Y_k = 0$  otherwise, one can write the number  $M$  of local minima in an MAP with iid random coefficients as the sum of indicator variables:

$$M = \sum_{k=1}^N Y_k, \quad \text{whence} \quad \mathbb{E}[M] = \sum_{k=1}^N \mathbb{P}[Y_k = 1], \quad (9)$$

where  $\mathbb{P}[Y_k = 1]$  is the probability that the cost of the  $k$ -th feasible solution does not exceed the cost of any of its neighbors. According to Proposition 1, any feasible solution has  $|\mathcal{N}_2| = d$  neighbors whose costs, are iid continuous random variables. Then, Proposition 2 implies that

$$\mathbb{P}[Y_k = 1] = \frac{1}{d+1},$$

which, upon substitution into (9), yields the statement of the theorem (8).  $\square$

Equality (8) implies that in a  $n = 2$ ,  $d \geq 3$  MAP the number of local minima  $\mathbb{E}[M]$  is *exponential* in  $d$  when the cost coefficients are independently drawn from *any* continuous distribution.

## 4 Expected number of local minima in a random MAP with normally distributed costs

Our ability to derive a closed-form expression (8) for the expected number of local minima  $\mathbb{E}[M]$  in the previous section has relied on the independence of feasible solution costs (3) in an  $n = 2$  MAP. As it is easy to verify directly, in the case  $n \geq 3$  the costs of feasible solutions are generally not independent. This complicates analysis significantly if an arbitrary continuous distribution for assignment costs  $c_{i_1 \dots i_d}$  in (2) is assumed. However, as we show below, one can derive upper and lower bounds for  $\mathbb{E}[M]$  in the case when the costs coefficients of (2) are independent normally distributed random variables. First, we develop bounds for the number of local minima  $\mathbb{E}[M_2]$  defined with respect to 2-exchange neighborhoods  $\mathcal{N}_2$  that are most widely used in practice.

### 4.1 2-exchange local neighborhoods

Noting that in the general case the number  $N$  of the feasible solutions to MAP (2) is equal to  $N = (n!)^{d-1}$ , rewrite the expression (9) for  $\mathbb{E}[M_2]$  in the form

$$\mathbb{E}[M_2] = \sum_{k=1}^N \mathbb{P} \left[ \bigcap_{j \in \mathcal{N}_2(k)} z_k - z_j \leq 0 \right], \quad (10)$$

where  $\mathcal{N}_2(k)$  is the 2-exchange neighborhood of the  $k$ -th feasible solution, and  $z_i$  is the cost of the  $i$ -th feasible solution,  $i = 1, \dots, N$ .

If we allow the  $n^d$  cost coefficients  $c_{i_1 \dots i_d}$  of the MAP to be independent standard normal  $N(0, 1)$  random variables, then for any two feasible solutions the difference of their costs  $Z_{ij} = z_i - z_j$  in (10) is a normal variable with mean zero, as follows from (3).

Without loss of generality, consider the  $k = 1$  feasible solution to (2) whose cost is

$$z_1 = c_{1\dots 1} + c_{2\dots 2} + \dots + c_{n\dots n}. \quad (11)$$

In the 2-exchange neighborhood  $\mathcal{N}_2(1)$ , the cost of a feasible solution differs from (11) by two cost coefficients, e.g.,

$$z_2 = c_{21\dots 1} + c_{12\dots 2} + c_{3\dots 3} + \dots + c_{n\dots n}.$$

Generally, the difference  $z_1 - z_l$  between the cost of (11) and that of any its neighbor  $l \in \mathcal{N}_2(1)$  has the form

$$Z_{rsq} = c_{r\dots r} + c_{s\dots s} - c_{r\dots rsr\dots r} - c_{s\dots srs\dots s}, \quad (12)$$

where the last two cost coefficients have “switched” indices in the  $q$ -th position,  $1 \leq q \leq d$ . Observing that

$$Z_{rsq} = Z_{srq} \quad \text{for } r, s = 1, \dots, n; \quad q = 1, \dots, d,$$

consider the  $|\mathcal{N}_2|$ -dimensional random vector

$$\mathbf{Z} = (Z_{121}, \dots, Z_{12d}, Z_{131}, \dots, Z_{13d}, \dots, \dots, Z_{rs1}, \dots, Z_{rsd}, \dots, Z_{n-1,n,1}, \dots, Z_{n-1,n,d}), \quad r < s. \quad (13)$$

Vector  $\mathbf{Z}$  has normal distribution  $N(\mathbf{0}, \Sigma)$ , with covariance matrix  $\Sigma$  defined as

$$\text{Cov}(Z_{rsq}, Z_{ijk}) = \begin{cases} 4, & \text{if } i = r, j = s, q = k, \\ 2, & \text{if } i = r, j = s, q \neq k, \\ 1, & \text{if } (i = r, j \neq s) \text{ or } (i \neq r, j = s), \\ 0, & \text{if } i \neq r, j \neq s. \end{cases} \quad (14)$$

For example, in the case  $n = 3, d = 3$  the covariance matrix  $\Sigma$  has the form

$$\Sigma = \begin{pmatrix} 4 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 4 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 4 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 4 \end{pmatrix}.$$

Now, the probability in (10) can be expressed as

$$\mathbf{P}\left[\bigcap_{j \in \mathcal{N}_2(k)} z_k - z_j \leq 0\right] = F_{\Sigma}(\mathbf{0}), \quad (15)$$

where  $F_{\Sigma}$  is the c.d.f. of the  $|\mathcal{N}_2|$ -dimensional multivariate normal distribution  $N(\mathbf{0}, \Sigma)$ . While the value of  $F_{\Sigma}(\mathbf{0})$  in (15) is difficult to compute exactly for large  $d$  and  $n$ , lower and upper bounds can be constructed using Slepian’s inequality Slepian (1962); Tong (1990). To this end, let us introduce covariance matrices  $\underline{\Sigma} = (\underline{\sigma}_{ij})$  and  $\bar{\Sigma} = (\bar{\sigma}_{ij})$  as

$$\underline{\sigma}_{ij} = \begin{cases} 4, & \text{if } i = j, \\ 2, & \text{if } i \neq j \text{ and } (i - 1) \text{ div } d = (j - 1) \text{ div } d, \\ 0, & \text{otherwise,} \end{cases} \quad (16a)$$

$$\bar{\sigma}_{ij} = \begin{cases} 4, & \text{if } i = j, \\ 2, & \text{otherwise,} \end{cases} \quad (16b)$$

so that  $\underline{\sigma}_{ij} \leq \sigma_{ij} \leq \bar{\sigma}_{ij}$  holds for all  $1 \leq i, j \leq |\mathcal{N}_2|$ , with  $\sigma_{ij}$  being the components of the covariance matrix  $\underline{\Sigma}$  (14). Then, Slepian's inequality claims that

$$F_{\underline{\Sigma}}(\mathbf{0}) \leq F_{\Sigma}(\mathbf{0}) \leq F_{\bar{\Sigma}}(\mathbf{0}), \quad (17)$$

where  $F_{\underline{\Sigma}}(\mathbf{0})$  and  $F_{\bar{\Sigma}}(\mathbf{0})$  are c.d.f.'s of random variables  $\mathbf{X}_{\underline{\Sigma}} \sim N(\mathbf{0}, \underline{\Sigma})$  and  $\mathbf{X}_{\bar{\Sigma}} \sim N(\mathbf{0}, \bar{\Sigma})$  respectively.

As the random vector  $\mathbf{X}_{\bar{\Sigma}}$  is equicorrelated, the upper bound in (17) can be expressed by the one-dimensional integral (see, e.g., Tong (1990))

$$\int_{-\infty}^{+\infty} [\Phi(az)]^{|\mathcal{N}_2|} d\Phi(z), \quad a = \sqrt{\frac{\rho}{1-\rho}}, \quad (18)$$

where  $\Phi(\cdot)$  is the c.d.f. of standard normal distribution:

$$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt,$$

and  $\rho = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$  is the correlation coefficient of distinct components of  $\mathbf{X}_{\bar{\Sigma}}$ . According to (16b), the correlation coefficient is  $\rho = \frac{1}{2}$ , which allows for a simple closed-form expression for the upper bound in (17)

$$F_{\bar{\Sigma}}(\mathbf{0}) = \int_{-\infty}^{+\infty} [\Phi(z)]^{d \binom{n}{2}} d\Phi(z) = \frac{1}{d \binom{n}{2} + 1}. \quad (19)$$

This immediately yields the value of the upper bound  $\overline{\mathbb{E}[M_2]}$  for the expected number of local minima  $\mathbb{E}[M_2]$ :

$$\overline{\mathbb{E}[M_2]} = \sum_{k=1}^N F_{\bar{\Sigma}}(\mathbf{0}) = \frac{2(n!)^{d-1}}{n(n-1)d+2}. \quad (20)$$

Let us now calculate the lower bound for  $\mathbb{E}[M]$  using (17). According to the covariance matrix  $\underline{\Sigma}$  (16a), the random vector  $\mathbf{X}_{\underline{\Sigma}}$  is comprised of  $\frac{n(n-1)}{2}$  blocks of variables, each consisting from  $d$  elements,

$$\mathbf{X}_{\underline{\Sigma}} = \left( X_1, \dots, X_d, \dots, X_{(n-1)/2-1)d+1}, \dots, X_{(n-1)/2}d \right),$$

such that the variables from different blocks are independent, whereas variables within a block have a  $d \times d$  covariance matrix defined as in (16b). Thus, one can express the lower bound  $F_{\underline{\Sigma}}(\mathbf{0})$  in (17) as a product

$$F_{\underline{\Sigma}}(\mathbf{0}) = \prod_{i=1}^{\frac{n(n-1)}{2}} \mathbb{P}[X_{(i-1)d+1} \leq 0, \dots, X_{id} \leq 0].$$

Since variables  $X_{(i-1)d+1}, \dots, X_{id}$  are equicorrelated with correlation coefficient  $\rho = \frac{1}{2}$ , each probability term in the last equality can be computed similarly to evaluation of the lower-bound probability (19), i.e.,

$$F_{\underline{\Sigma}}(\mathbf{0}) = \prod_{i=1}^{\frac{n(n-1)}{2}} \int_{-\infty}^{+\infty} [\Phi(z)]^d d\Phi(z) = \left( \frac{1}{d+1} \right)^{\frac{n(n-1)}{2}}.$$

This finally leads to a lower bound for the number of expected minima:

$$\underline{\mathbb{E}}[M_2] = \sum_{k=1}^N F_{\underline{\Sigma}}(\mathbf{0}) = \frac{(n!)^{d-1}}{(d+1)^{n(n-1)/2}}.$$

The presented arguments can be readily extended to the case when cost coefficients  $c_{i_1 \dots i_d}$  are drawn independently from an arbitrary  $N(\mu, \sigma^2)$  normal distribution. It suffices to note that in this case variables  $Z_{rsq}$  (12) have marginal distributions  $N(0, 4\sigma^2)$  and joint distribution  $N(\mathbf{0}, \Sigma_\sigma)$ , where covariance matrix  $\Sigma_\sigma$  is obtained from matrix  $\Sigma$  (14) by multiplying all its elements by  $\sigma^2$ :

$$\Sigma_\sigma = \sigma^2 \Sigma.$$

Similarly, covariance matrices  $\underline{\Sigma}_\sigma = \sigma^2 \underline{\Sigma}$  and  $\overline{\Sigma}_\sigma = \sigma^2 \overline{\Sigma}$  can be used to construct the lower and upper bounds for probability  $F_{\Sigma_\sigma}(\mathbf{0})$  as in (17). Observe that, for instance, components of vector  $\mathbf{X}_{\overline{\Sigma}_\sigma}$  with c.d.f.  $F_{\overline{\Sigma}_\sigma}$  are still equicorrelated with  $\rho = \frac{1}{2}$ , which yields  $F_{\overline{\Sigma}_\sigma}(\mathbf{0}) = [d \binom{n}{2} + 1]^{-1}$  by (19). Hence, the obtained upper bound (20) for  $\mathbb{E}[M_2]$  is valid in MAP with iid normal  $N(\mu, \sigma^2)$  coefficients. It also can be demonstrated that the corresponding lower bound holds as well. In such a way, we have proved the following

**Theorem 2** *In a  $n \geq 3$ ,  $d \geq 3$  MAP with iid normal cost coefficients, the expected number of 2-exchange local minima is bounded as*

$$\frac{(n!)^{d-1}}{(d+1)^{n(n-1)/2}} \leq \mathbb{E}[M_2] \leq \frac{2(n!)^{d-1}}{n(n-1)d+2}. \quad (21)$$

**Remark 2.1** Both the lower and upper bounds in (21) coincide with the exact expression (8) for  $\mathbb{E}[M_2]$  in the case  $n = 2$ .

From (21) it follows that for fixed  $n \geq 3$ , the expected number of local minima is *exponential* in the number of dimensions  $d$  for a fixed  $n$ .

## 4.2 Higher-order neighborhoods ( $p \geq 3$ )

The outlined approach is applicable to general  $p$ -exchange neighborhoods. For convenience, here we consider the neighborhoods  $\mathcal{N}_p^*$  as defined in Section 2, i.e., the neighborhoods obtained from a given feasible solution by permuting *exactly*  $p$  elements in one of the  $d$  dimensions, so that for any feasible solution  $i = \{i_1^{(1)} \dots i_d^{(1)}, \dots, i_1^{(n)} \dots i_d^{(n)}\}$  and its  $p$ -exchange neighbor  $j = \{j_1^{(1)} \dots j_d^{(1)}, \dots, j_1^{(n)} \dots j_d^{(n)}\} \in \mathcal{N}_p^*(i)$  one has (compare to (5))

$$\sum_{r=1}^n \bar{\delta}_{i_{k_0}^{(r)} j_{k_0}^{(r)}} = p, \quad k_0 \in \{1, \dots, d\}, \quad \text{and} \quad \sum_{r=1}^n \bar{\delta}_{i_k^{(r)} j_k^{(r)}} = 0 \quad \text{for all } k \in \{1, \dots, d\} \setminus k_0. \quad (22)$$

Then, upper and lower bounds for the expected number of local minima  $\mathbb{E}[M_p^*]$  defined with respect to  $p$ -exchange neighborhoods  $\mathcal{N}_p^*$  can be derived in a similar fashion. As before,

consider the neighborhood of the  $i = 1$ , or the “diagonal” solution  $\{1 \cdots 1, \dots, n \cdots n\}$ . Then the difference between its cost and the cost of any of its neighbors from  $\mathcal{N}_p^*(1)$  has the form

$$Z_i = z_1 - z_i = \sum_{k=1}^p c_{i_k \cdots i_k} - \sum_{i=1}^p c_{i_k \cdots \pi_p^*(i_k) \cdots i_k}. \quad (23)$$

Here  $\{i_1, \dots, i_p\} \subseteq \{1, \dots, n\}$  is the set of  $p$  elements that is permuted (deranged), and  $\pi_p^*$  is the corresponding derangement. If the assignment costs of the MAP (2) are standard normal  $N(0, 1)$  random variables, then the differences  $Z_i = z_1 - z_i$ ,  $i \in \mathcal{N}_p^*(1)$ , are also normal  $N(0, 2p)$  random variables. Following the approach of section 4, let us partition the  $|\mathcal{N}_p^*|$ -dimensional vector of cost differences (23) into  $\binom{n}{p}$  blocks, each block corresponding to a fixed selection  $\{i_1, \dots, i_p\}$  of the  $p$  elements being permuted. Within each block, there are

$$dD(p) = d \sum_{j=0}^p (-1)^{p-j} \binom{p}{j} j!$$

variables that differ by the second summation term in (23). Next, observe that the minimum covariance between two variables within a block is  $p$ , which happens when these variables do not have any common summands in the second summation term in the representation (23). This is the case if, for example, the corresponding derangements  $\pi_p^*$  are distinct cyclic permutations.

If  $p \geq 4$ , the maximum covariance between two variables in a block is equal to  $2p - 2$ , i.e., there may be at most  $p - 2$  common elements in the second summation term (23). Examples of the corresponding derangements are  $\{1, 2, 3, \dots, k\} \mapsto \{k, 1, 2, 3, \dots, k - 1\}$  and  $\{1, 2, 3, \dots, k\} \mapsto \{2, 1, k, 3, \dots, k - 1\}$ . Note that in the case  $p = 3$  all variables within a block are equicorrelated with covariance equal to  $p = 3$ .

Similarly to the above, introduce two matrices  $\bar{\Sigma}_p, \underline{\Sigma}_p \in \mathbb{R}^{|\mathcal{N}_p^*| \times |\mathcal{N}_p^*|}$  such that

$$(\bar{\Sigma}_p)_{ij} = \begin{cases} 2p, & \text{if } i = j, \\ 2p - 2, & \text{if } i \neq j, \end{cases} \quad (24a)$$

$$(\underline{\Sigma}_p)_{ij} = \begin{cases} 2p, & \text{if } i = j, \\ p, & \text{if } i \neq j \text{ and } (i - 1) \operatorname{div}(dD(p)) = (j - 1) \operatorname{div}(dD(p)), \\ 0, & \text{otherwise.} \end{cases} \quad (24b)$$

Then  $(\underline{\Sigma}_p)_{ij} \leq (\Sigma_p)_{ij} \leq (\bar{\Sigma}_p)_{ij}$ , where  $\Sigma_p$  is the covariance matrix of the  $|\mathcal{N}_p^*|$ -dimensional random vector of the cost differences (23). Then the sought probability

$$\mathbb{P} \left[ \bigcap_{i \in \mathcal{N}_p^*(k)} z_k - z_i \leq 0 \right] = F_{\Sigma_p}(\mathbf{0})$$

can be bounded as  $F_{\underline{\Sigma}_p}(\mathbf{0}) \leq F_{\Sigma_p}(\mathbf{0}) \leq F_{\bar{\Sigma}_p}(\mathbf{0})$ , which yields

$$\frac{(n!)^{d-1}}{[dD(p) + 1]^{\binom{n}{p}}} \leq \mathbb{E}[M_p^*] \leq (n!)^{d-1} \int_{-\infty}^{+\infty} [\Phi(\sqrt{p-1}z)]^{d \binom{n}{p} D(p)} d\Phi(z), \quad (25)$$

where  $M_p^*$  is the number of local minima with respect to  $\mathcal{N}_p^*$  neighborhoods. Note that (21) is a special case of (25) for  $p = 2$ . The integral in the right-hand side of (25) does not seem to

allow for an explicit-form solution, except for the case  $p = 2$ . Yet, the asymptotical behavior of this integral for large  $n$  or  $d$  can be estimated using the standard methods of asymptotic analysis Olver (1997). In the case  $p = 3$ , however, the upper bound in (25) can be simplified due to the above observations by defining the matrix  $\bar{\Sigma}_3$  as

$$(\bar{\Sigma}_3)_{ij} = \begin{cases} 6, & \text{if } i = j, \\ 3, & \text{if } i \neq j, \end{cases}$$

which leads to an upper bound for  $E[M_3^*]$  of the form

$$E[M_3^*] \leq \frac{3(n!)^{d-1}}{n(n-1)(n-2)d+3}. \quad (26)$$

Following the same track of arguments as used in the proof of Theorem 2, it can be shown that the developed bounds (25), (26) hold for MAP with arbitrary normal  $N(\mu, \sigma^2)$  iid assignment costs. We summarize the results of this subsection in

**Theorem 3** *In a  $n \geq 3$ ,  $d \geq 3$  MAP with iid normal cost coefficients, the expected number of local minima  $M_p^*$  with respect to  $p$ -exchange local neighborhoods  $\mathcal{N}_p^*$  is bounded as in (25). For 3-exchange neighborhoods  $\mathcal{N}_3^*$ , an improved upper bound (26) holds.*

It is interesting to note that for a fixed  $p$  the ratio of number of local minima to the number of feasible solutions becomes infinitely small as the dimensions of the problem increase (see (8), (21), and (25)).

## 5 Numerical results

In this section we report the numerical results on the number of local minima in random MAPs. The experimentally determined number of local minima in random MAPs indicate that the developed bounds for  $E[M_2]$  are applicable in the cases when the assignment costs have distributions other than normal. Also, the effect of the number of local minima on various (heuristic) solution algorithms is discussed.

### 5.1 Experimentally determined number of local minima

We can evaluate the goodness of the developed bounds by empirically determining the number of local minima in randomly generated MAPs. The assignment costs  $c_{i_1 \dots i_d}$  for each problem instance have been drawn from one of the three distributions: the uniform  $U[0, 1]$ , exponential with mean  $\lambda = 1$ , and standard normal,  $N(0, 1)$ . In the case of exponential distribution, the assignment costs have been generated as  $c_{i_1 \dots i_d} = -\ln U$ , and for the standard normal distribution we used the well-known polar method (see, e.g., Law and Kelton, 1991).

Table 1 shows the average number of local minima with respect to 2-exchange neighborhoods  $\mathcal{N}_2$ . For small sized problems, the number of local minima was computed exactly by complete enumeration of all feasible solutions in a given instance of the problem; the values in Table 1 have been obtained by averaging over 100 problem instances. For larger problems (indicated

by \* in the table), the average number of local minima was estimated by examining a large number<sup>1</sup> of randomly generated problem instances. For each instance, we randomly selected a feasible solution and determined whether it was a local minimum. This yields an estimate of the probability that any feasible solution is a local minimum, which was then used to estimate the average number of local minima by multiplying this probability by the number of feasible solutions. This technique proved to have results consistent with the complete enumeration method mentioned above for small problems. Regardless of the distribution that cost coefficients were drawn from, standard deviations of 40% and 5% have observed for problems of sizes  $d = 3, n = 3$  and  $d = 5, n = 5$ , respectively. It is clear from the tables that  $E[M_2]$  is effected by the distribution from which assignment costs are drawn.

Table 2 shows similar results for the 3-exchange local neighborhood when the cost coefficients are iid standard normal. As expected, experimental evidence indicates that  $E[M_3] \leq E[M_2]$ , i.e., the expected number of local minima with respect to 3-exchange neighborhoods is smaller than that for 2-exchange neighborhoods for same-sized problems.

Figure 1 visualizes the experimentally determined number of local minima in random MAPs as presented in Table 1, along with the developed bounds (17), for instances with fixed  $d = 6$  (left) and  $n = 6$  (right) and assignment costs drawn from the above three distributions. The logarithmic scale is used due to the exponential growth of the corresponding values with  $n$  and  $d$ . In particular, Fig. 1 indicates that although the bounds (17) have been developed in assumption of normally distributed assignment costs, they are also valid for other distributions.

Number of local minima, Uniform on [0,1]				
$n \setminus d$	3	4	5	6
2	1	1.68	2.66	4.56
3	2	7.69	33.5	159
4	5.60	77.8	1230	2.1E+4
5	21.0	1355	9.58E+04*	7.60E+6*
6	116	3.62E+04*	1.30E+07*	6.56E+9*

Number of local minima, Exponential $\lambda = 1$				
$n \setminus d$	3	4	5	6
2	1	1.54	2.66	4.71
3	2.06	7.84	35.8	165
4	5.47	80.6	1290	2.18E+4
5	22.7	1400	1.01E+5*	7.67E+06*
6	122	3.91E+4*	1.53E+07*	6.26E+09*

Number of local minima, Standard Normal costs				
$n \setminus d$	3	4	5	6
2	1	1.56	2.58	4.66
3	1.82	7.23	30.3	141
4	4.54	62.2	949	1.58E+04
5	16.3	939	6.5E+4*	4.75E+06*
6	75.6	2.36E+4*	7.90E+06*	3.48E+09*

Table 1: Average number of local minima with respect to 2-exchange local neighborhoods in random MAPs with iid assignment costs

<sup>1</sup>The number examined depends on problem size. The number ranged from  $10^6$  to  $10^7$

$n \backslash d$	3	4	5	6
3	1.55	5.98	26.0	124
4	3.27	43.1	670	1.11E+4
5	8.27	516	3.48E+4*	2.65E+06*
6	28.8	8710*	3.06E+06*	1.22E+09*

Table 2: Average number of local minima with respect to 3-exchange local neighborhoods in random MAPs with iid standard normal assignment costs

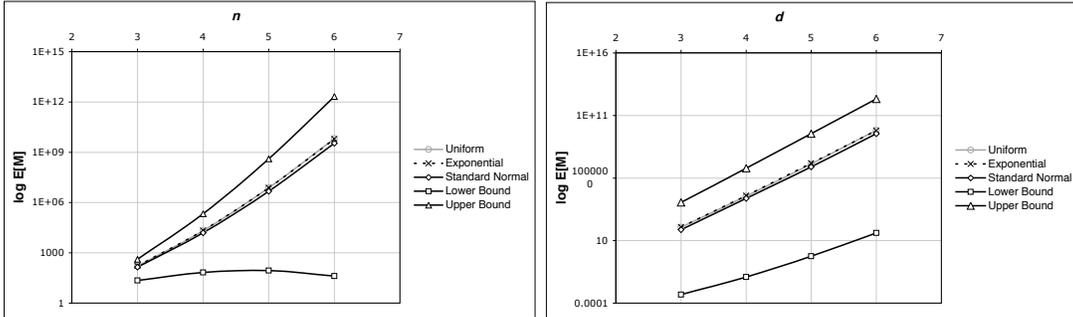


Figure 1: Experimentally determined  $E[M_2]$  and bounds (17) in random MAPs with fixed  $d = 6$  (left) and  $n = 6$  (right) and uniform, exponential, and standard normal assignment costs (in logarithmic scale)

## 5.2 Effect of $M$ on solution algorithms

In this subsection we examine the impact of the number of local minima in random MAPs on the performance of several heuristic solution algorithms that rely, at least in part, on local neighborhood searches. In particular, we consider three heuristics: random local search, Greedy Randomized Adaptive Search (GRASP), and Simulated Annealing. These heuristics were exercised against various-sized problems randomly generated from the standard normal distribution.

**Random Local Search:** The random local search procedure simply steps through a given number of iterations. Each iteration begins by selecting a random starting solution. The algorithm then conducts a local search until no better solution can be found. The algorithm captures the overall best solution and reports it after executing the maximum number of iterations.

**GRASP:** A greedy randomized adaptive search procedure (GRASP) (see Feo and Resende, 1989, 1995; Festa and Resende, 2001) is a multi-start or iterative process, in which each iteration consists of two phases. In a construction phase, a random adaptive rule is used to build a feasible solution one component at a time. In a local search phase, a local optimum in the neighborhood of the constructed solution is sought. The best overall solution is kept as the result. The neighborhood search conducted is similar to that in the random local search algorithm above. That is, neighbors of a current solution are examined one at a time and if an improving solution is found, it is adopted as the current solution and local search starts again. Local search ends when no improving solution is

found. GRASP has been used in many applications and specifically in solving the MAP by Aiex et al. (2005) and Murphey et al. (1998).

**Simulated Annealing:** Simulated Annealing is a popular heuristic used in solving a variety of problems (Kirkpatrick et al., 1983; Metropolis et al., 1953). Simulated annealing uses a local search procedure, but the process allows *uphill* moves. The probability of moving uphill is higher at high *temperatures*, but as the process *cools*, there is less probability of moving up hill. The specific steps for simulated annealing used in this paper are taken from Gosavi (2003). Simulated annealing has been applied to the MAP by Clemons et al. (2003).

The heuristic solution quality  $Q$ , which is the relative difference from the optimal value, is reported and compared for problems with the same size, and assignment costs independently drawn from the standard normal distribution. The purpose of our analysis is not to compare the efficiency of the heuristic algorithms, but to determine the extent to which the number of local minima affects the performance of these algorithms. Each run of an experiment involved the following general steps:

1. Generate random MAP instance with cost coefficients that are iid standard normal random variables
2. Obtain  $M$  by checking each feasible solution for being a local minimum
3. Solve the generated MAP instance using each of the above heuristics 100 times and return the average solution quality,  $\overline{Q}$ , for each heuristic method

Problem sizes were chosen so as to allot a practically reasonable amount of time to determine  $M$  (since counting  $M$  is the obvious bottleneck in the experiment). Four problem sizes chosen were  $d = 3, n = 6$ ;  $d = 4, n = 5$ ;  $d = 4, n = 6$ ; and  $d = 6, n = 4$ . For problem size  $d = 4, n = 6$ , which has the largest number of feasible solutions  $N$ , a single run took approximately four hours on a 2.2 GHz Pentium 4 machine. The number of runs for each experiment varied for each problem size with fewer runs for larger problems. The number of runs were 100, 100, 30, and 50, respectively, for the problem sizes listed above.

Figure 2 displays the average solution quality for each of the three heuristics versus the number of local minima. We report the results for problem size  $d = 4, n = 5$ , and the results for other problem sizes are similar. Included in each plot is a best-fit linear least-squares line that provides some insight on the effect of  $M$  on solution quality. A close examination of the figures indicates that the solution quality improves with smaller  $M$  for each heuristic solution method. This conclusion was verified using a regression analysis to determine that the effect of  $M$  on average solution quality is statistically significant ( $p$ -value averaged approximately 0.01).

We have also investigated the 3-exchange neighborhood. Figure 3 displays plots of the average solution quality of the three heuristics versus the number of local minima when using the larger neighborhood. The parameters, such as number of iterations, were kept the same for each heuristic. The only change made was the neighborhood definition. The plots for random local search and GRASP indicate that the effect of  $M$  on solution quality is statistically significant (the average  $p$ -value of the regression analysis is 0.05). However, the effect of  $M$  is not

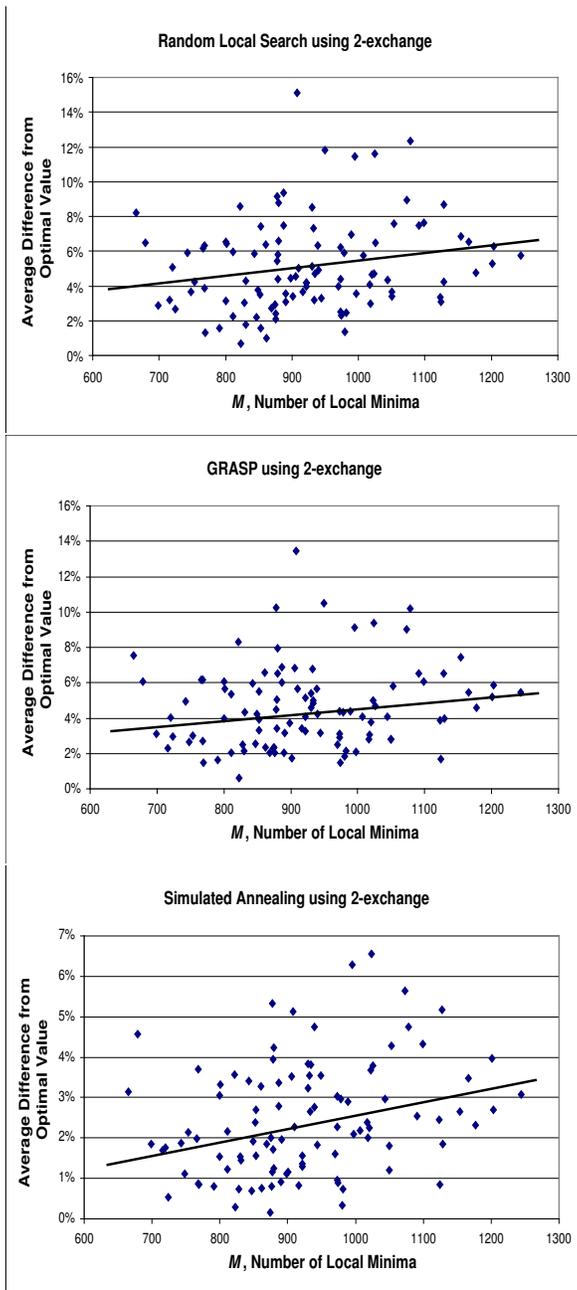


Figure 2: Plots of solution quality versus number of local minima when using the 2-exchange neighborhood. The MAP has a size of  $d = 4$ ,  $n = 5$  with cost coefficients that are iid standard normal.

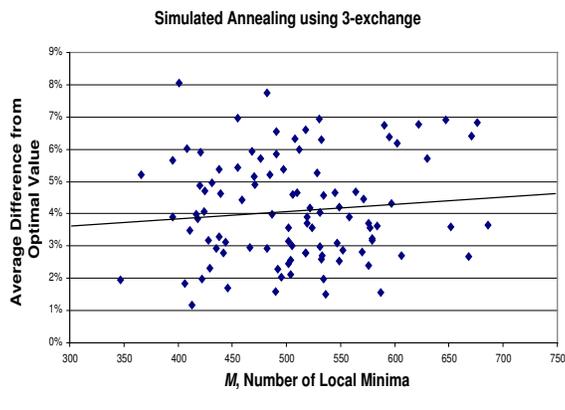
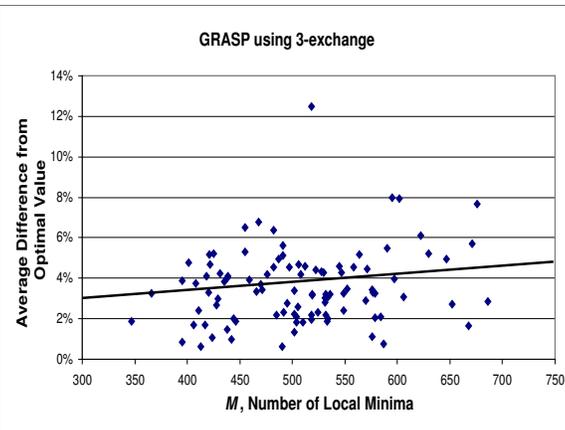
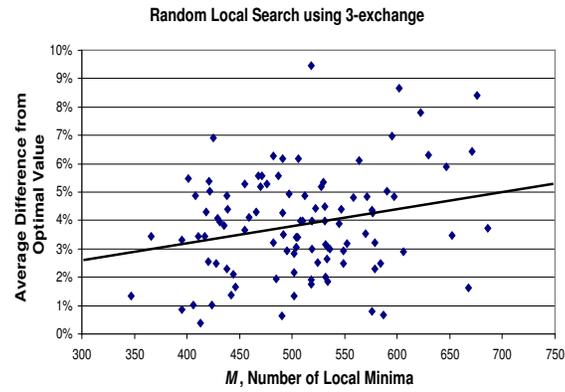


Figure 3: Plots of solution quality versus number of local minima when using a 3-exchange neighborhood. The MAP has a size of  $d = 4$ ,  $n = 5$  with cost coefficients that are iid standard normal.

statistically significant in the case of simulated annealing when the 3-exchange neighborhood is used ( $p$ -value of 0.4).

We note a couple of interesting aspects when comparing Figures 2 and 3. The solution quality obtained by random local search or GRASP improves when using the larger neighborhood. This is to be expected, but at the expense of longer solution times. We observed that, on average, the random local search took approximately 30% longer and GRASP took about 15% longer to complete the same number of iterations. Simulated annealing’s performance in terms of solution quality dropped when the neighborhood size was increased. This is not surprising as the optimal starting temperature and cooling rate depend on the problem-instance characteristics such as size, neighborhood definition, cost coefficient values, etc. Our experimental results for simulated annealing reiterate the necessity for properly tuning heuristic parameters when the neighborhood definition is changed.

## 6 Conclusions

In this paper we report some analytical and experimental results concerning the number of local minima in the MAP whose assignment costs are realizations of independent identically distributed random variables. Experimental evidence suggests that the expected number of local minima,  $E[M]$ , is exponential with respect to the dimensions of the problem when cost coefficients are iid continuous random variables. This was proved true for the case  $n = 2$  and  $d \geq 3$ , and costs being drawn from an *arbitrary* continuous distribution. For the case  $n \geq 3$ ,  $d \geq 3$ , we were able to develop lower and upper bounds for  $E[M]$  provided that cost coefficients are independent standard normal random variables. Considering the lower bound, we found that, for fixed  $n \geq 3$ ,  $E[M]$  is exponential in  $d$ . An interesting consequence of these results is that the ratio of  $E[M]$  to the number of feasible solutions becomes infinitely small with increasing problem size.

Through experimental procedures we found that increasing  $M$  has a statistically significant negative effect on solution quality for three heuristic solution algorithms that rely on local neighborhood searches (random local search, GRASP, and simulated annealing).

## References

- Aiex, R., Resende, M., Pardalos, P. M., and Toraldo, G. (2005) “GRASP with path relinking for the three-index assignment problem,” *INFORMS Journal on Computing*, **17** (2), 224–247.
- Andrijich, S. M. and Caccetta, L. (2001) “Solving the multisensor data association problem,” *Nonlinear Analysis*, **47**, 5525–5536.
- Angel, E. and Zissimopoulos, V. (2001) “On the landscape ruggedness of the quadratic assignment problem,” *Theoretical Computer Science*, **263**, 159–172.
- Balas, E. and Saltzman, M. J. (1991) “An algorithm for the three-index assignment problem,” *Operations Research*, **39**, 150–161.

- Clemons, W., Grundel, D., and Jeffcoat, D. (2003) “Applying simulated annealing on the multidimensional assignment problem,” in: “Proceedings of the 2nd Cooperative Control and Optimization Conference,” .
- Feo, T. A. and Resende, M. G. C. (1989) “A probabilistic heuristic for a computationally difficult set covering problem,” *Operations Research Letters*, **8**, 67–71.
- Feo, T. A. and Resende, M. G. C. (1995) “Greedy randomized adaptive search procedures,” *Journal of Global Optimization*, **6**, 109–133.
- Festa, P. and Resende, M. (2001) “GRASP: An annotated bibliography,” in: P. Hansen and C. C. Ribeiro (Eds.) “Essays and Surveys on Metaheuristics,” 325–367, Kluwer Academic Publishers.
- Garey, M. R. and Johnson, D. S. (1979) *Computers and Intractability: A Guide to the Theory of NP-completeness*, W.H. Freeman and Company.
- Gosavi, A. (2003) *Simulation-Based Optimization: Parametric Optimization Techniques and Reinforcement Learning*, Kluwer Academic Publishers.
- Grundel, D. A., Oliveira, C. A. S., Pardalos, P. M., and Pasiliao, E. L. (2005) “Asymptotic Results for Random Multidimensional Assignment Problems,” *Computational Optimization and Applications*, **31** (3), 275–293.
- Kirkpatrick, S., Gelatt, C. D., and Vecchi, M. P. (1983) “Optimization by Simulated Annealing,” *Science*, **220**, 671–680.
- Law, A. and Kelton, W. (1991) *Simulation Modeling and Analysis*, McGraw-Hill, Inc., New York, 2nd edition.
- Lin, S. and Kernighan, B. W. (1973) “An effective heuristic algorithm for the traveling salesman problem,” *Operations Research*, **21**, 498–516.
- Metropolis, N., Rosenbluth, A., Rosenbluth, M., Teller, A., and Teller, E. (1953) “Equation of State Calculations by Fast Computing Machines,” *Journal of Chemical Physics*, **21** (6), 1087–1092.
- Murphey, R., Pardalos, P., and Pitsoulis, L. (1998) “A greedy randomized adaptive search procedure for the multitarget multisensor tracking problem,” in: “DIMACS Series Vol. 40,” 277–302, American Mathematical Society.
- Olver, F. W. (1997) *Asymptotics and Special Functions*, AK Peters Ltd, Wellesley, MA, 2nd edition.
- Palmer, R. (1991) “Optimization on rugged landscapes,” in: A. Perelson and S. Kauffman (Eds.) “Molecular Evolution on Rugged Landscapes: Proteins, RNA, and the Immune System,” 3–25, Addison Wesley, Redwood City, CA.
- Pardalos, P. M. and Pitsoulis, L. (Eds.) (2000) *Nonlinear Assignment: Problems, Algorithms and Applications*, Kluwer Academic Publishers, Dordrecht.

- Pasilio, E. L. (2003) *Algorithms for Multidimensional Assignment Problems*, Ph.D. thesis, Department of Industrial and Systems Engineering, University of Florida.
- Slepian, D. (1962) "The one-sided barrier problem for Gaussian noise," *Bell System Technical Journal*, **41**, 463–501.
- Stanley, R. (1986) *Enumerative Combinatorics*, Wadsworth & Brooks, Belmont, CA.
- Tong, Y. L. (1990) *The Multivariate Normal Distribution*, Springer Verlag, Berlin.
- Veenman, C. J., Hendriks, E. A., and Reinders, M. J. T. (1998) "A fast and robust point tracking algorithm," in: "Proceedings of the Fifth IEEE International Conference on Image Processing," 653–657, Chicago, USA.
- Yong, L. and Pardalos, P. M. (1992) "Generating quadratic assignment test problems with known optimal permutations," *Computational Optimization and Applications*, **1** (2), 163–184.