Random Assignment Problems

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Abstract

Analysis of random instances of optimization problems is instrumental for understanding of the behavior and properties of problem's solutions, feasible region, optimal values, especially in large-scale cases. A class of problems that have been studied extensively in the literature using the methods of probabilistic analysis is represented by the assignment problems, and many important problems in operations research and computer science can be formulated as assignment problems. This paper presents an overview of the recent results and developments in the area of probabilistic assignment problems, including the linear and multidimensional assignment problems, quadratic assignment problem, etc.

Keywords: Random assignment problems, linear assignment problem, quadratic assignment problem, multidimensional assignment problem, bottleneck assignment problem, probabilistic analysis, asymptotic analysis, fitness landscape analysis.

1 Introduction

Analysis of random instances of optimization problems is instrumental for elucidating the properties of problem's solutions, feasible region, optimal values, especially in large-scale cases. The probabilistic framework, where it is assumed that the problem's data is drawn from some probability distribution, provides a facility for maintaining a consistent structure among problem instances of different sizes, thus enabling one to analyze the problem's optimal value, solutions, etc., as functions of the problem size. A particular class of combinatorial optimization problems that have been studied extensively in the probabilistic context is represented by the assignment, or matching problems.

In an assignment problem, one is looking to find an assignment, or matching, between the elements of two (or more) sets, such that the total cost of all matched pairs (tuples) is minimized. Depending on the particular structure of the sets being matched, the form of the cost function, the matching rule, and so on, the assignment problems are categorized into linear, quadratic, bottleneck, multidimensional, etc. The assignment problems can be stated in a variety of forms, including mathematical programming, combinatorial, or graph-theoretic formulations, and constitute one of the most important and fundamental objects in the areas of computer science, operations research, and discrete mathematics. Besides these areas of their "natural

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habitat," assignment problems have found numerous applications in other disciplines of science and engineering, including chemistry, biology, physics, archeology, electrical engineering, sports, and others (for a comprehensive review of the subject of assignment problems, their formulations and applications, see, for instance, Pardalos and Pitsoulis (2000), Burkard (2002), Pentico (2007), and references therein).

The studies of random instances of assignment problems date back to as early as (Donath, 1969), who investigated the limiting behavior of the linear assignment problem by solving small (by today's standards) randomly generated instances. In retrospect, probabilistic analysis of the linear assignment problem, whose deterministic instances are considered as being computationally "easy", has turned out to be quite challenging to researchers. Until recently, much more has been known in the literature about the behavior of many random combinatorial optimization problems, including assignment problems, that are regarded as "difficult" and belong to the NP-hard class (e.g., the quadratic assignment problem, traveling salesman problem, etc.) than about the random instances of the "easy" linear assignment problem. This situation has changed only in the last few years, when a number of powerful results have been obtained in the scope of the random linear assignment problem, including results for *finite-size* instances that juxtapose the predominantly *asymptotic* analyses contained in the literature on random optimization problems.

It also must be noted that studies of random assignment problems have led to important findings in the context of other optimization problems (and vice versa). The striking insight of Burkard and Fincke (1982a) that random large-scale instances of the quadratic assignment problem, which is considered one of the hardest combinatorial optimization problems, can be solved almost to optimality by practically any heuristic algorithm with high probability, has led to discovery of an entire class of combinatorial optimization problems with similar properties (Burkard and Fincke, 1985; Szpankowski, 1995). Similarly, Karp's (1987) derivation of an upper bound on the expected optimal value of the linear assignment problem was generalized to a broad class of random linear programming problems (Dyer et al., 1986), etc.

In this survey we present a detailed exposition of the state-of-the art results in the area of random assignment problems, including the recent developments in the context of the random linear assignment problem (section 2) and its higher-dimensional generalization, the multidimensional assignment problem (section 3), the quadratic assignment problem (section 4), and the bottleneck assignment problem (section 5). Finally, section 6 contains results on assignment problems that do not fall into the broad classes mentioned above, but for which probabilistic analysis has been conducted in the literature.

Our main focus is on the analytical results pertinent to various aspects of the corresponding random assignment problems: the optimal value, its convergence and limits, properties of the problem's landscape, local extrema, etc. The computational and algorithmic sides of the probabilistic analysis of assignment problems are covered only briefly, as a number of comprehensive surveys on solution methods for various classes of assignment problems, which also discuss probabilistic-based algorithms, are available in the literature (see, e.g. Burkard and Çela (1999), Burkard (2002), Anstreicher (2003), Loiola et al. (2007), and others).

As many authors point out (see e.g., Rhee, 1988, Albrecher et al., 2006), the methods employed by different researchers to tackle the random optimization problems are often problem-specific and may be quite intricate and involved as well.[1] This, as well as the diversity of mathematical tools employed by different authors in their studies, makes it impractical to present detailed derivations of all the findings discussed here. Instead, we tried to summarize shortly one's approach, where appropriate, while maintaining the rigorousness in formulations of the corresponding results.

[1] As W. Rhee acknowledges, one of her papers (Rhee, 1988) has been partially motivated by an attempt to classify the arguments employed by Frenk, van Houweninge, and Rinnooy Kan (1985) in their study of the convergence of the optimal value of random quadratic assignment problem.

2 Linear assignment problem

The linear assignment problem (LAP)[2] is one of the basic and fundamental models in operations research, computer science, and discrete mathematics. In its most familiar interpretation, it answers the question of finding an assignment of n workers to n jobs that has the lowest total cost, if the cost of assigning worker i to task j equals c_{ij}. Apart from the straightforward applications, such as personnel assignment problems, the LAP frequently arises as a part of other optimization problems, such as quadratic assignment problem, multidimensional assignment problem, traveling salesman problem, etc. Other applications of the LAP, including earth-satellite systems with TDMA protocol, and tracking objects in space are considered in Burkard (1985) and Brogan (1989); for a more comprehensive discussion of the applications of the LAP, refer to, e.g., Burkard and Çela (1999).

The mathematical programming formulation of the LAP has the form

$$\begin{aligned} L_n = \quad & \min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \\ & \text{s. t.} \sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, \ldots, n, \\ & \sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, \ldots, n, \end{aligned} \tag{1a}$$

where the decision variables x_{ij} can be taken as either binary: $x_{ij} \in \{0, 1\}$, or non-negative: $x_{ij} \geq 0$, leading to an integer programming (IP) or linear programming (LP) formulations of the LAP, respectively. In the graph-theoretical setting, the LAP corresponds to finding a minimum-cost perfect matching in an edge-weighed bipartite graph; another useful interpretation of the LAP presents it as finding such a permutation of rows and columns of the cost matrix $\mathbf{C} = (c_{ij})$ that minimizes the sum of the elements on the diagonal. The latter observation leads to the permutation formulation of the LAP:

$$L_n = \min_{\pi \in \Pi_n} \sum_{i=1}^{n} c_{i,\pi(i)}, \tag{1b}$$

where Π_n denotes the set of all permutations π of the set $\{1, \ldots, n\}$, i.e., the one-to-one mappings $\pi : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}$. The LAP is long known to be polynomially solvable: Kuhn (1955) published one of the first effective methods for solving LAP, the primal-dual Hungarian method; currently, the best worst-case complexity of sequential algorithms for the LAP is $\mathcal{O}(n^3)$ (see, for example, Burkard (2002) for details).

Studies of the properties of random LAP and, in particular, its expected optimal value, began as early as in the sixties (Kurtzberg, 1962; Donath, 1969), but the most powerful results have been obtained only in the last several years.

2.1 Bounds on the cost of optimal assignment

One of the first results concerning the behavior of random LAP was obtained by Walkup (1979), who established an upper bound for the expected cost of optimal assignment (1) assuming that the cost coefficients

[2]Many authors use a more precise term *linear sum assignment problem* (LSAP) to distinguish it from other problems where the total cost of assignment is linear, but not equal to the sum of the costs of individual assignments (matchings): e.g., linear bottleneck assignment problem, etc.

c_{ij} are iid uniform [0, 1] random variables:

$$\mathsf{E}[L_n] \leq 3. \tag{2}$$

A key role in D. Walkup's arguments played his result (Walkup, 1980), which was later utilized in studies of other random assignment problems, on the probability of existence of a perfect matching in a graph selected uniformly from the class $\mathcal{G}(n, d)$ of bipartite digraphs with n nodes in each class and outward degree d at each node. Namely, such a probability vanishes with n when $d = 1$, but for $d \geq 2$ it approaches unity as $n \to \infty$. Then, in order to apply this observation and circumvent possible dependence in selecting one of two cheapest edges forming a $\mathcal{G}(n, d)$ digraph, Walkup (1979) represents $c_{ij} = \min\{a_{ij}, b_{ij}\}$, where a_{ij} and b_{ij} are iid random variables with a distribution specially selected to make c_{ij} iid uniform on [0, 1]. This technique was later adopted also by other authors, see, e.g., Pferschy (1996).

D. Walkup's upper bound (2) for a random LAP with iid uniform [0, 1] costs was improved by Karp (1987): $\mathsf{E}[L_n] \leq 2$. Karp's arguments utilized LP duality and were subsequently[3] generalized by Dyer et al. (1986) to other instances of random LPs. It is of interest to note that the upper bounds due to Walkup (1979) and Karp (1987) have been derived in a non-constructive way; this is contrast to lower bounds on $\mathsf{E}[L_n]$ (see below), which are obtained by evaluating the expected cost of some feasible solution of the dual LP problem of (1a). The first constructive upper bound for $\mathsf{E}[L_n]$ was developed by Alvis and Lai (1988) using a heuristic that provided a solution of random LAP with expected cost of less than $6 + o(1)$. Karp, Rinnooy Kan, and Vohra (1994) constructively proved that $L_n \leq 3 + o(1)$ holds with probability $1 - o(1)$. A constructive upper bound of 1.94, which improves upon the results of Karp (1987) and Walkup (1979), has been furnished by Coppersmith and Sorkin (1999) via application of an assignment algorithm to large cost matrices whose entries are iid exponential with mean 1.

The first lower bound on the value of an optimal assignment in (1), again in assumption of iid uniform [0, 1] assignment costs, was developed by Lazarus (1993): $\mathsf{E}[L_n] \geq 1 + 1/e + \mathcal{O}(n^{-1+\varepsilon}) \approx 1.368$. This lower bound was derived using the dual LP formulation of (1a) and evaluating the dual objective after row and column reductions due to the Hungarian method. A dual heuristic proposed by Goemans and Kodialam (1993) allowed them to improve the dual solution of Lazarus (1993), and, consequently, increase the lower bound for the expected cost of minimum assignment to $\alpha - o(1)$, where

$$\alpha = 3 - e + \sum_{k=1}^{\infty} \frac{1}{kk!} \sum_{l=0}^{k-1} (k-l) \frac{(k/e)^l e^{-k/e}}{l!} > 1.441.$$

Olin (1992) has further improved this result by showing that $\mathsf{E}[L_n] \geq 1.51$.

A random LAP whose coefficients are not restricted to be exponential or uniform was considered by Frenk et al. (1987); the distribution F of the cost coefficients of the LAP was assumed to satisfy some mild assumptions on the behavior in the left-end point of the support. Combining the results of Walkup (1980) with an analysis of order statistics, they obtained the following asymptotic bounds on the expected minimum cost $\mathsf{E}[L_n]$ in the case when the distribution F is defined on $(-\infty, +\infty)$:

$$\left(1 - \frac{3}{2e^{1/2}}\right)^2 \leq \liminf_{n\to\infty} \frac{\mathsf{E}[L_n]}{nF^{-1}(1/n)} \leq \limsup_{n\to\infty} \frac{\mathsf{E}[L_n]}{nF^{-1}(1/n)} < \infty. \tag{3}$$

[3] The upper bound of 2 for the optimal cost of random LAP with uniform [0,1] costs was reported by R. Karp as early as in 1983 in his lecture at the NIHE Summer School on Combinatorial Optimization in Dublin.

When F has support $(0, \infty)$, the lower bound in (3) is replaced by 0. Under some stronger assumptions on the distribution F with support $(0, \infty)$, Olin (1992) has derived bounds for the (random) optimal cost L_n that hold almost surely (a.s.):

$$\frac{1+e^{-1}}{f(0+)} \leq \liminf_{n\to\infty} L_n \leq \limsup_{n\to\infty} L_n \leq \frac{4}{f(0+)} \quad \text{a.s.},$$

where f is the (continuous) density of F that is positive in the vicinity of 0.

Although the ideas underlying the derivation of the lower and upper bounds for the minimum cost of random LAP (1) are still of interest and value as evidenced by some of the most recent works (see, e.g., Alm and Sorkin, 2002, etc), the bounds themselves have been superseded by the recent advances in evaluating the exact expectations of the optimal value of LAP, both in the asymptotic ($n \to \infty$) and finite cases, which are discussed next.

2.2 Asymptotic and exact results for expected minimum cost assignment

Conjectures of Mézard and Parisi Perhaps, the most widely known result in this area is the conjecture by Mézard and Parisi (1985) that the expected optimal value of a random LAP with iid uniform [0, 1] or exponential with mean 1 cost coefficients satisfies

$$\lim_{n\to\infty} \mathsf{E}[L_n] = \frac{\pi^2}{6} \approx 1.645. \tag{4}$$

The existence of the limit in (4) was argued by the authors using a non-rigorous *replica method* (see, e.g., Mézard et al., 1987; Dotsenko, 2001). Experimental evidence in support of the Mézard-Parisi conjecture traces back to Donath (1969), who noticed that $\mathsf{E}[L_n]$ increases with n and approaches a limit close to 1.6. Further evidence in support of (4) was obtained by Olin (1992), who carried out numerical experiments with problem sizes of up to $n = 500$ and by Pardalos and Ramakrishnan (1993), where large-scale ($n \leq 10{,}000$) instances of the LAP with iid uniform [0, 1] and exponential costs have been solved to confirm (4).

A rigorous verification of the existence of a limit in (4) was done by Aldous (1992). By developing the concept of a "limit random object", an infinite tree with random edge costs that represents the limit at $n \to \infty$ of the random cost matrix $\mathbf{C} = (c_{ij})$, D. Aldous has demonstrated that the expected optimal cost $\mathsf{E}[L_n]$ of the random LAP converges to a limiting value, which is defined by the density of the distribution of c_{ij}'s at 0; however, a numerical value of this limit was not identified. Several years later, Aldous (2001) has completed the proof of the Mézard-Parisi conjecture (4) by constructing an optimal matching on the infinite tree and demonstrating that the limiting value of $\mathsf{E}[L_n]$ is indeed equal to $\pi^2/6$.

In the mean time, a number of new, stronger properties of random assignment problems have been conjectured in the literature. One of them is the conjecture due to Parisi (1998), which represents a restriction of (4) to random LAPs of finite size n:

$$\mathsf{E}[L_n] = \sum_{i=1}^{n} \frac{1}{i^2}, \tag{5}$$

where, as before, L_n is the optimal cost of a random LAP of size n with iid exponential(1) cost coefficients. Even stronger conjectures that generalize (5) have been formulated by Coppersmith and Sorkin (1999) and Buck, Chan, and Robbins (2002) with respect to the so-called *k-assignment problem*, a linear assignment problem whose cost matrix is not square.

Minimum k-assignment and Coppersmith-Sorkin and Buck-Chan-Robbins conjectures The k-assignment problem is formulated as follows: given an $m \times n$ matrix $\mathbf{C} = (c_{ij})$, for a fixed integer $k \leq \min\{m, n\}$ let $L_{m,n}(k)$ be the smallest possible sum of k elements of $\mathbf{C}$ such that no two elements share a row or a column. The value $L_{m,n}(k)$ can be expressed via LP:

$$\begin{aligned} L_{m,n}(k) \;=\; & \min_{x_{ij} \geq 0} \; \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \\ & \text{s. t.} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} = k, \\ & \qquad \sum_{j=1}^{n} x_{ij} \leq 1, \qquad i = 1, \ldots, m, \\ & \qquad \sum_{i=1}^{m} x_{ij} \leq 1, \qquad j = 1, \ldots, n. \end{aligned} \tag{6}$$

Then, Coppersmith and Sorkin (1999) have conjectured that if the elements c_{ij} of $\mathbf{C}$ are iid exponential with mean 1, the expected optimal value of (6) has the form

$$\mathsf{E}\big[L_{m,n}(k)\big] = \sum_{\substack{i,j \geq 0 \\ i+j<k}} \frac{1}{(m-i)(n-j)}, \tag{7}$$

and verified this equality for the cases $k = 1$, $k = 2$, $k = m = 3$, $k = m = n = 4$. Alm and Sorkin (2002) have validated (7) for $k \leq 4$, $k = m = 5$, and $k = m = n = 6$.

Buck, Chan, and Robbins (2002) further generalized the conjecture of Coppersmith and Sorkin (7) by considering random $m \times n$ assignment matrices whose elements c_{ij} were independent random variables distributed exponentially with parameter λ_{ij}: $\mathsf{P}[c_{ij} \leq t] = 1 - e^{-\lambda_{ij} t}$, where $\lambda_{ij} = \alpha_i \beta_j$. Given a random assignment matrix with the described structure, M. Buck, C. Chan, and D. Robbins have conjectured that the expected minimum k-assignment cost equals to

$$\mathsf{E}[L_{m,n}(k)] = \sum_{I,J} \binom{m+n-1-|I|-|J|}{k-1-|I|-|J|} \frac{(-1)^{k-1-|I|-|J|}}{\big(\sum_{i \notin I} \alpha_i\big)\big(\sum_{j \notin J} \beta_j\big)}, \tag{8}$$

where the summation is over sets $I \subset \{1, \ldots, m\}$ and $J \subset \{1, \ldots, n\}$ such that $|I| + |J| < k$.

Quite remarkably, conjectures (5), (7), and (8) have been proven within very short period of time. In 2003, proofs of both the Parisi (5) and Coppersmith–Sorkin (7) conjectures have been obtained independently by Linusson and Wästlund (2004) and Nair, Prabhakar, and Sharma (2005). The paper of Nair et al. (2005) actually contains two related proof methods, both of which involve analyses of matchings in a submatrix of $\mathbf{C}$ obtained by removing its i-th column. To establish equalities (5) and (7), the authors demonstrate that the negative differences between costs of the smallest and second-smallest, second- and third-smallest, and so on, matchings, are exponentially distributed random variables, which, depending on the type of matching used in one of these approaches, can be also independent.

The proof of S. Linusson and J. Wästlund relies on the following result, which constitutes yet another generalization of (7), and was conjectured in their earlier work (Linusson and Wästlund, 2000): if the entries of the $m \times n$ assignment matrix $\mathbf{C}$ are either zeros or independent exponential with mean 1 random variables, the corresponding expected cost of minimum k-assignment is given by

$$\mathsf{E}[L_{m,n}(k)] = \frac{1}{mn} \sum_{i,j} \frac{d_{ijk}}{\binom{m-1}{i}\binom{n-1}{j}}, \tag{9}$$

where d_{ijk} is an integer defined in terms of combinatorics of the set of zero elements of **C**. In particular, when **C** has no zeros, $d_{ijk} = \binom{m}{i}\binom{n}{j}$, yielding (7). Generalizing (9) to the case of matrix **C** comprised of either zeros or independent random variables distributed exponentially with parameters $\lambda_{ij} = \alpha_i \beta_j$, Wästlund (2005a) has validated the Buck-Chan-Robbins conjecture (8) as well.[4]

LAP with discrete random costs Traditionally, probabilistic analysis of optimization problems is conducted in assumption that the problem's data (in our case, the assignment costs) are random variables with continuous distributions. An immediate benefit of such an assumption is that all feasible solutions can be considered to have (almost surely) different costs. An interesting departure from this general setup is found in the paper by Parviainen (2004), where an LAP with discretely distributed random assignment costs is discussed. Making use of some of the results of Aldous (2001), R. Parviainen considers the following cases of random LAP (1):

(i) Each row of matrix $\mathbf{C} = (c_{ij})$ is an independent random permutation of $\{1, \ldots, n\}$, chosen uniformly from the set of all permutations.

(ii) Each element of **C** is an independent random number, chosen uniformly from $\{1, \ldots, n\}$.

(iii) **C** is a random permutation of $\{1, \ldots, n^2\}$ chosen uniformly.

(iv) Each element of **C** is an independent random number, chosen uniformly from $\{1, \ldots, n^2\}$.

Under such conditions, the objective of the LAP needs normalization; namely, it is normalized by n in the cases (i) and (ii), and by n^2 in the cases (iii) and (iv). In effect, the following problems are considered:

$$L_n^{(p)} = \min_{\pi \in \Pi_n} \frac{1}{n} \sum_{i=1}^{n} c_{i,\pi(i)}, \quad L_n^{(q)} = \min_{\pi \in \Pi_n} \frac{1}{n^2} \sum_{i=1}^{n} c_{i,\pi(i)}, \quad p = \text{i}, \text{ii}, \ q = \text{iii}, \text{iv}.$$

Then, Parviainen (2004) shows that the limiting expected optimal value $\mathsf{E}[L^{(\cdot)}] = \lim_{n\to\infty} \mathsf{E}[L_n^{(\cdot)}]$ of the random LAP with the described above structure of the cost matrix satisfies

$$\frac{\pi^2}{6} \le \mathsf{E}[L^{(\text{i})}] \le 2, \quad \frac{\pi^2}{6} + \frac{12}{24} \le \mathsf{E}[L^{(\text{ii})}] \le \frac{\pi^2}{6} + \frac{13}{24}, \quad \mathsf{E}[L^{(\text{iii})}] = \mathsf{E}[L^{(\text{iv})}] = \frac{\pi^2}{6}. \tag{10}$$

Relations (10) are obtained by utilizing a correspondence between random permutations and ranks of elements in an iid uniform cost matrix and then invoking the existing results for random LAPs with iid uniform costs (e.g., Aldous, 2001).

2.3 Properties of the optimal assignments and other results

The properties of random LAP have proved to be of value in studies of other optimization problems, and a number of interesting results have been obtained in this context. In particular, Karp (1979) showed that an optimal solution to LAP (1) can be transformed into a good solution of asymmetric traveling salesman problem (ATSP) using an $\mathcal{O}(n^3)$ *patching heuristic*; moreover, if the assignment costs c_{ij} are iid uniform

[4] The methods used in Linusson and Wästlund (2004) and Nair et al. (2005) to verify the Parisi and Coppersmith-Sorkin conjectures are rather intricate; a much simplified proof of both results was presented by J. Wästlund in a follow-up paper (Wästlund, 2005b).

[0, 1], the expected costs of optimal solutions of both problems have been found to be asymptotically equal (Karp, 1979; Karp and Steele, 1985; Dyer and Frieze, 1990). Working on the same problem, Frieze and Sorkin (2007) determined some interesting properties of optimal assignments in random LAPs. Namely, they have demonstrated that the maximum summand $c_{\max}(n)$ in the optimal assignment cost L_n of the LAP satisfies

$$\big(1 - o(1)\big)\frac{\ln n}{n} \leq c_{\max}(n) \leq C_1 \frac{\ln n}{n}$$

with probability $(1-o(1))$, which improved an earlier bound $c_{\max}(n) \leq \mathcal{O}(n^{-1}\ln^2 n)$ due to Karp and Steele (1985). Moreover, they constructed bounds for the difference Δ_1 between the cheapest and second-cheapest assignments:

$$\big(1 - o(1)\big)\frac{1}{n^2} \leq \mathsf{E}[\Delta_1] \leq C_2 \frac{\ln n}{n^2},$$

where C_1, C_2 are certain constants. Another property of optimal assignments, which proved to be instrumental in elucidating the behavior of random LAP, concerns the probability that the minimum-cost assignment uses a given row's ith smallest element. Aldous (2001) has shown that for iid exponential(1) cost coefficients, the limiting value of this probability approaches 2^{-i}, which confirmed an earlier conjecture made by Olin (1992). Linusson and Wästlund (2004) generalized this expression to the case of k-assignment problem whose matrix may contain zeros. As a special case of this more general result, it follows that for finite square matrices with iid exponential(1) entries the probability that a given row's smallest element belongs to an optimal assignment is $\frac{1}{2} + \frac{1}{n}$. Using a connection between the random LAP (1) and the shortest path problem in a complete graph with exponentially distributed edge weights, Wästlund (2006) showed that ideas contained in Nair et al. (2005) allowed for computing the exact expressions of the aforementioned probabilities. In particular, he presents an exact (although rather complicated) formula for the probability that the second smallest entry in a row belongs to the minimum cost assignment.

Along with the first moment $\mathsf{E}[L_n]$, the variance $\sigma^2(L_n)$ of the minimum assignment cost has been discussed in the literature. Alm and Sorkin (2002) studied this question in the setting of the k-assignment problem (6), but here we mention only their results for the case $k = m = n$. Namely, they developed an asymptotic lower bound for $\sigma^2(L_n)$, which complemented an earlier upper bound due to Talagrand (1995):

$$\left(1 + \frac{1}{e}\right)\frac{1}{n} + \mathcal{O}\left(\frac{1}{n}\sqrt{\frac{\ln n}{n}}\right) \leq \sigma^2(L_n) \leq \mathcal{O}\left(\frac{(\ln n)^4}{n(\ln\ln n)^2}\right),$$

and conjectured that $\sigma^2(L_n) = 2/n + \mathcal{O}\big(n^{-2}\ln n\big)$; later, same conjecture was made by Nair (2005). Wästlund (2005c) has derived an exact expression for $\sigma^2(L_n)$, much in the spirit of Mézard and Parisi's formula for $\mathsf{E}[L_n]$:

$$\sigma^2(L_n) = 5\sum_{i=1}^{n}\frac{1}{i^4} - 2\left(\sum_{i=1}^{n}\frac{1}{i^2}\right)^2 - \frac{4}{n+1}\sum_{i=1}^{n}\frac{1}{i^3}. \tag{11}$$

In the limiting case $n \to \infty$, due to the known identity $5\zeta(4) = 2\zeta^2(2) = \pi^4/18$ for the Riemann zeta function

$$\zeta(s) = \sum_{i=1}^{\infty}\frac{1}{i^s},$$

the first two terms in the right-hand side of (11) cancel out and an asymptotic expression for $\sigma^2(L_n)$ reads as

$$\sigma^2(L_n) = \frac{4\zeta(2) - 4\zeta(3)}{n} + \mathcal{O}\big(n^{-2}\big). \tag{12}$$

Observe that the numerator in (12) equals to $4\zeta(2) - 4\zeta(3) \approx 1.77$, which is less than that conjectured by Alm and Sorkin (2002) and Nair (2005).

Finally, we would like to mention that a number of algorithms, both exact and heuristic, have been developed in the literature for solving random LAPs. The motivation behind these developments was to exploit the special structure of the assignment matrix in order to devise solution methods that would outperform, in a certain sense, the worst-case bound of $\mathcal{O}(n^3)$ on the running time for the deterministic LAP. For instance, Karp (1980) has developed an exact algorithm with expected running time $\mathcal{O}(n^2 \ln n)$ for random LAPs with iid uniform [0, 1] coefficients. Heuristic algorithms for random LAPs have aimed at producing a guaranteed-quality solution in a faster running time, or economized on space (Alvis and Devroye, 1985; Alvis and Lai, 1988; Karp et al., 1994). For a detailed discussion of the algorithms for random LAPs, see a survey by Burkard and Çela (1999).

3 Multidimensional assignment problem

The multidimensional assignment problem (MAP) is a higher dimensional version of the (two-dimensional) LAP (1). If a classical textbook formulation of the LAP is to find an optimal assignment of n jobs to n workers, then, for example, the 3-dimensional assignment problem can be interpreted as finding an optimal assignment of n jobs to n workers on n machines, etc. In general, given d sets of n elements each, the objective is to find an optimal assignment of the elements of each set to n d-tuples, such that the total cost of the d-tuples is minimized. In the graph-theoretic setting, this corresponds to finding a minimum-cost perfect matching in an edge-weighted d-partite hypergraph.

The d-dimensional axial MAP with n elements per dimension can be formulated as an integer programming problem with n^d binary variables and nd constraints

$$\begin{aligned}
Z_{d,n} \;=\; & \min_{x_{i_1\cdots i_d}\in\{0,1\}} && \sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} c_{i_1\cdots i_d}\, x_{i_1\cdots i_d} && \\
& \text{s. t.} && \sum_{i_2=1}^{n}\cdots\sum_{i_d=1}^{n} x_{i_1\cdots i_d} = 1, && i_1 = 1,\ldots,n, \\
& && \sum_{i_1=1}^{n}\cdots\sum_{i_{k-1}=1}^{n}\;\sum_{i_{k+1}=1}^{n}\cdots\sum_{i_d=1}^{n} x_{i_1\cdots i_d} = 1, && i_k = 1,\ldots,n, \\
& && && k = 2,\ldots,d-1, \\
& && \sum_{i_1=1}^{n}\cdots\sum_{i_{d-1}=1}^{n} x_{i_1\cdots i_d} = 1, && i_d = 1,\ldots,n.
\end{aligned} \tag{13a}$$

Alternatively, the MAP admits a permutation-based formulation

$$Z_{d,n} \;=\; \min_{\pi_1,\ldots,\pi_{d-1}\in\Pi^n} \sum_{i=1}^{n} c_{i,\pi_1(i),\ldots,\pi_{d-1}(i)}. \tag{13b}$$

The MAP was first introduced by Pierskalla (1968), and since then has found numerous applications in the areas of data association (Andrijich and Caccetta, 2001), image recognition (Veenman et al., 1998), multisensor multitarget tracking (Murphey et al., 1998; Poore, 1994), tracking of elementary particles (Pusztaszeri et al., 1996), etc. For a discussion of the MAP and its applications see, for example, Burkard and Çela (1999), Burkard (2002), and references therein. In contrast to the LAP that is polynomially solvable, the MAP with $d \geq 3$ is generally NP-hard, a fact first established by Karp (1972) for the 3-dimensional assignment problem ($d = 3$).

3.1 Expected optimal value of random MAP

It is interesting to note that in the special case $n = 2$, $d \geq 3$, the probabilistic analysis of the MAP becomes trivial due to the fact that the costs of feasible solutions are iid random variables with distribution $F_2 = F * F$, where F is the distribution of the cost coefficients in (13) (see, e.g., Grundel et al., 2005b). This reduces the analysis of virtually any aspect of random MAP to the corresponding problem for order statistics, making it a useful "testing ground" for conjectures, etc. In the general case $d \geq 3, n \geq 3$, however, the stochastic independence of solution costs no longer holds, which restores the (anticipated) challenge of the problem.

Dyer, Frieze, and McDiarmid (1986) obtained an upper bound on the expected optimal cost of the LP relaxation of (13a) in the case of uniform iid [0, 1] costs. By generalizing the arguments of Karp (1987), they derived an upper bound on the expected value $\mathsf{E}[z^*]$ of a linear program with m deterministic constraints and a random objective with uniform [0, 1] iid costs:

$$\mathsf{E}[z^*] \leq m \max\{\hat{x}_i\},$$

where $\hat{x}_i$ are the elements of some feasible solution. By selecting a feasible solution of the LP relaxation of (13a) as $\hat{x}_{i_1 \cdots i_d} = 1/n^{d-1}$, Dyer et al. (1986) obtained that the expected optimal value of the LP relaxation of (13a) is bounded from above as

$$\mathsf{E}[Z_{d,n}^{\text{LP}}] \leq d/n^{d-2}.$$

Martin, Mézard, and Rivoire (2005) applied the *cavity method* of statistical physics to analyze random instances of the MAP. In the asymptotic case, they arrived at relations that generalize the corresponding conjecture of Mézard and Parisi (1985) for the random LAP, but for which no explicit form was provided.

The convergence and corresponding limiting values of the expected optimal cost of random MAP were established by Grundel, Oliveira, and Pardalos (2004) and Krokhmal, Grundel, and Pardalos (2007). The approach of Grundel et al. (2004) and Krokhmal et al. (2007) to studying the expected optimal value $\mathsf{E}[Z_{d,n}]$ of random MAP (13) with iid cost coefficients relied on the analysis of the so-called *index tree* (Pierskalla, 1968), where each of the $(n!)^{d-1}$ feasible solutions of the MAP was represented by a path from the root node to a leaf node. By uniformly choosing a subgraph that is expected to contain at least one feasible path, Grundel, Oliveira, and Pardalos (2004) have determined that the expected optimal value of MAP with iid uniform [0, 1] or exponential with mean 1 cost coefficients satisfies

$$\lim_{n\to\infty} \mathsf{E}[Z_{d,n}] = \lim_{d\to\infty} \mathsf{E}[Z_{d,n}] = 0,$$

whereas in the case of normally $N(0, 1)$ distributed cost coefficients, the expected optimal value of the MAP is unbounded from below: $\lim \mathsf{E}[Z_{d,n}] = -\infty$, with either n or d approaching infinity. These results have been generalized by Krokhmal, Grundel, and Pardalos (2007), who demonstrate that the expected optimal value of large-scale instances of a random MAP with iid coefficients drawn from a broad class of distributions F is determined by the location of the left-end point of the support of the distribution, $F^{-1}(0+) = \inf\{x \mid F(x) > 0\}$. Namely, for a random MAP (13) with $d \geq 3$, $n \geq 2$, and with cost coefficients that are iid random variables from an absolutely continuous distribution F with existing first moment that fulfills one of the following conditions:

(i) $F^{-1}(u) = F^{-1}(0+) + \mathcal{O}(u^{\beta})$, $u \to 0+$, $\beta > 0$,

(ii) $F^{-1}(u) \sim -\nu u^{-\beta_1}\big(\ln \frac{1}{u}\big)^{\beta_2}$, $u \to 0+$, $0 \leq \beta_1 < 1$, $\beta_2 \geq 0$, $\beta_1 + \beta_2 > 0$, $\nu > 0$,

the expected optimal value of the MAP satisfies

$$\lim \mathsf{E}[Z_{d,n}] = \lim\, nF^{-1}(0+),$$

where both limits are taken at either $n \to \infty$ or $d \to \infty$.

The technique employed in (Krokhmal et al., 2007) for establishing the limiting values for the expected optimal value of the MAP (13) also allowed the authors to develop tight asymptotical bounds for $\mathsf{E}[Z_{d,n}]$. Namely, if the distribution F of the cost coefficients of a random MAP with $d \geq 3$, $n \geq 2$ has a finite left endpoint a of the support, and the inverse F^{-1} of the distribution function admits the following asymptotic expansion in the vicinity of 0,

$$F^{-1}(u) \sim a + \sum_{s=1}^{\infty} a_s u^{s/\mu}, \quad u \to 0+, \quad \mu > 0, \tag{14}$$

then, for any integer $m \geq 1$, lower and upper bounds for the expected optimal cost $\mathsf{E}[Z_{d,n}]$ of the MAP can be asymptotically evaluated as

$$\begin{aligned} \mathsf{E}[Z_{d,n}] &\geq na + \sum_{s=1}^{m-1} a_s \frac{n\,\Gamma(\kappa+1)\,\Gamma\big(\frac{s}{\mu}+1\big)}{\Gamma\big(\kappa+\frac{s}{\mu}+1\big)} + \mathcal{O}\left(n\frac{\Gamma(\kappa+1)\,\Gamma\big(\frac{m}{\mu}+1\big)}{\Gamma\big(\kappa+\frac{m}{\mu}+1\big)}\right), \\ \mathsf{E}[Z_{d,n}] &\leq na + \sum_{s=1}^{m-1} a_s \frac{n\,\Gamma(\kappa+1)\,\Gamma\big(\frac{s}{\mu}+\alpha\big)}{\Gamma(\alpha)\,\Gamma\big(\kappa+\frac{s}{\mu}+1\big)} + \mathcal{O}\left(n\frac{\Gamma(\kappa+1)\,\Gamma\big(\frac{m}{\mu}+\alpha\big)}{\Gamma(\alpha)\,\Gamma\big(\kappa+\frac{m}{\mu}+1\big)}\right), \end{aligned} \tag{15}$$

where $\Gamma(z)$ is the Gamma function and

$$\kappa = n^{d-1} \text{ and } \alpha = \left\lceil \frac{n^{d-1}}{n!^{\frac{d-1}{n}}} \right\rceil.$$

The asymptotic expressions (15) hold for either $d \gg 1$ or $n \gg 1$; moreover, the distance between bounds (15) vanishes for both $d \to \infty$ or $n \to \infty$. In the case when the cost coefficients of the MAP (13) are distributed exponentially with mean 1 or uniformly on $[0, 1]$, expressions (15) ensure that the expected optimal value of the MAP converges to zero as

$$\mathsf{E}[Z_{d,n}] = \mathcal{O}\left(\frac{1}{n^{d-2}}\right), \quad n \to \infty.$$

This confirms an earlier conclusion of Grundel, Oliveira, Pardalos, and Pasiliao (2005b) who conducted numerical experiments to approximate the values of $\mathsf{E}[Z_{d,n}]$ for finite n and d.

Although the asymptotic behavior of the expected optimal cost of assignment problems in both cases of $d = 2$ (LAP) and $d \geq 3$ (MAP) depends on the properties of the costs distribution in the left-end point $F^{-1}(0+)$ of the support, in the $d = 2$ case it is determined by the value of the density of the distribution (if it exists and positive, see Aldous (2001)), whereas in the multidimensional case it depends only on the location of $F^{-1}(0+)$. In this respect, the asymptotic behavior of the MAP is similar to that of the linear bottleneck assignment problem (see section 5.1).

3.2 Landscape of random MAP

Since many of the exact and heuristic algorithms for combinatorial optimization problems rely, at least partly, on repetitive local searches in neighborhoods of feasible solutions, the number of local extrema in the problem's landscape may have a significant impact on the performance of these algorithms. An optimization problem that is relatively difficult to solve (even approximately) with local search-based algorithms is said to have a *rugged* landscape (see also section 4.2). A solution landscape may be considered rugged if the number of local minima is exponential in the problem's dimensions (Palmer, 1991).

Grundel et al. (2005a, 2007) studied the expected number of local minima in random MAPs. For any $p = 2, \ldots, n$, the p-exchange local neighborhood $\mathcal{N}_p$ of a feasible solution to the MAP may be defined as the set of solutions obtained from the given one by permuting exactly p elements in one of the dimensions $1, \ldots, d$. The 2-exchange neigborhood $\mathcal{N}_2$, which corresponds to switching two positions in one of the dimensions of a given solution, is one of the most widely used in practice for solving the MAP (see, e.g., Aiex et al. (2005); Balas and Saltzman (1991); Pasiliao (2003), and others). The size $|\mathcal{N}_p|$ of a p-exchange local neighborhood in a $d \geq 3$, $n \geq 2$ MAP equals to

$$|\mathcal{N}_p| = d\binom{n}{p}D(p), \quad \text{where} \quad D(k) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j!. \tag{16}$$

In the special case of $n = 2$, the expected number $\mathsf{E}[M]$ of local minima in a random MAP can be computed exactly for any continuous distribution of the assignment costs (Grundel et al., 2005a):

$$\mathsf{E}[M] = \frac{2^{d-1}}{d+1}. \tag{17}$$

Observe that in the case $n = 2$ the largest local neighborhood is the 2-exchange neighborhood $\mathcal{N}_2$, hence (17) determines the expected value of the total number of local minima in the problem, which turns out to be exponential in the number of dimensions d. In the case $n \geq 3$ Grundel et al. (2007) have constructed bounds for $\mathsf{E}[M_p]$ in the assumption of independent normally distributed assignment costs by bounding the covariance matrix of the solutions within a single neighborhood using Slepian's inequality (Slepian (1962), see also Tong (1990)). With respect to the 2- and 3-exchange neighborhoods, one has

$$\frac{(n!)^{d-1}}{\big[dD(p)+1\big]^{\binom{n}{p}}} \;\leq\; \mathsf{E}[M_p] \;\leq\; p(n!)^{d-1}\left[d\prod_{k=0}^{p-1}(n-k)+p\right]^{-1}, \quad p = 2, 3. \tag{18}$$

For p-exchange neighborhoods with $p > 3$, the lower bound stays the same, whereas the upper bound takes the form

$$\mathsf{E}[M_p] \;\leq\; (n!)^{d-1}\int_{-\infty}^{+\infty}\Big[\Phi\big(\sqrt{p-1}\,z\big)\Big]^{d\binom{n}{p}D(p)}\,\mathrm{d}\Phi(z), \tag{19}$$

where $\Phi(z)$ is the c.d.f. of the standard normal $N(0,1)$ distribution. Although bounds (18) have been computed in the assumption of iid normal $N(\mu,\sigma^2)$ cost coefficients, numerical studies in Grundel et al. (2007) demonstrate that these bounds hold for other distributions of cost coefficients, such as uniform [0, 1] or exponential with mean 1. Grundel et al. (2007) also observe that, indeed, the number of local minima does have a statistically significant adverse impact on the quality of solutions produced by heuristic algorithms that employ local neighborhood searches (random local search, GRASP, and simulated annealing).

4 Quadratic Assignment Problem

Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be two square $n \times n$ matrices. Then the quadratic assignment problem (QAP), as introduced by Koopmans and Beckmann (1957), reads as

$$\begin{aligned} Q_n \;=\; & \min_{x_{ij}\in\{0,1\}} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{ij} b_{kl} x_{ik} x_{jl} \\ & \text{s. t.} \quad \sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1,\ldots,n, \\ & \qquad\quad \sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1,\ldots,n. \end{aligned} \tag{20a}$$

Similarly to the linear assignment problem (1), the set of feasible solutions of QAP can be represented as the set of permutations of $\{1,\ldots,n\}$, which allows for (20a) to be rewritten in the permutation form:

$$Q_n(\pi_*) = \min_{\pi\in\Pi_n} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} b_{\pi(i)\pi(j)}, \tag{20b}$$

where $Q_n(\pi_*) = Q_n$ is the optimal value of the QAP, and $\pi_* \in \Pi_n$ is a permutation that delivers minimum to (20b).

The QAP (20) has been first formulated by Koopmans and Beckmann (1957) in the context of facility layout, with A being the matrix of distances between sites, and B representing the flow of goods. Since then, the QAP has found numerous applications in various areas of science and engineering, including archeology, chemistry, numerical and statistical analysis, etc.; for a comprehensive survey of the applications of the QAP, see, for example, Burkard (1984); Pardalos and Wolkowitz (1994); Çela (1998); Burkard et al. (1998); Burkard (2002); Pentico (2007); Loiola et al. (2007).

The QAP is known to be NP-hard and non-approximable (Sahni and Gonzales, 1976), which makes the properties of its large-scale instances all the more surprising (see next). An overview of the recent advances in solution algorithms for the QAP, both exact and heuristic, can be found in Burkard (2002); Anstreicher (2003); Loiola et al. (2007), and others.

4.1 Asymptotic behavior of optimal value and solutions of random QAP

The behavior of random QAP is, under quite general conditions, distinctly different from that of the LAP or MAP. One of the most well-known characteristics of random QAPs is that with increase of the problem size n, the difference between the cost of any feasible solution and that of an optimal solution becomes negligible. To formulate more rigorous statements, it is useful to consider, along with the "best" solution $Q_n(\pi_*)$ of the QAP (20), its "worst" solution:

$$Q_n(\pi^*) = \max_{\pi \in \Pi_n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\pi(i)\pi(j)}. \tag{21}$$

Burkard and Fincke (1982a) presented the first asymptotic results for the expected optimal value of random QAP. In their analysis, R. Burkard and U. Fincke consider the cases of a "planar" QAP, where, in accordance with the original Koopmans and Beckmann's (1957) formulation, the elements of matrix **A** represent distances between points on the plane, and a "general" random QAP, with the elements of matrices **A** and **B** being iid random variables. In the case of a "planar" random QAP, the elements of matrix **A** have been selected in the form $a_{ij} = \|X_i - X_j\|_q$, where the random variables X_i are iid uniform on $[0,1] \times [0,1]$; the elements of the flow matrix B have been assumed to be independently distributed on [0, 1] with $\mathsf{E}[b_{ij}] = E_n > b > 0$. Under these conditions Burkard and Fincke (1982a) established that the cost ratio of the best and worst solution converged to 1 in probability, although the rate of convergence was rather slow. Namely, for any $\alpha > 0$,

$$\lim_{n \to \infty} \mathsf{P} \left\{ \frac{Q_n(\pi^*) - Q_n(\pi_*)}{Q_n(\pi_*)} \leq \frac{3 + \alpha}{b n^{0.18}} \right\} = 1. \tag{22}$$

A similar result holds for the "general" random QAP, when a_{ij} and b_{kl} are independent random variables on [0, 1] such that $\mathsf{E}[a_{ij}] = E_n > a > 0$ and $\sum_{k=1}^{n} \sum_{l=1}^{n} \mathsf{E}[b_{kl}] \geq cn^2$ for some $c > 0$. In (Burkard and Fincke, 1982a) it was shown that in such a case, for any $\alpha > 0$,

$$\lim_{n \to \infty} \mathsf{P} \left\{ \frac{Q_n(\pi^*) - Q_n(\pi_*)}{Q_n(\pi_*)} \leq \frac{2 + \alpha}{acn^{0.225}} \right\} = 1. \tag{23}$$

Expressions (22) and (23) essentially say that for large enough instances of the QAP, practically any algorithm will produce a nearly optimal solution with high probability – a rather unexpected conclusion, given the computational complexity of the QAP!

Frenk, van Houweninge, and Rinnooy Kan (1985) strengthened this statement by showing that it holds not only in probability, but almost everywhere (a.e.). In addition, they derived an asymptotic value of the optimal cost Q_n of a random QAP. Following Burkard and Fincke (1982a), they considered a planar random QAP with $a_{ij} = \|X_i - X_j\|_q^k$ and X_i defined as before; the elements b_{ij} of the flow matrix **B** were assumed to be iid on $(0, +\infty)$ with positive finite first moment and such that $\mathsf{E}\left[e^{-\lambda b_{ij}}\right] < \infty$ for values of λ from some neighborhood of 0. Under these conditions Frenk et al. (1985) have shown that there exists $M > 0$ such that the following holds almost everywhere for both $\pi = \pi_*$ and $\pi = \pi^*$:

$$\limsup_{n \to \infty} \frac{n^{1/4}}{\ln n} \left| \frac{Q_n(\pi)}{n^2 \mathsf{E}[a_{ij}] \mathsf{E}[b_{ij}]} - 1 \right| \leq M \quad \text{a.e.} \tag{24}$$

The counterpart of (24) for the case of the general QAP has been established by Frenk et al. (1985) for series a_{ij}, b_{ij} that are mutually independent and identically distributed on $(0, +\infty)$, and further satisfy $\mathsf{E}[a_{ij} b_{ij}] > 0$ and $\mathsf{E}\left[e^{-\lambda a_{ij} b_{ij}}\right] < \infty$ in some neighborhood of 0, which then guarantees that, for $\pi = \pi_*$ and $\pi = \pi^*$,

$$\limsup_{n \to \infty} \sqrt{\frac{n}{\ln n}} \left| \frac{Q_n(\pi)}{n^2 \mathsf{E}[a_{ij} b_{ij}]} - 1 \right| \leq M \quad \text{a.e.} \tag{25}$$

Equations (24) and (25) maintain that costs of all feasible solutions of a planar (respectively, general) random QAP converge almost surely to the optimal cost. Moreover, the asymptotic value of the optimal cost of a planar random QAP of size n equals $n^2\mathsf{E}[a_{ij}]\mathsf{E}[b_{ij}]$, whereas in the case of a general QAP the corresponding asymptotic value is $n^2\mathsf{E}[a_{ij}b_{ij}]$.

Rhee (1988) used the toolkit of U-statistics (see, e.g., Serfling, 1980) to further sharpen the results of Burkard and Fincke (1982a) and Frenk et al. (1985) for the planar random QAP. Working in a more general setting, W. Rhee considered a_{ij} to be a U-sequence, i.e., a sequence of random variables measurable with respect to the σ-field generated by X_i and X_j, where X_i is a sequence of independent random variables, and, moreover, such that $a_{ii} = 0$ for all $i \le n$. Then, assuming that a_{ij} are bounded: $|a_{ij}| \le M$, and b_{ij} satisfy the conditions given by Frenk et al. (1985), she defined

$$Y = \sum_{i,j=1}^{n} a_{ij} b_{\pi(i),\pi(j)} - \mathsf{E}[b_{11}] \sum_{i,j=1}^{n} \mathsf{E}[a_{ij}],$$

and demonstrated that there exists K_1 such that for n large enough and any $u \ge 1$,

$$\mathsf{P}\Big\{ |Y| \ge u K_1 M n^{3/2} (\ln n)^{1/2} \Big\} \le n^{-u^2}. \tag{26}$$

In particular, (26) implies that the factor $n^{1/4}/\ln n$ in (24) can be replaced by $n^{1/2}/\ln n$, thus improving the convergence rate.

Chebyshev's inequality for random QAP Later, Rhee (1991) have proposed a greedy algorithm for random QAPs that yields $\mathsf{E}[Y]$ of the same order of magnitude as $\mathsf{E}[Y_*]$, where $Y_* = \min_\pi Y = Y(\pi_*)$, and developed bounds for $\mathsf{E}[Y_*]$. Namely, for mutually independent sequences a_{ij} and b_{ij} that are iid with existing moments of order α and β, respectively, where $\alpha + \beta \le 1/2$, there exists a number K_2 such that

$$\frac{1}{K_2} n^{3/2} (\ln n)^{1/2} \le \mathsf{E}[Y_*] \le K_2 n^{3/2} (\ln n)^{1/2}. \tag{27}$$

For a_{ij} and b_{ij} that are also bounded, Rhee (1991) established an analog of Chebyshev's inequality for random QAP:

$$\mathsf{P}\Big\{ \big|Y_* - \mathsf{E}[Y_*]\big| \ge t \Big\} \le 2 \exp\left(\frac{-t^2}{4n^2 \|\mathbf{A}\|_\infty^2 \|\mathbf{B}\|_\infty^2} \right), \tag{28}$$

where $\|\mathbf{A}\|_\infty$ is the so-called row sum norm of a matrix: $\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$.

Generalization to other combinatorial optimization problems It is important to note that ascertaining the remarkable properties of the large-scale random QAPs has led to discovery of an entire class of combinatorial optimization problems with similar behavior. In general, a member of this class can be described as a sequence of problems indexed by $n \in \mathbb{N}$, which have the form

$$Z_n(S_*) = \min_{S \in \mathcal{S}_n} \sum_{e \in S} c_n(e). \tag{29}$$

In (29) S is a feasible solution that is defined as a set of several elements $e \in E_n$ of a finite ground set E_n; $\mathcal{S}_n$ is the set of feasible solutions (a class of subsets of E_n); and $c_n(e)$ is the (non-negative) cost of an element $e \in E_n$, so that $\sum_{e \in S} c_n(e)$ is the cost of a feasible solution $S \in \mathcal{S}_n$. Next, it is assumed that all feasible solutions have the same cardinality $|S| = s_n$ for a given $n \in \mathbb{N}$, and the cardinalities of the feasible set $|\mathcal{S}_n|$ and feasible solution s_n both approach infinity as n increases. Burkard and Fincke (1985) have shown that if costs $c_n(e)$ are identically distributed on $[0, 1]$ random variables with existing first and second moments, and, moreover, $c_n(e)$ are independent for $e \in S$ in each fixed $S \in \mathcal{S}_n$, then the cost ratio $Z_n(S^*)/Z_n(S_*)$ converges to 1 in probability as $n \to \infty$,

$$\lim_{n\to\infty} \mathsf{P}\left\{ \frac{Z_n(S^*)}{Z_n(S_*)} \le 1 + \varepsilon \right\} = 1 \quad \text{for all} \quad \varepsilon > 0, \tag{30}$$

provided that the cardinalities $|\mathcal{S}_n|$ and s_n of the feasible set and feasible solutions satisfy the following asymptotical relation for some $\lambda > 0$

$$\lambda s_n - \ln |\mathcal{S}_n| \to \infty, \quad n \to \infty. \tag{31}$$

The above characterization of the class of problems that satisfy (30) was first obtained by Burkard and Fincke (1985) soon after publication of their initial insight into the properties of random QAPs, and since then had been employed in studies of other random assignment problems (e.g., biquadratic assignment problem, communication assignment problem – see below).

Relation (32) has been obtained independently by Szpankowski (1995), who further strengthened this result by showing that the optimal solution $Z_n(S_*)$ of (29) (and the cost of any other feasible solution for that matter) converged almost surely to s_n times the expected cost of an element e:

$$\lim_{n\to\infty} \frac{Z_n(S_*)}{s_n \mathsf{E}[c_n(e)]} = 1 \quad \text{a.s.} \tag{32}$$

Remarkably, W. Szpankowski's description of the class of problems that satisfy (32) is almost the same as that given by Burkard and Fincke (1985), but the key relation between the cardinalities of the feasible set and a feasible solution has a sharper form:

$$\ln |\mathcal{S}_n| = o(s_n), \quad n \to \infty. \tag{33}$$

His proof allows $c_n(e)$ to have a distribution with a support not necessarily bounded (but a finite third moment is required). To guarantee the almost sure convergence in (32), he imposes the *monotonicity* conditions: the objective of (29) must be non-increasing in n, and $|\mathcal{S}_n| \le |\mathcal{S}_{n+1}|$ must hold. W. Szpankowski observes that for his approach, the monotonicity conditions are essential: if they are dropped, then the limit (32) holds only in probability.

Albrecher, Burkard, and Çela (2006) revisited this problem using the formalism of statistical mechanics; in their paper they correct and generalize the approach of Bonomi and Lutton (1986) who applied the statistical mechanics methods to study random QAPs but whose proof, unfortunately, has been found to suffer from some technical inaccuracies. They show that, instead of being a product of purely probabilistic arguments in (Burkard and Fincke, 1985) and (Szpankowski, 1995), relation (32) can be obtained using a general framework based on the analogy between combinatorial optimization problems and thermodynamics, which also serves as the foundation for the simulated annealing heuristic. It turned out however, that for the statistical mechanics approach the boundedness of the support of distribution of $c_n(e)$ seemed to be important. Albrecher et al. (2006) have relaxed the monotonicity conditions of Szpankowski (1995), but instead introduced another cardinality condition:

$$\lim_{n\to\infty} \frac{s_n}{\ln n} = \infty \qquad \text{and} \qquad \forall e \in E_n : \left|\mathcal{S}_n(e)\right| = \eta_n, \tag{34}$$

where $\mathcal{S}_n(e)$ is the set of all feasible solutions that contain element e.

Similar results have been obtained in the context of bottleneck combinatorial optimization problems, which are discussed in section 5.

4.2 Landscape of the QAP

The properties of the landscape of the quadratic assignment problem have been investigated by Angel and Zissimopoulos (1998, 2001), Merz and Freisleben (2000), Stützle and Hoos (2000). Typically, analysis of the problem's *fitness landscape* (Weinberger, 1990; Stadler, 1996), which comprises the set of feasible solutions, their fitness values (e.g., costs) $f(\cdot)$, and a measure $d(\cdot,\cdot)$ of distance between solutions, can yield useful insights into the performance and tuning of various heuristic algorithms. An aggregate characteristic of a fitness landscape that reflects its suitability for search algorithms is its *ruggedness*: in general, a rugged landscape has more peaks (local extrema) and is more difficult for search algorithms to tackle; a landscape with fewer peaks is considered to be flat and more amenable to heuristic solution methods.

The metrics that have been applied in studies of the QAP include the *correlation length* of the landscape (Weinberger, 1990; Stadler, 1996)

$$\ell = -\frac{1}{\ln|r(1)|}, \quad r(s) = \frac{1}{\sigma^2(f)(m-s)} \sum_{k=1}^{m-s} \big(f(\pi_t) - \bar{f}\big)\big(f(\pi_{t+s}) - \bar{f}\big), \tag{35}$$

and the *fitness distance correlation coefficient* (FDC) (Jones and Forrest, 1995)

$$\varrho(f, d_{\text{opt}}) = \frac{\operatorname{cov}(f, d_{\text{opt}})}{\sigma(f)\,\sigma(d_{\text{opt}})}. \tag{36}$$

In (35), $r(s)$ is the correlation function of two points s steps apart in a random walk $f(\pi_t)$, $t = 1, \ldots, m$ of length m performed along the landscape; smaller values of the correlation length ℓ indicate more rugged landscapes. The FDC measures the correlation between the fitness value of a solution and its distance d_{opt} to a global optimum. If the fitness value generally improves (i.e., cost decreases) for solutions that are closer to a global optimum ($0 < \varrho \leq 1$), the search promises to be relatively easy.

Merz and Freisleben (2000) used the correlation length and FDC to characterize the suitability of the QAP to memetic algorithms. They considered several types of QAP instances: real-world and randomly generated instances from QAPLIB (Burkard et al., 1991), instances obtained by transformation of other combinatorial problem (TSP, graph bipartitioning problem), and generated instances with known optimal solutions (Li and Pardalos, 1992). It follows from their study that QAP instances have landscapes that are rather unstructured ($\varrho \approx 0$), with correlation length ℓ varying between $n/4.0$ and $n/2.86$. They have also considered randomly generated instances whose matrices obey the triangle inequality, a property which is often found in applications, at least for the distance matrix. This structure of matrices translates into better structured fitness landscapes ($\varrho \geq 0.2$ in most cases). For structured random instances, it has been observed that the ability of local search procedures to produce good solutions can be captured using the *flow dominance* characteristic (Vollmann and Buffa, 1966) of the flow matrix $\mathbf{B}$

$$\operatorname{dom}(\mathbf{B}) = 100\frac{\sigma(\mathbf{B})}{\overline{\mathbf{B}}}, \tag{37}$$

where $\sigma(\mathbf{B})$ and $\overline{\mathbf{B}}$ are the standard deviation and the average of the elements b_{ij} of the matrix $\mathbf{B}$, correspondingly. The flow dominance is high when a few elements constitute a large part of the total flow, and it is low when all elements of **B** have similar values. Merz and Freisleben (2000) conclude that local search-based heuristics may work better on instances of the QAP that are structured and have low flow dominance.

A study by Stützle and Hoos (2000) has also involved several types of QAP instances, including the randomly and uniformly generated ones, real-life instances of QAPLIB, and "real-life like" random instances whose matrices resemble those of real-life problems (Taillard, 1995). The authors observe that in the case of randomly generated instances, the FDC values are close to zero, indicating little relationship between solution quality and distance to the global minimum, whereas for real-life and real-life like instances, the FDC achieves values of about 0.2–0.3, on average.

Angel and Zissimopoulos (1998) developed an upper bound on the cost of a local minimum in a 2-exchange local neighborhood of the QAP, or, in other words, the cost of a solution that can be reached by a local search where each next solution is obtained by exchanging two positions in the permutation representation of the current solution:

$$C_{\text{loc}} \leq \varkappa_{\mathbf{AB}} \langle C \rangle.$$

Above, C_{loc} is the cost of a local minimum, $\langle C \rangle$ is the average of costs of all feasible solutions, and the coefficient $\varkappa_{\mathbf{AB}}$ is determined by the elements of matrices **A** and **B** and satisfies $\varkappa_{\mathbf{AB}} \leq 1/2$. In a subsequent paper, Angel and Zissimopoulos (1998) considered the *autocorrelation coefficient* as a measure of ruggedness of problem's landscape:

$$\xi = \frac{\langle (C(s) - C(t))^2 \rangle}{\langle (C(s) - C(t))^2 \rangle_{d(s,t)=1}}, \tag{38}$$

where $C(s)$ is the cost of solution s, and the numerator and denominator in (38) are equal to the average values of $C(s) - C(t)$ over all pairs $\{s, t\}$, and all pairs $\{s, t\}$ that belong to the same neighborhood, correspondingly. The autocorrelation coefficient (38) is similar to the correlation length (35), in that smaller values of ξ indicate a rugged landscape, one where the cost differences between adjacent solutions are rather significant, whereas larger values of ξ suggest a flat landscape, amenable to local search-based metaheuristics. For a QAP with matrices **A** and **B** that are symmetric with null diagonal, E. Angel and V. Zissimopoulos derived an expression for exact computation of the autocorrelation coefficient ξ in polynomial time, and obtained a sharp lower bound for ξ:

$$\xi \geq n/4, \tag{39}$$

and, based on their experimental results, conjectured an upper bound: $\xi \leq n/2$. Note that (39) implies that the landscape of the QAP is not highly rugged, comparing to other combinatorial problems, which explains the fact that metaheuristics based on local search are often able to find good solutions for the QAP, whereas it remains very difficult to find exact solutions. This relatively low ruggedness of the QAP landscape has been acknowledged also by other authors; in particular, Burkard (2002) observes that, while potentially beneficial for heuristic algorithms, it may have detrimental effect on exact solution methods. In fact, Dyer et al. (1986) have proven that a branch-and-bound algorithm for random QAP that assigns 1 index per step and employs a Gilmore-Lawler bound will have at least $n^{(1-o(1))n/4}$ branched nodes explored with probability $1 - o(1)$ as $n \to \infty$.

5 Bottleneck assignment problems

The bottleneck assignment problems are closely related to the assignment problems with sum objective; if the latter minimize the total cost of all assignments, then in the corresponding bottleneck problem the cost of the most expensive assignment is minimized. Interestingly, both the linear sum assignment problem and linear bottleneck assignment problem can be viewed as special cases of the general *algebraic assignment problem*

$$\min_{\pi} \left(c_{1\pi(1)} * c_{2\pi(2)} * \cdots * c_{n\pi(n)}\right),$$

defined on a totally ordered commutative semigroup with composition operation $*$ and a linear order relation (see Burkard, Hahn, and Zimmermann (1977) for details). Then, the LAP (1) is obtained by setting $a * b := a + b$, and the linear bottleneck assignment problem is obtained by choosing $a * b := \max\{a, b\}$. The general algebraic assignment problems can be solved in $\mathcal{O}(n^4)$ or, under additional assumptions, $\mathcal{O}(n^3)$ steps by only adding, subtracting, and comparing the cost coefficients (Burkard et al., 1977).

5.1 Linear bottleneck assignment problem

Continuing the terminology of job-machine assignment, the linear bottleneck assignment problem (LBAP) is concerned with such an assignment of tasks to parallel machines that the latest task completion time is minimized. In the graph-theoretic setting, the LBAP corresponds to finding a perfect matching in a weighted bipartite graph that minimizes the maximum weight of all matched edges. A mathematical programming formulation of the linear bottleneck assignment problem reads as

$$\begin{aligned} Z_n = \min_{x_{ij}\in\{0,1\}} \; & \max_{i,j} \; c_{ij}\, x_{ij} \\ \text{s. t.} \; & \sum_{i=1}^{n} x_{ij} = 1, \qquad j = 1, \ldots, n, \\ & \sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, \ldots, n, \end{aligned} \tag{40a}$$

with the corresponding permutation formulation having the form

$$Z_n = \min_{\pi\in\Pi_n} \max_{1\le i\le n} c_{i,\pi(i)}. \tag{40b}$$

One of the first applications of bottleneck assignment problems can be found in Fulkerson, Glickberg, and Gross (1953); applications to time slot assignment assignment problem are presented in Balas and Randweer (1983); bus drivers' rostering using LBAPs was introduced in Carraresi and Gallo (1984); for further discussion see Burkard and Çela (1999).

Pferschy (1996) considered the asymptotical behavior of random LBAPs (40) with iid cost coefficients drawn from a continuous distribution F. Using D. Walkup's (1980) bounds on the probability of a perfect matching in a graph chosen from the class $\mathcal{G}(n, d)$, and proceeding along the lines of Walkup (1979) and Olin (1992), Pferschy (1996) demonstrated that as the size n of the LBAP increases, the expected optimal value $\mathsf{E}[Z_n]$ of the LBAP approaches the left endpoint of the support of distribution F:

$$\lim_{n\to\infty} \mathsf{E}[Z_n] = \inf\{x \mid F(x) > 0\}, \tag{41}$$

provided that the support of F is bounded from above: $\sup\{x \mid F(x) < 1\} < +\infty$. In the special case when cost coefficients of the LBAP are iid uniform [0, 1] random variables, U. Pferschy has also derived asymptotic lower and upper bounds on the expected cost of optimal assignment:

$$\mathsf{E}[Z_n] < 1 - \left(\frac{2}{n(n+2)}\right)^{2/n} \frac{n}{n+2} + \frac{123}{610\,n}, \quad \text{for } n > 78, \tag{42}$$

$$\mathsf{E}[Z_n] \geq 1 - n\mathrm{B}\left(n, 1 + \frac{1}{n}\right) = \frac{\ln n + 0.5749\cdots}{n} + \mathcal{O}\left(\frac{\ln^2 n}{n^2}\right), \tag{43}$$

where $\mathrm{B}(\cdot\,,\cdot)$ is the Beta function. In addition, Pferschy (1996) presents an efficient algorithm for the LBAP with expected running time $\mathcal{O}(n^2)$ (i.e., linear in the size of the LBAP). The algorithm proceeds in three steps: first, a subgraph containing $2n \ln n$ of the cheapest edges is selected from the original bipartite graph; then, the LBAP is solved on this subgraph using the method of Gabow and Tarjan (1988); if, in an event of low probability, a perfect matching is not found in the subgraph, the original graph is used to complete the matching.

5.2 Quadratic Bottleneck Assignment Problem

The quadratic bottleneck assignment (QBAP) problem was first considered by Steinberg (1961) in application to backboard wiring problem, and had been later studied by Burkard (1974), Kellerer and Wirsching (1998), see also (Burkard, 2002). The QBAP can be written down by replacing the sum in the objective function of the QAP (20) with maximization:

$$Z_n = \min_{\pi \in \Pi_n} \max_{i,j} \; a_{ij} b_{\pi(i),\pi(j)}. \tag{44}$$

Like the quadratic sum assignment problem (20), the quadratic bottleneck assignment problem is NP-hard, and, furthermore, it exhibits a very similar asymptotic behavior in large-scale instances. Burkard and Fincke (1982b) investigated the asymptotic behavior of random instances of the QBAP in both cases of a "planar" QBAP, when the matrix (a_{ij}) represents distances: $a_{ij} = \|X_i - X_j\|_q$, and in the "general" case when a_{ij} are iid uniform [0, 1] random variables that are independent from another iid uniform [0, 1] sequence b_{ij}.

Denoting, as before, $Z_n = Z_n(\pi_*)$ as the best (optimal) solution of the BQAP (52), and $Z_n(\pi^*) = \max_{\pi \in \Pi_n} \max_{i,j} a_{ij} b_{\pi(i),\pi(j)}$ as its worst solution, we formulate here the results obtained by Burkard and Fincke (1982b), which say that the cost ratio of worst to best solutions of the "planar" BQAP (52) converges to unity in probability:

$$\lim_{n\to\infty} \mathsf{P}\left\{ \frac{Z_n(\pi^*) - Z_n(\pi_*)}{Z_n(\pi_*)} \leq \frac{3}{n^{0.2} - 3} \right\} = 1. \tag{45}$$

Observe that the convergence rate in (45) is better than the one originally obtained by the same authors for the planar QAP (22). Same is true for the case of "general" QBAP, which has been shown to satisfy

$$\lim_{n\to\infty} \mathsf{P}\left\{ \frac{Z_n(\pi^*) - Z_n(\pi_*)}{Z_n(\pi_*)} \leq \frac{1}{\sqrt{n/(2\ln n)} - 1} \right\} = 1. \tag{46}$$

Stronger results for random QBAP, including convergence with probability 1, can be obtained using the general asymptotic theory for bottleneck type combinatorial optimization problems that has been developed in the papers by Burkard and Fincke (1985) and Szpankowski (1995).

General bottleneck problems Along with the class of problems with sum objective (29) whose large scale instances have costs of all feasible solutions asymptotically equal to the optimal cost, both Burkard and Fincke (1985) and Szpankowski (1995) characterize a class of random bottleneck-type problems,

$$Z_n(S_*) = \min_{S \in \mathcal{S}_n} \max_{e \in S} c_n(e) \tag{47}$$

that exhibit essentially the same asymptotic behavior. Burkard and Fincke (1985) show that if $\mathcal{S}_n$, s_n, $c_n(e)$ satisfy the same conditions as for the problem with sum objective (29), with an additional requirement

$$\mathsf{P}\Big\{c_n(e) < 1 - \tfrac{\varepsilon}{2}\Big\} \le 1 - \lambda, \tag{48}$$

then the limit in probability (30) holds for the worst and best solutions costs $Z_n(S^*)$, $Z_n(S_*)$ of the bottleneck problem (47) as well. To establish the convergence with probability 1 for the optimal value of the bottleneck problem (47), Szpankowski (1995) modifies the conditions for costs $c_n(e)$ adopted from the sum objective problem (29) by requiring a strictly increasing continuous distribution F. In such a case, the optimal value $Z_n(S_*)$ of (47) converges to $F^{-1}(1)$ almost surely; in fact, a sharper result holds:

$$F^{-1}\big(1 - \ln|\mathcal{S}_n|/s_n\big) \le Z_n(S_*) \le F^{-1}\big(1 - 1/s_n\big) \quad \text{a.s.} \tag{49}$$

Albrecher (2005) reports a further improvement upon the results of Burkard and Fincke (1982b) and Szpankowski (1995) regarding random bottleneck problems. He considers problems (47) having the same structure as described above, with cost coefficients $c_n(e)$ that are identically distributed on $[0, M]$ and independent for $e \in S$ in each fixed $S \in \mathcal{S}_n$, and whose distribution further satisfies

$$\lim_{n\to\infty} \left(\mathsf{P}\Big\{c_n(e) \le \tfrac{M}{1+g(n)}\Big\}\right)^{s_n} |\mathcal{S}_n| = 0 \qquad \text{or} \qquad \sum_{n=1}^{\infty} \left(\mathsf{P}\Big\{c_n(e) \le \tfrac{M}{1+g(n)}\Big\}\right)^{s_n} |\mathcal{S}_n| < \infty, \tag{50}$$

for some function $g(n) > 0$. Then, H. Albrecher demonstrates that for the worst and the best costs of the bottleneck problem (47) one has

$$\lim_{n\to\infty} \mathsf{P}\left\{\frac{Z_n(S^*) - Z_n(S_*)}{Z_n(S_*)} \le g(n)\right\} = 1 \qquad \text{or} \qquad \frac{Z_n(S^*) - Z_n(S_*)}{Z_n(S_*)} \le g(n) \quad \text{a.s.}, \tag{51}$$

respectively. Under a suitable choice of $g(n)$ (namely, for $g(n)$ that vanish as $n \to \infty$), expressions (51) imply that the ratio $Z_n(S^*)/Z_n(S_*)$ converges to 1 in probability, or almost surely, correspondingly. Albrecher (2005) uses the above results to sharpen the convergence rates obtained by Burkard and Fincke (1982b) for the quadratic bottleneck assignment problem (44) with iid uniform [0, 1] costs:

$$\frac{Z_n(\pi^*)}{Z_n(\pi_*)} \le 1 + \sqrt{\frac{2\ln n}{n}}\left(1 - \frac{1}{2\ln n} - \frac{1}{8(\ln n)^2}\right) \quad \text{a.s.}$$

6 Other assignment problems

Here we present results of probabilistic analysis of combinatorial assignment problems whose probabilistic behavior has been investigated in the literature but which do not belong to the discussed above classes of assignment problems.

Biquadratic assignment problem The biquadratic assignment problem (BQAP) was introduced by Burkard, Çela, and Klinz (1994) in connection with VLSI synthesis (see also Burkard and Çela, 1995; Burkard, 2002):

$$\begin{aligned} Q_n^{(2)}(\pi_*) \;=\; & \min_{x_{ij}\in\{0,1\}} \sum_{i,j,k,l,m,p,s,t\in\{1,\dots,n\}} a_{ijkl} b_{mpst} x_{im} x_{jp} x_{ks} x_{lt} \\ & \text{s. t.} \quad \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1,\dots,n, \\ & \qquad\;\; \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,\dots,n. \end{aligned} \tag{52a}$$

The corresponding permutation form of the BQAP reads as

$$Q_n^{(2)}(\pi_*) = \min_{\pi\in\Pi_n} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{ijkl} b_{\pi(i),\pi(j),\pi(k),\pi(l)}, \tag{52b}$$

where a_{ijkl}, $b_{mpst} \in \mathbb{R}^4$ are arrays of n^4 elements each.

The authors have adopted the approach of Burkard and Fincke (1985) to determine the asymptotic behavior of large-scale BQAPs in the case when the entries of the 4-dimensional arrays **A** and **B** are random variables on [0, 1], with an important distinction that, possibly, some of the entries can be zeros. Namely, let I_n be a subset of $\{1,\dots,n\}^4$ of cardinality $|I_n|$ such that

$$\lim_{n\to\infty} \frac{|I_n|}{n \ln n} = \infty,$$

and a_{ijkl} are iid on [0, 1] for $(i, j, k, l) \in I_n$ and $a_{ijkl} = 0$ for $(i, j, k, l) \notin I_n$. If b_{mpst} are iid on [0, 1], pairwise independent from variables a, then

$$\lim_{n\to\infty} \mathsf{P}\left\{ \frac{Q_n^{(2)}(\pi_*) - Q_n^{(2)}(\pi^*)}{Q_n^{(2)}(\pi^*)} < \varepsilon \right\} = 1 \quad \text{for all } \varepsilon > 0. \tag{53}$$

As before, π_* and π^* in (53) denote the best and the worst solutions of the BQAP (52), correspondingly. Burkard and Fincke (1985) also show that relation (53) holds when some of the elements of b_{ijkl} are zeros as well: consider, in addition to the set I_n, a set $J_n \subseteq \{1,\dots,n\}^4$ such that b_{ijkl} are fixed at zero for $(i, j, k, l) \notin J_n$ and are iid on [0, 1] with positive variance $\sigma^2(b) > 0$ otherwise. Further, denote

$$\alpha_{ijkl} = \begin{cases} 1, & (i,j,k,l)\in I_n, \\ 0, & \text{otherwise}, \end{cases} \qquad \beta_{mpst} = \begin{cases} 1, & (m,p,s,t)\in J_n, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\gamma(n) = \min_{\pi\in\Pi_n} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \alpha_{ijkl} \beta_{\pi(i),\pi(j),\pi(k),\pi(l)}.$$

Then, relation (53) holds true provided that

$$\lim_{n\to\infty} \frac{\gamma(n)}{n \ln n} = \infty. \tag{54}$$

In addition, Burkard et al. (1994) furnish examples demonstrating that condition (54) and the finite-variance condition for the elements b are essential.

Communication assignment problem Burkard, Çela, and Woeginger (1995) introduced the following communication assignment problem: consider n communication centers $C_1, \dots, C_n$ that are to be embedded in, or assigned to the vertices of, an undirected network (V, E) with $|V| = n$ vertices. Each communication node C_i transmits packets of information to node C_j at a rate t_{ij}. When there is no edge between C_i and C_j, the information has to be routed through other communication nodes. Then, for a fixed assignment ϵ of the communication centers to the vertices of the network and a fixed routing pattern ρ, let $N_{\epsilon,\rho}(C_i)$ denote the traffic through the communication center C_i. The communication assignment problem (CAP) then consists in finding such an assignment and routing pattern that minimizes the maximum intermediate traffic:

$$\min_{\epsilon,\rho} \max_{1\le i\le n} N_{\epsilon,\rho}(C_i). \tag{55}$$

Burkard et al. (1995) have demonstrated that the CAP (55) is NP-hard for such networks (graphs) as paths, cycles, and stars with branch length 3. In a subsequent development, Burkard, Çela, and Dudás (1997) applied a simulated annealing heuristic for solving the CAP, and also presented asymptotical results for random CAPs. Namely, it turns out that CAPs on trees fall into the class of problems described in Burkard and Fincke (1985) and Szpankowski (1995). For a sequence of random CAPs on trees where the elements t_{ij} are iid random variables of $[0, M]$, the ratio between the worst and the best solutions of the CAP approaches unity in probability.

Summary

Analysis of random instances of optimization problems provides valuable insights into the problem's behavior, particularly in large-scale instances. One class of problems that has been studied intensively by the methods of probabilistic analysis is the assignment problems, which represent basic models in operations research and other adjacent areas of science and engineering, and are frequently encountered in various applications. In this paper we have discussed the state-of-the art results concerning the properties of random assignment problems, including the recent advances in the context of random linear assignment problem, multidimensional assignment problem, quadratic assignment problem, etc.

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