

# Risk-Averse Optimization and Resilient Network Flows

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## Abstract

We propose an approach to constructing metrics of network resilience, where resilience is understood as the network’s amenability to restoring its optimal or near-optimal operations subsequent to unforeseen (stochastic) disruptions of its topology or operational parameters, and illustrated it on the examples of the resilient maximum network flow problem and the resilient minimum cost network problem. Specifically, the network flows in these problems are designed for resilience against unpredictable losses of network carrying capacity, and the mechanism of attaining a degree of resilience is through preallocation of resources toward (at least partial) restoration of the capacities of the arcs. The obtained formulations of resilient network flow problems possess a number of useful properties, e.g., similarly to the standard network flow problems, the network flow is integral if the arc capacities, costs, etc., are integral. It is also shown that the proposed formulations of resilient network flow problems can be viewed as “network measures of risk”, similar in properties and behavior to convex measures of risk. Efficient decomposition algorithms have been proposed for both the resilient maximum network flow problem and the resilient minimum cost network flow problem, and a study of the network flow resilience as a function of network’s structure has been conducted on networks with three types of topology: that of uniform random graphs, scale-free graphs, and grid graphs.

**Keywords:** Resilient maximum network flow problem, resilient minimum cost network flow problem, stochastic network, convex risk measures, Benders decomposition

## 1 Introduction and Motivation

Network models and representations find applications in a variety of fields, including physics, biology, transportation and logistics, energy, telecommunications, social media, and so on. A simple network consists of a set of nodes, which are the principal components of the system, and a set of arcs that represent the relations and/or connections between the nodes. Industrial networked systems are most often designed with the purpose of transmitting material goods or information among their nodes. As such, a networked system may be assessed with respect to how efficient it is at performing its intended function, and a variety of network performance metrics can be used to describe the various aspects of network functionality.

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Typical characterizations of real-life networked and/or distributed systems that can be found in the literature include *robustness*, *vulnerability*, *resilience*, *redundancy*, *scalability*, *connectivity*, and so on [34]. Remarkably, despite the widespread usage of these and other terms in technical literature, the quantitative or mathematical models designed *specifically* for quantification of these performance metrics are rather scarce. Most commonly, various graph-theoretical constructs have been used as proxies for the aforementioned network performance metrics. For example, the *strength*  $\sigma(G)$  of a graph  $G$  [17, 32] can be regarded as a measure of network’s connectivity due to the Tutte-Nash-Williams theorem, according to which  $\lfloor \sigma(G) \rfloor$  is the *tree packing number* of  $G$ , i.e., the maximum number of edge-disjoint spanning trees that can be contained in  $G$ .

The goal of the present paper is to outline a general approach to construction of mathematically rigorous formulations for the aforementioned network performance metrics, and illustrate it by constructing a measure of network’s *resilience* with respect to network flow operations. We note that “resilience” and many other characterizations of network performance, including those mentioned above, have common (and implicit) context: namely, they refer to the *network’s ability to sustain operations under unforeseen future changes in operating conditions, network’s own topology, etc.* Within this common context, we can further identify the following two aspects:

- *uncertainty aspect*: the uncertainties in data and/or parameters must be explicitly taken into account to obtain a quantifiable metric of network’s performance;
- *decision aspect*: response to unforeseen changes necessarily involves decision making.

This interplay of uncertainty and decision making lends itself naturally to the framework of stochastic optimization. Of course, the stochastic optimization literature contains an extremely large volume of models and algorithms that deal with networks under uncertainties, some of which are discussed below. The present endeavor is focused on the construction of mathematical programming models of network resilience that can fit the description of a “performance metric”, i.e., are comparatively simple from a computational point of view yet methodologically meaningful. Furthermore, we base our approach on analogies with the methodologies and formalism of modern risk theory and risk-averse optimization.

Specifically, in this work we concern ourselves with the development of metrics of *resilience of network flow operations*, where “resilience of network flow” is understood as the network’s amenability to restoring its optimal or near-optimal flow operations subsequent to unforeseen (stochastic) disruptions. This interpretation of the concept of resilience, which focuses on the ability to recover after an adverse event, is adopted across a number of disciplines, including engineering, social, economic, and organizational domains [33, 35]. In the context of network science and engineering, definitions of resilience with an emphasis on the recovery aspect have been adopted by the National Infrastructure Advisory Council [50], Department of Homeland Security [56], National Institute of Standards and Technology [66], and others.

Network flows, and more generally, network optimization [1, 2, 23] is an integral part of operations research discipline that studies how network systems can be successfully designed and managed. Network optimization provides the basic models that is used in transportation, supply chain management, and logistic [7, 8] and so on. In particular, the classical maximum network flow problem consists in determining the maximum volume of flow that can be sent from a given source node  $s$  to another prescribed sink node  $t$ , while the minimum cost network flow problem seeks to satisfy the system of flow demand and supply values at the network’s nodes at the lowest possible cost. The importance of these two classical network flow problems is underlined by the fact that they contain a number of other well known network problems as special cases. Among such examples are the shortest path problem, the transportation problem, and the assignment problem. The shortest path problem consists in finding the minimum cost path between two particular nodes, which is a special case of the minimum cost network problem in the case when the arc capacities are equal to one and there is just one supply node and one demand node with a unit of demand. The transportation problem features  $n$  supply nodes and  $m$  demand nodes, while the arcs in the network have unbounded capacities. The assignment problem can be viewed as a transportation problem in which every node has demand or supply of one unit. The maximum network flow problem appears as a subproblem or a building block in many operations research and computer science models, and while the first (pseudo) polynomial algorithm for the maximum network flow problem was proposed by Ford and Fulkerson [27], the problem continues to attract the attention of researchers who have since then put forward a number of efficient

maximum network flow algorithms [1, 4, 30].

As it has been mentioned, the concept of resilience is considered in the present work with respect to unforeseen disruptions in network's operation, and the construction of network flow resilience metrics is pursued within the framework of risk-averse stochastic optimization based on the formalism of risk measures [42]. In general, exogenous uncertainties in network models, such as those affecting the overall topology, capacity, flow distribution and cost, and so on, have been studied extensively in the context of network problems, within the frameworks of stochastic optimization [5, 22, 25, 28, 41, 44, 55], robust optimization [5, 12, 63], and chance programming [20, 47, 60]. Specifically, resilience is a well-known and widely used term in the supply chain literature, where it is understood as the ability to resist, respond, and recover from disruptions in order to fulfill consumer demand and assure performance has been studied in the context of supply chain networks [36]. Context-specific strategies for creation of disruption-resilient structures, such as multiple sourcing rather than single sourcing [53, 59], backup suppliers [38, 39], keeping extra inventory [29], minimizing flow complexity [5], and so on have been suggested in the literature, see comprehensive reviews in [36]. While the majority of studies considered resilience in a risk-neutral setting, a number of papers in this area have used the Conditional Value-at-Risk (CVaR) measure [15, 18, 46, 62, 72, 74] to obtain risk-averse strategies. Transportation problems are another area where resilience of network operations have been considered, particularly in the context of routing of hazardous materials [21, 64]. Risk-averse CVaR-based approaches to evaluation of random flows on networks with stochastic arc failures or costs in the context of Stackelberg games [71] have been considered in [45, 52]. Application of the CVaR measure for construction of risk-averse network flows has been discussed in [14, 73].

With respect to the above, the authors view this work's contributions to the literature as follows:

- (i) We propose the resilient maximum network flow and resilient minimum cost network flow models that quantify the network's amenability to restoring the optimal network flow following unforeseen stochastic losses to network's throughput capacities. The developed models are sufficiently simple to serve as "metrics" of resilience and possess a number of useful and meaningful methodological properties. The models are generally nonlinear and employ the concept of utility function for quantification of the effects of stochastic disruptions on the network performance. The rationale behind formulation of the proposed network flow resilience models follows the ideas used for construction of a class of statistical functionals known as "certainty equivalent measures of risk."
- (ii) It is demonstrated that the developed models of resilient network flows possess properties analogous to those of a class of *convex measures of risk*, and as such can be considered as "network measures of risk", which quantify the risk exposure of a network to a set of stochastic factors with nontrivial interactions.
- (iii) Efficient cutting plane (Benders decomposition) methods are proposed for the proposed mathematical programming models of network flow resilience that offer substantial computational advantage over state-of-the-art off-the-shelf solvers.
- (iv) Lastly, we conduct a numerical study elucidating the effects of network topology on the resilience of maximum network flow.

The remainder of the paper is organized as follows. Section 2 provides the requisite background on the concepts of measures of risk that serve as a motivation for development of the models of network resilience for maximum network flow and minimum cost network flow in sections that are introduced and analyzed in the rest of this section. Section 3 presents several decomposition algorithms for the constructed mathematical programming formulations of resilient network flow models. Section 4 contains numerical studies of the computational efficiency of developed algorithms as well as an investigation of the influence of a network's structure, or topology, on its resilience with respect to stochastic disruptions of its throughput capacity. Finally, conclusions and acknowledgements are presented in Sections 5 and 6, respectively.

## 2 A Risk-Based Approach to Constructing Resilience Metrics

In this section we delineate the proposed approach to constructing data-driven metrics of network performance that is based on risk-averse stochastic optimization, and in particular, uses analogies with construction of certain families of statistical functionals known as *measures of risk*, such as the *optimized certainty equivalents* [10] and *certainty equivalent measures of risk* [69].

### 2.1 Certainty Equivalent Measures of Risk

The concept of a *measure of risk* as a quantification of the adverse realizations of uncertainties with respect to the outcome of a decision making process traces back to the seminal work of Markowitz [48]. His idea of bi-criteria, mean-variance optimization in portfolio selection has since then flourished into the *risk-reward paradigm* of decision science, which maintains that every decision under uncertainties can be viewed in terms of a tradeoff between the risks and the rewards it incurs. The risk-reward paradigm provides a fertile modeling ground for a vast array of applications, and represents an alternative to another pillar of decision making science, the utility theory. The utility theory, originated by D. Bernoulli [11], has received its current formal axiomatic foundation in the work of von Neumann and Morgenstern [70], where exact conditions (“axioms”) on the *preferences* of a decision maker were formulated that guaranteed the existence of a utility function which equivalently described these preferences. The rigorous mathematical framework of the utility theory is one of its appealing features; a similar axiomatic approach to the risk-reward paradigm in the context of quantification of risk-averse preferences was pioneered by Artzner et al. [3] with the introduction of the *coherent measures of risk*, and was later extended to *convex measures of risk* [26], *deviation measures* [58], *error measures* [68], etc. In this work, we are primarily concerned with convex measures of risk, which include coherent measures of risk as a special case.

Given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , a measure of risk is a mapping  $\rho : \mathcal{X} \mapsto \mathbb{R}$ , where  $\mathcal{X}$  is a space of  $\mathcal{F}$ -measurable functions  $X : \Omega \mapsto \mathbb{R}$ , which further satisfies a number of specific properties, or axioms. Depending on whether the random element  $X = X(\omega)$ ,  $\omega \in \Omega$ , represents wealth or gain, or a cost or a loss (in other words, whether the larger or smaller realizations of  $X(\omega)$  are preferable), the form of some of these axioms will change. In this work, we assume that  $X(\omega)$  constitutes a loss or a cost incurred due to a random event  $\omega \in \Omega$ , whereby the axioms of convex measures of risk take the form

- (C1) *monotonicity*:  $\rho(X_1) \leq \rho(X_2)$  whenever  $X_1 \leq X_2$ ,  $X_{1,2} \in \mathcal{X}$ ;
- (C2) *convexity*:  $\rho(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda\rho(X_1) + (1 - \lambda)\rho(X_2)$  for any  $\lambda \in [0, 1]$  and all  $X_1, X_2 \in \mathcal{X}$ ;
- (C3) *translation invariance*:  $\rho(X + a) = \rho(X) + a$  for any  $a \in \mathbb{R}$ ,  $X \in \mathcal{X}$ .

The monotonicity axiom provides that lower costs or losses imply lower risks. The convexity axiom is of fundamental importance from both mathematical and methodological points of view, for it enables efficient optimization and control of risks, as well as reduction of risk via diversification, a staple strategy in risk management. In the context of the latter, (C2) guarantees that it is impossible to create a combination of stochastic factors  $X_1, X_2$  such that its risk exceeds the corresponding individual risks, i.e., the inequality

$$\rho(\lambda X_1 + (1 - \lambda)X_2) > \max\{\rho(X_1), \rho(X_2)\}$$

is impossible if  $\rho$  is convex, but may become satisfiable if the convexity requirement of risk measure  $\rho$  is relaxed. The translational invariance property (C3) relates to the financial setting that was considered in the original work [3], where it guaranteed that adding a certain amount of risk-free asset to a portfolio can eliminate the portfolio’s risk. More generally, the translation invariance axiom postulates that adding a constant amount to a stochastic loss factor changes the risk of that factor by the same amount. Coherent measures of risk constitute a special case of convex risk measures that further are positively homogeneous,

- (C4) *positive homogeneity*:  $\rho(\lambda X) = \lambda\rho(X)$  for any  $\lambda > 0$ ,  $X \in \mathcal{X}$ ,

or, in other words, that guarantee that the risk scales linearly with losses.

As it was stated in the introduction, in this work we pursue the goal of constructing metrics of network resilience based on the analogies with representations for a certain class of risk measures. Specifically, we consider the *infimal convolution* representation for convex and coherent measures of risk proposed in [69]:

$$\rho(X) = \min_{\eta \in \mathbb{R}} \eta + \phi(X - \eta), \quad (1)$$

where  $\phi : \mathcal{X} \mapsto \mathbb{R}$  is a functional satisfying (C1)–(C2) (additionally, (C4) in the case of coherent measures of risk) along with some other technical conditions.

The insight that motivates the present endeavor is that the risk measure  $\rho(X)$  in representation (1) can be viewed as a measure of *resilience* to stochastic loss  $X$ , particularly through the lens of resource allocation needed to recover from the loss  $X$ . Namely, the value of  $\eta$  in (1) can be regarded as the cost of resources preallocated to offset the loss  $X$ , and the term  $\phi(X - \eta)$  can be viewed as the cost of recovering from loss  $X$ , given the preallocation  $\eta$ .

This intuitive interpretation admits a definite explication in the case of *certainty equivalent measures of risk* [69], which constitute a special case of (1):

$$\rho(X) = \min_{\eta \in \mathbb{R}} \eta + (1 - \alpha)^{-1} v^{-1} \mathbb{E} v(X - \eta)_+, \quad \alpha \in [0, 1). \quad (2)$$

Since the rationale behind expression (2) will be used in development of resilient network flows models presented in the sequel, we discuss (2) in more detail. Let  $X = X(\omega)$  be the amount of a cost or a loss that may occur sometime in the future, and suppose that right now it is possible to preallocate a fixed amount  $\eta$  worth of resources to be used for offsetting the future cost  $X$ . Then, at the time when the actual realization of  $X$  is observed, the decision maker will have to deal with the amount of losses not covered by  $\eta$ , namely the amount  $(X - \eta)_+ = \max\{0, X - \eta\}$ . To quantify the uncertain future cost  $(X - \eta)_+$  deterministically, one may resort to the concept of *certainty equivalents*, well known in utility theory [70]. Namely, a rational decision maker with utility function  $u : \mathbb{R} \mapsto \mathbb{R}$  will be indifferent between a random wealth or reward amount  $Y(\omega)$  and the corresponding deterministic, or certainty equivalent amount  $\text{CE}(Y) = u^{-1}(\mathbb{E}[u(Y)])$ .

In this work we follow the convention that the random element  $X$  represents costs or losses, whose lower values are preferred, and thus employ the *deutility function*  $v(t) = -u(-t)$ , or utility function adopted to losses. In other words, while the utility function  $u(Y)$  quantifies the decision maker's "satisfaction" with wealth amount  $Y$ , the deutility function  $v(X)$  quantifies her "dissatisfaction" with loss  $X = -Y$ . Consequently, given a deutility function  $v(t)$ , the decision maker's certainty equivalent of the random cost  $(X - \eta)_+$  is equal to

$$v^{-1} \mathbb{E} v(X - \eta)_+ \equiv v^{-1} (\mathbb{E}[v((X - \eta)_+)]).$$

Then, the total deterministic cost associated with a future random loss  $X$  amounts to

$$\eta + (1 - \alpha)^{-1} v^{-1} \mathbb{E} v(X - \eta)_+,$$

where a "safety" factor  $(1 - \alpha)^{-1} \geq 1$ ,  $\alpha \in [0, 1)$ , is added. Since the choice of the preallocation amount  $\eta$  is at the discretion of the decision maker, the risk  $\rho(X)$  of the uncertain future cost/loss  $X$  can be associated with the minimum deterministic cost as given above, leading to expression (2). The risk measures of the form (2) were introduced in [69]; similar arguments were proposed in [10] in the discussion of *optimized certainty equivalents*.

The conditions under which the risk measure in (2) satisfies the axioms for convex measures of risk (C1)–(C3) were discussed in [69]. In particular, due to the fact that loss  $(X - \eta)_+$  is nonnegative, it suffices to consider the so-called one-sided deutility functions, which are also convex and nondecreasing:

$$v(t) = v(t_+) = \begin{cases} v(t) > 0, & t > 0, \\ 0, & t \leq 0. \end{cases} \quad (3)$$

Conditions that guarantee the convexity of risk measure  $\rho$  in (2) can be obtained from the sufficient conditions for convexity of certainty equivalents given in [10, 49]. Namely, if function  $v \in C^3(\mathbb{R})$  is strictly convex and  $v'/v''$  is convex, then the certainty equivalent  $v^{-1} \mathbb{E} v$  is also convex.

In particular, the choice of a linear deutility function  $v(t) = t_+$  yields the celebrated Conditional Value-at-Risk (CVaR) measure [57],

$$\text{CVaR}_\alpha(X) = \min_{\eta \in \mathbb{R}} \eta + (1 - \alpha)^{-1} \mathbb{E}(X - \eta)_+, \quad \alpha \in [0, 1), \quad (4)$$

whereas selecting a power deutility function  $v(t) = (t_+)^p$  with  $p \geq 1$  leads to the Higher-Moment Coherent Measures of Risk (HMCR) [43]

$$\text{HMCR}_{p,\alpha}(X) = \min_{\eta \in \mathbb{R}} \eta + (1 - \alpha)^{-1} \|(X - \eta)_+\|_p, \quad p \in [1, \infty], \alpha \in [0, 1), \quad (5)$$

and an exponential deutility function  $v(t) = \lambda^{t_+} - 1$  for  $\lambda > 1$  results in the family of Log-Exponential Convex Measures of Risk (LogExpCR) [69]:

$$\text{LogExpCR}_{\lambda,\alpha}(X) = \min_{\eta \in \mathbb{R}} \eta + (1 - \alpha)^{-1} \log_\lambda \mathbb{E} \lambda^{(X - \eta)_+}, \quad \lambda > 1, \alpha \in [0, 1). \quad (6)$$

The successes of the formal axiomatic approach to development of risk averse decision making have manifested themselves in the development of a number of classes of axiomatically defined statistical functionals that quantify various aspects of decision making under uncertainties, see comprehensive reviews in [42, 68], and applications ranging from finance to healthcare, energy, etc. [19, 37, 40, 65].

Returning to the notion of interpreting expression (1) as a measure of resilience to loss  $X$ , we observe that the outlined above interpretation of its special case (2) carries the familiar “action-recourse,” or “action-reaction” connotation, well known in two-stage stochastic optimization [13], and which lends itself naturally to the context of *resilience*, particularly when a system’s resilience is evaluated in terms of the resource (pre)allocation needed to restore the system’s functionality in the aftermath of a disruptive event. Next, we consider two network flow models to flesh out an approach to constructing metrics of resilience that is based on the analogy with the described interpretation of (2) and demonstrate that such an approach yields metrics that possess properties analogous to those of convex measures of risk.

## 2.2 Resilient Maximum Network Flow Model

In this section, we illustrate the proposed approach to construction of resilience metrics of network operations by developing a model of resilient maximum network flow. The standard deterministic maximum network flow problem is usually formulated as follows. Let  $G = (V, E)$  be a directed graph, where  $V$  is the set of nodes (vertices) and  $E$  is the set of arcs (edges) as ordered pairs  $(i, j)$  for distinct nodes  $i, j \in V$ . For each arc  $(i, j) \in E$ , let  $u_{ij}$  denote the maximum capacity of that arc, and  $x_{ij}$  denote the amount of flow along this arc. Let  $A^+ \in \mathbb{R}^{|V| \times |E|}$  be the node-arc incidence matrix of the digraph  $G$ :

$$a_{ij} = \begin{cases} +1, & \text{if arc } j \text{ leaves node } i, \\ -1, & \text{if arc } j \text{ enters node } i, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the maximum flow problem can be written as

$$\begin{aligned} \max \quad & \mathbf{a}_s^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{0} \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{u}, \end{aligned} \quad (7)$$

where  $\mathbf{x} \in \mathbb{R}^{|E|}$  is the vector of flows along the arcs in  $E$ , vector  $\mathbf{a}_s^\top$  is the row of matrix  $A^+$  corresponding to source node  $s$ , and matrix  $A$  is obtained from  $A^+$  by removing the rows corresponding to the source node  $s$  and sink node  $t$ .

Now, assume that operational or topological disruptions within the network manifest themselves through stochastic reductions of the arc capacities:

$$\mathbf{0} \leq \mathbf{x} \leq \mathbf{u} - \mathbf{X}(\omega), \quad (8)$$

where  $\mathbf{X}(\omega) = (X_{ij}(\omega))_{(i,j) \in E}$  is the vector of “capacity losses” such that

$$\mathbf{0} \leq \mathbf{X}(\omega) \leq \mathbf{u}, \quad \omega \in \Omega, \quad (9)$$

and whose joint distribution is known. Regarding the set  $\Omega$ , we generally assume that it is a bounded and closed, i.e., a compact set in some Euclidean space. This assumption also ensures integrability of the bounded vector  $\mathbf{X}$ .

Note that the upper bound on the network flow in expression (8) is ill-defined at the moment since it does not specify which  $\omega \in \Omega$  it is valid for. A well-defined description of constraints in (8) and, ultimately, a meaningful formulation of the maximum network flow problem with randomized flow bounds can be obtained in a number of different ways. Perhaps, the most straightforward and rather uninteresting solution is to require the upper bound in (8) to hold “on average”, or in expectation, which immediately yields the standard deterministic maximum network flow formulation (7) with the flow bounds  $\mathbf{0} \leq \mathbf{x} \leq \mathbf{u} - \mathbb{E}[\mathbf{X}(\omega)]$ . Also, under certain conditions, it could be of value to consider maximization of the expected flow by allowing the network flow to respond stochastically to the stochastic capacity losses,  $\mathbf{x} = \mathbf{x}(\omega)$ , whereby the problem could be written as

$$\begin{aligned} \max \quad & \mathbb{E}[\mathbf{a}_s^\top \mathbf{x}(\omega)] \\ \text{s.t.} \quad & A\mathbf{x}(\omega) = \mathbf{0}, \quad \omega \in \Omega \\ & \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - \mathbf{X}(\omega), \quad \omega \in \Omega. \end{aligned}$$

The above formulation is known in the stochastic optimization literature as the “wait-and-see” approach to decision under uncertainties and can be reduced to solving deterministic *scenario-based* maximum flow problems of the form

$$\left. \begin{aligned} \max \quad & \mathbf{a}_s^\top \mathbf{x}(\omega) \\ \text{s.t.} \quad & A\mathbf{x}(\omega) = \mathbf{0} \\ & \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - \mathbf{X}(\omega) \end{aligned} \right\}, \quad \omega \in \Omega \quad (10)$$

for each  $\omega \in \Omega$ . We will return to formulation (10) later in this section.

Among the approaches that offer more significant interest and challenge, robust optimization as well as chance-constrained programming can be employed, requiring that the upper bound in (8) holds with a prescribed probability  $\beta$ , i.e.,  $\mathbb{P}(\mathbf{x} \leq \mathbf{u} - \mathbf{X}(\omega)) \geq \beta$ , where  $\beta = 1$  or  $\beta \in (0, 1)$ , respectively. In both these cases, an optimal network flow solution of the corresponding optimization problem could be characterized by its “robustness” or “reliability” with respect to the randomized losses of capacity, in the sense of how likely the network be able to sustain the flow (without changing it) under the conditions of reduced network capacities.

The approach proposed and pursued in this work aims at quantifying the network flow’s “resilience,” generally understood as an ability to restore, perhaps partially, the flow in the network, subsequent to the flow’s reduction due to some unforeseen reductions in the network’s throughput capacities. Noting that certain expenditures or investments are typically required in order to restore the functionality of a compromised system, let us suppose that it is possible to preallocate in advance  $\boldsymbol{\eta} \in \mathbb{R}_+^{|E|}$  worth of resources to be used towards offsetting the observed realizations of the stochastic capacity losses  $\mathbf{X}(\omega)$ , thereby aiding in recovery of the network flow after the disruption caused by the capacity reductions. Upon application of the preallocated resources to negate the observed capacity losses, the corresponding realization  $\mathbf{x}(\omega)$  of the flow in the network has to satisfy the constraints

$$\mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\omega) - \boldsymbol{\eta})_+, \quad \omega \in \Omega, \quad (11)$$

as well as the flow balance equations  $A\mathbf{x}(\omega) = \mathbf{0}$ ,  $\omega \in \Omega$ . Then, the *resilient maximum network flow problem* can

be formulated as

$$Z_\alpha^*(\mathbf{X}) = \max - (1 - \alpha) \mathbf{1}^\top \boldsymbol{\eta} + u^{-1} \mathbb{E} u(\mathbf{a}_s^\top \mathbf{x}(\omega)) \quad (12a)$$

$$\text{s.t. } A\mathbf{x}(\omega) = \mathbf{0}, \quad \omega \in \Omega \quad (12b)$$

$$\mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\omega) - \boldsymbol{\eta})_+, \quad \omega \in \Omega \quad (12c)$$

$$\boldsymbol{\eta} \geq \mathbf{0}, \quad (12d)$$

where  $\mathbf{1}$  is the vector of ones,  $\mathbf{1} = (1, \dots, 1)^\top$ . In the objective function (12), the total amount  $\mathbf{1}^\top \boldsymbol{\eta}$  of capacity preallocations across the network includes the “discount”  $\alpha \in [0, 1]$  associated with a preemptive action in the face of possible future losses, which can also be viewed as an incentive to take a preemptive action. Given a utility function  $u(\cdot)$ , the certainty equivalent term  $u^{-1} \mathbb{E} u(\cdot)$  in (12a) represents the current deterministic value of a future uncertain network flow, and the term  $(1 - \alpha) \mathbf{1}^\top \boldsymbol{\eta}$  is taken with a negative sign since preallocation of the capacity resources prevents the utilization of these resources at the current moment.

To maintain the computational tractability of model (12), we assume that the utility function  $u$  in (12) is such that the certainty equivalent  $u^{-1} \mathbb{E} u(\cdot)$  is increasing and concave. It is known that for functions  $u \in C^{(3)}$  that are strictly increasing and strictly concave, the certainty equivalent  $u^{-1} \mathbb{E} u(\cdot)$  is concave if and only if the function  $u'/u''$  is convex, see [9, 49]. Given the concavity of the certainty equivalent in (12), the compactness of  $\Omega$ , and the boundedness of  $\mathbf{X}(\omega)$ , the objective function (12a) is finite and problem (12) is well-defined. In general, the certainty equivalent term in (12) can be replaced by a function  $\Phi : \mathcal{X} \mapsto \mathbb{R}$  that is increasing and concave.

Formulation (12) can be viewed as a two-stage stochastic programming problem [13, 54], where the preallocation capacities  $\boldsymbol{\eta}$  and the stochastic flows  $\mathbf{x}(\omega)$  represent the first- and second-stage variables, respectively. As such, problem (12) quantifies the *resilience* of network flows in  $G$  by providing a measure of the degree to which the amount of the network flow can be restored, through resource preallocation, after observing a stochastic reduction  $\mathbf{X}(\omega)$  of capacities of the network arcs in  $E$ . This resilient maximum network flow problem has a number of properties that further illuminate its methodological and practical merits.

First, we note that the resilient maximum network flow model (12) inherits the well-known integrality property of the standard maximum network flow problem (7), namely that an optimal network flow is integer-valued if the arc capacities are integer-valued.

**Proposition 2.1 (Integrality)** *Consider problem (12) where the uncertainty set  $\Omega$  is finite and the utility function  $u$  is linear. If both the arc capacity vector  $\mathbf{u}$  and the stochastic capacity loss vector  $\mathbf{X}(\omega)$  are integral for any  $\omega \in \Omega$ , then there exists an optimal solution  $(\boldsymbol{\eta}^*, \mathbf{x}^*(\omega))$  of the resilient maximum network flow model (12) that is also integral.*

**Proof:** Rewriting (12) as

$$Z_\alpha^*(\mathbf{X}) = \max - (1 - \alpha) \mathbf{1}^\top \boldsymbol{\eta} + u^{-1} \mathbb{E} u(\mathbf{a}_s^\top \mathbf{x}(\omega)) \quad (13a)$$

$$\text{s.t. } A\mathbf{x}(\omega) = \mathbf{0}, \quad \omega \in \Omega \quad (13b)$$

$$\mathbf{x}(\omega) + \mathbf{w}(\omega) \leq \mathbf{u}, \quad \omega \in \Omega \quad (13c)$$

$$\mathbf{w}(\omega) + \boldsymbol{\eta} \geq \mathbf{X}(\omega), \quad \omega \in \Omega \quad (13d)$$

$$\boldsymbol{\eta}, \mathbf{x}(\omega), \mathbf{w}(\omega) \geq \mathbf{0}, \quad \omega \in \Omega \quad (13e)$$

and taking into account the finiteness of set  $\Omega$  and linearity of the utility function  $u$ , we obtain that (13) is a linear programming problem whose feasible region can be presented in the form

$$\{\mathbf{z} : B\mathbf{z} \leq \mathbf{b}, \mathbf{z} \geq \mathbf{0}\},$$

where  $\mathbf{z} = (\boldsymbol{\eta}^\top, \mathbf{x}(\omega_1)^\top, \mathbf{w}(\omega_1)^\top, \dots, \mathbf{x}(\omega_N)^\top, \mathbf{w}(\omega_N)^\top)^\top$ , and matrix  $B$  consists of blocks  $A$ ,  $-A$ ,  $I$ ,  $-I$ , and  $O$ , where  $I$  is an identity matrix, and  $O$  is a zero matrix of appropriate dimensions. Given that matrix  $A$  is totally unimodular, matrix  $B$  can be constructed from blocks  $\pm A$ ,  $\pm I$ ,  $O$  using the basic operations that preserve total unimodularity, see, e.g., [51]. Hence, the vertices of polyhedron  $\{B\mathbf{z} \leq \mathbf{b}, \mathbf{z} \geq \mathbf{0}\}$  are integral if  $\mathbf{b}$  is integral.  $\square$



The statement of Proposition 2.1 will also hold when the utility function  $u$  is strictly convex and  $u'/u''$  is convex, i.e., such that the certainty equivalent  $u^{-1}\mathbb{E}u(\cdot)$  is convex [67].

It is worth noting that an alternative formulation of resilient maximum network problem may be considered, where only the total amount  $M$  of capacity preallocations is decided in advance, to be distributed among arcs  $(i, j) \in E$  in an optimal way after observing the actual realizations  $X_{ij}(\omega)$  of the capacity losses in the network:

$$\begin{aligned} Z_\alpha^*(\mathbf{X}) = \max \quad & -(1-\alpha)M + u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}(\omega)) \\ \text{s.t.} \quad & A\mathbf{x}(\omega) = \mathbf{0}, \quad \omega \in \Omega \\ & \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\omega) - \boldsymbol{\eta}(\omega))_+, \quad \omega \in \Omega \\ & \mathbf{1}^\top \boldsymbol{\eta}(\omega) = M, \quad \omega \in \Omega \\ & \boldsymbol{\eta}(\omega) \geq \mathbf{0}, \quad \omega \in \Omega \end{aligned} \quad (14)$$

In contrast to problem (12), the above formulation (14) does not preserve the integrality of an optimal solution.

Next, it is easy to see that in general problem (12) is bounded and its optimal objective value is nonnegative:

$$0 \leq Z_\alpha^*(\mathbf{X}) < \infty. \quad (15)$$

Indeed, note that  $(\boldsymbol{\eta}', \mathbf{x}'(\omega))$ , where  $\boldsymbol{\eta}' = \mathbf{0}$  and  $\mathbf{x}'(\omega) = \mathbf{0}$  is a feasible solution of (12) with the corresponding objective value of 0, whence  $Z_\alpha^*(\mathbf{X}) \geq 0$ . It is obvious that  $Z_\alpha^*(\mathbf{X}) < \infty$  since  $\boldsymbol{\eta} \geq \mathbf{0}$  and  $\mathbf{x}$  is bounded. Further, the following observation on the monotonicity properties of the optimal objective value of (12) holds:

**Proposition 2.2 (Monotonicity with respect to  $\alpha$ )** *The optimal value of problem (12) is non-decreasing as a function of  $\alpha \in [0, 1]$ :  $Z_{\alpha_1}^*(\mathbf{X}) \leq Z_{\alpha_2}^*(\mathbf{X})$ ,  $\alpha_1 < \alpha_2$ . Moreover, both the total capacity preallocations  $\mathbf{1}^\top \boldsymbol{\eta}$  and the certainty equivalents of the network flow  $u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}(\omega))$  are non-decreasing in  $\alpha$  at optimality.*

**Proof:** Let  $(\boldsymbol{\eta}^*, \mathbf{x}^*(\omega))$  and  $(\boldsymbol{\eta}^{**}, \mathbf{x}^{**}(\omega))$  be optimal solutions of problem (12) with  $\alpha = \alpha_1$  and  $\alpha = \alpha_2$ , respectively, such that  $\alpha_1 < \alpha_2$ . Observing that  $(\boldsymbol{\eta}^*, \mathbf{x}^*(\omega))$  is a feasible solution of (12) with  $\alpha = \alpha_2$ , we have

$$\begin{aligned} Z_{\alpha_2}^*(\mathbf{X}) &= -(1-\alpha_2)\mathbf{1}^\top \boldsymbol{\eta}^{**} + u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^{**}(\omega)) \geq -(1-\alpha_2)\mathbf{1}^\top \boldsymbol{\eta}^* + u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^*(\omega)) \\ &= -(1-\alpha_1)\mathbf{1}^\top \boldsymbol{\eta}^* + u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^*(\omega)) + (\alpha_2 - \alpha_1)\mathbf{1}^\top \boldsymbol{\eta}^* \geq Z_{\alpha_1}^*(\mathbf{X}), \end{aligned}$$

where the last inequality is due to  $\alpha_2 > \alpha_1$  and  $\boldsymbol{\eta}^* \geq \mathbf{0}$ . Similarly, since  $(\boldsymbol{\eta}^{**}, \mathbf{x}^{**}(\omega))$  is a feasible solution of (12) with  $\alpha = \alpha_1$ , we have

$$-(1-\alpha_1)\mathbf{1}^\top \boldsymbol{\eta}^* + u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^*(\omega)) \geq -(1-\alpha_1)\mathbf{1}^\top \boldsymbol{\eta}^{**} + u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^{**}(\omega))$$

and

$$-(1-\alpha_2)\mathbf{1}^\top \boldsymbol{\eta}^{**} + u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^{**}(\omega)) \geq -(1-\alpha_2)\mathbf{1}^\top \boldsymbol{\eta}^* + u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^*(\omega)).$$

Rearranging terms in the above expressions, we obtain

$$(1-\alpha_2)\mathbf{1}^\top (\boldsymbol{\eta}^{**} - \boldsymbol{\eta}^*) \leq u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^{**}(\omega)) - u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^*(\omega)) \leq (1-\alpha_1)\mathbf{1}^\top (\boldsymbol{\eta}^{**} - \boldsymbol{\eta}^*)$$

which immediately yields

$$(\alpha_2 - \alpha_1)\mathbf{1}^\top (\boldsymbol{\eta}^{**} - \boldsymbol{\eta}^*) \geq 0.$$

For  $\alpha_1 < \alpha_2$  the latter inequality then yields  $\mathbf{1}^\top \boldsymbol{\eta}^* \leq \mathbf{1}^\top \boldsymbol{\eta}^{**}$ , while the former inequality provides that

$$u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^*(\omega)) \leq u^{-1}\mathbb{E}u(\mathbf{a}_s^\top \mathbf{x}^{**}(\omega)),$$

which verifies the statements of the proposition.  $\square$

Proposition 2.2 implies that the largest possible objective value in problem (12) is achieved when  $\alpha = 1$ , i.e., the resources  $\boldsymbol{\eta}$  to counteract the losses  $\mathbf{X}(\omega)$  can be preallocated for free. In such a case, obviously, one can select  $\eta_{ij} = \max_\omega X_{ij}(\omega)$ , whereby the resilient maximum flow problem (12) reduces to its deterministic version.

**Corollary 2.3** *If  $\alpha = 1$  and the utility function is linear, there exists an optimal solution  $(\eta^*, \mathbf{x}^*(\omega))$  of (12) such that*

$$\eta_{ij}^* = \max_{\omega} X_{ij}(\omega) \quad \forall (i, j) \in E \quad \text{and} \quad \mathbf{x}^*(\omega) = \mathbf{x}^* \in \arg \max \{ \mathbf{a}_s^\top \mathbf{x} \mid A\mathbf{x} = \mathbf{0}, \mathbf{0} \leq \mathbf{x} \leq \mathbf{u} \},$$

*i.e.,  $\mathbf{x}^*(\omega) = \mathbf{x}^*, \omega \in \Omega$ , is a solution of the deterministic maximum flow problem without capacity losses (7).*

**Proof:** When  $\alpha = 1$ , problem (12) with a linear utility function becomes

$$\begin{aligned} \max \quad & \mathbb{E}[\mathbf{a}_s^\top \mathbf{x}(\omega)] \\ \text{s.t.} \quad & A\mathbf{x}(\omega) = \mathbf{0}, \quad \omega \in \Omega \\ & \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\omega) - \eta)_+, \quad \omega \in \Omega \\ & \eta \geq \mathbf{0}. \end{aligned} \tag{16}$$

Obviously, the optimal objective value of (16) attains its largest possible magnitude when the flow bound constraints take the form  $\mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u}$ ,  $\omega \in \Omega$ , which can be achieved by selecting  $\eta_{ij} = \max_{\omega} X_{ij}(\omega)$ ,  $(i, j) \in E$  in (16). In this case, the optimal values of network flows become  $\mathbf{x}^*(\omega) = \mathbf{x}^*, \omega \in \Omega$ , where  $\mathbf{x}^*$  is an optimal solution of the deterministic maximum network flow problem (7).  $\square$

In this case of “free” resources, the network exhibits maximum resilience, in the sense that the network flow can be completely restored to its pre-disruption levels given by a solution of the deterministic problem (7). In other words, *if there is no cost to preallocating resources, any network flow is fully resilient.*

On the other hand, in the case when there is no “discount” for preallocating the resources,  $\alpha = 0$ , then forgoing the preallocation,  $\eta = \mathbf{0}$ , is optimal, provided that the utility function is linear. An optimal flow,  $\mathbf{x}^*(\omega)$ , is in this case given by the solutions of the scenario based maximum flow problems with stochastic arc capacities (10):

**Corollary 2.4** *If the utility function is linear, then forgoing the resource preallocation is optimal when  $\alpha = 0$ . Specifically, there exists an optimal solution  $(\eta^*, \mathbf{x}^*(\omega))$  of problem (12) with  $\alpha = 0$ , where  $\eta^* = \mathbf{0}$  and*

$$\mathbf{x}^*(\omega) \in \arg \max \{ \mathbf{a}_s^\top \mathbf{x}(\omega) \mid A\mathbf{x}(\omega) = \mathbf{0}, \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - \mathbf{X}(\omega) \}, \quad \omega \in \Omega,$$

*i.e.,  $\mathbf{x}^*(\omega)$  is a solution of the maximum expected flow problem with stochastic arc capacities (10).*

**Proof:** Consider problem (12) with a linear utility function, the value of  $\alpha = 0$ , and with an additional constraint  $\eta = \mathbf{0}$ , corresponding to the case when capacity preallocations are not permitted. Let  $P_0$  denote an instance of such a problem, and  $Z_0^{**}(\mathbf{X})$  and  $\tilde{\mathbf{x}}(\omega)$  be its optimal value and an optimal solution (flow). Clearly, one has  $Z_0^{**}(\mathbf{X}) = \mathbb{E}(\mathbf{a}_s^\top \tilde{\mathbf{x}}(\omega))$ . For any  $\omega \in \Omega$  let

$$C(\omega) = \arg \max \{ \mathbf{a}_s^\top \mathbf{x}(\omega) \mid A\mathbf{x}(\omega) = \mathbf{0}, \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - \mathbf{X}(\omega) \}, \quad \omega \in \Omega.$$

Obviously, for any  $\mathbf{x}^{**}(\omega) \in C(\omega)$  one has  $\mathbf{a}_s^\top \tilde{\mathbf{x}}(\omega) \leq \mathbf{a}_s^\top \mathbf{x}^{**}(\omega)$  and thus  $\mathbb{E}(\mathbf{a}_s^\top \mathbf{x}^{**}(\omega)) \leq \mathbb{E}(\mathbf{a}_s^\top \tilde{\mathbf{x}}(\omega))$ , whereby  $\mathbf{x}^{**}(\omega)$  is also an optimal solution of  $P_0$ . Now, let  $Z_0^*(\mathbf{X})$  and  $(\eta^*, \mathbf{x}^*(\omega))$  denote the optimal value and an optimal solution of (12) with  $\alpha = 0$  and a linear  $u$ , i.e., such that capacity preallocations are permitted. It is clear that  $(\eta^{**}, \mathbf{x}^{**}(\omega))$ , where  $\eta^{**} = \mathbf{0}$ , is a feasible solution of (12), whereby  $Z_0^{**}(\mathbf{X}) \leq Z_0^*(\mathbf{X})$ . Next, observe that adding  $M = \mathbf{1}^\top \eta^*$  units of capacity preallocations can increase the “no-preallocations” feasible network flow  $\mathbf{a}_s^\top \mathbf{x}^{**}(\omega)$  by at most  $M$  units for any given  $\omega \in \Omega$ , from which it follows that

$$\mathbf{a}_s^\top \mathbf{x}^*(\omega) \leq \mathbf{a}_s^\top \mathbf{x}^{**}(\omega) + M, \quad \forall \omega \in \Omega.$$

Therefore, we obtain

$$\begin{aligned} Z_0^{**}(\mathbf{X}) &\leq Z_0^*(\mathbf{X}) = -\mathbf{1}^\top \eta^* + \mathbb{E}(\mathbf{a}_s^\top \mathbf{x}^*(\omega)) \\ &\leq -M + \mathbb{E}(\mathbf{a}_s^\top \mathbf{x}^{**}(\omega) + M) = Z_0^{**}(\mathbf{X}), \end{aligned}$$

which establishes the statement of the corollary.  $\square$

This observation reinforces the proposed interpretation of resilience as the capability of (at least partially) recovering the flows in a network after an unforeseen reduction in the network capacity, depending on the amount of available resources. *Without additional resources (or, as in the above case, without an incentive for preemptive allocation of resources), resilience cannot be achieved.*

### 2.3 Resilient Maximum Network Flow Model as a Measure of Risk

Next, we show that the optimal value  $Z_\alpha^*(\mathbf{X})$  of the resilient network flow model (12) as a function of the vector of stochastic capacity losses  $\mathbf{X} = \mathbf{X}(\omega)$  possesses properties analogous to those of convex measures of risk. In order to examine the properties of model (12) as a measure of risk associated with stochastic losses  $\mathbf{X}(\omega)$ , we make the following two observations.

**Remark 2.1** Note that we explicitly imposed the nonnegativity constraints  $\boldsymbol{\eta} \geq \mathbf{0}$  due to modeling considerations: as the stochastic capacity disruptions  $\mathbf{X}(\omega)$  are assumed to be nonnegative, the resources to offset these disruptions should also be nonnegative. Mathematically, however, nonnegativity of  $\boldsymbol{\eta}$  is not required for the problem to remain meaningful; for example, note that if  $\boldsymbol{\eta} \in \mathbb{R}^{|E|}$  in (12), the problem remains bounded, since as soon as for some  $(i, j) \in E$  the value of  $\eta_{ij}$  is such that  $(X_{ij}(\omega) - \eta_{ij})_+ > u_{ij}$ , the problem becomes infeasible. Therefore, in the context of considering model (12) as a measure of risk, we relax the nonnegativity constraint (12d) and allow the vector of resource preallocations to take values  $\boldsymbol{\eta} \in \mathbb{R}^{|E|}$ .

**Remark 2.2** Observe that the possibility of resource preallocations in the resilient maximum flow model (12) allows for relaxing the upper and lower bounds on the capacity losses  $\mathbf{X}(\omega)$  in (9):

$$-\infty < X_{ij}(\omega) < \infty, \quad \omega \in \Omega, \quad (i, j) \in E, \quad (17)$$

in the sense that values of  $X_{ij}(\omega) > u_{ij}$  for some  $(i, j) \in E$  do not induce infeasibility in the resilient maximum flow problem (12), unlike in the stochastic maximum flow problem (10). Therefore, in the remainder of this section we will generally presume that the stochastic capacity reductions  $\mathbf{X}(\omega)$  are unbounded as in (17).

With these two considerations in mind, it can then be shown that the optimal value of (12) is monotonic with respect to capacity losses, namely that higher capacity losses  $\mathbf{X}(\omega)$  lead to lower values of  $Z_\alpha^*(\mathbf{X})$ .

**Proposition 2.5 (Monotonicity with respect to capacity losses)** *Let the two realizations  $\mathbf{X}^1(\omega)$ ,  $\mathbf{X}^2(\omega)$  of the stochastic capacity losses be such that*

$$\mathbf{X}^1(\omega) \leq \mathbf{X}^2(\omega) \quad \forall \omega \in \Omega, \quad (18)$$

*then the corresponding optimal values of (12) satisfy*

$$Z_\alpha^*(\mathbf{X}^1) \geq Z_\alpha^*(\mathbf{X}^2). \quad (19)$$

**Proof:** Let  $P(\mathbf{X}^1)$  and  $P(\mathbf{X}^2)$  denote the instances of problem (12) corresponding to the values of stochastic capacity losses  $\mathbf{X}^1(\omega)$  and  $\mathbf{X}^2(\omega)$ , respectively. Let  $(\boldsymbol{\eta}^*, \mathbf{x}^*(\omega))$  be an optimal solution of  $P(\mathbf{X}^2)$ . Observe that if we show that an optimal solution of  $P(\mathbf{X}^2)$  is also a feasible solution for  $P(\mathbf{X}^1)$ , it will prove that  $Z_\alpha^*(\mathbf{X}^1) \geq Z_\alpha^*(\mathbf{X}^2)$ . Indeed, since  $(\boldsymbol{\eta}^*, \mathbf{x}^*(\omega))$  is optimal for  $P(\mathbf{X}^2)$ , it satisfies

$$\mathbf{x}^*(\omega) + (\mathbf{X}^2(\omega) - \boldsymbol{\eta}^*)_+ \leq \mathbf{u} \quad \forall \omega \in \Omega.$$

Then from (18) it follows that

$$(\mathbf{X}^1(\omega) - \boldsymbol{\eta}^*)_+ \leq (\mathbf{X}^2(\omega) - \boldsymbol{\eta}^*)_+ \quad \forall \omega \in \Omega,$$

whereby

$$\mathbf{x}^*(\omega) + (\mathbf{X}^1(\omega) - \boldsymbol{\eta}^*)_+ \leq \mathbf{u} \quad \forall \omega \in \Omega,$$

i.e.,  $(\boldsymbol{\eta}^*, \mathbf{x}^*(\omega))$  is feasible for  $P(\mathbf{X}^1)$ .  $\square$

**Proposition 2.6 (Concavity with respect to capacity losses)** *The optimal value of (12) is concave as a function of stochastic capacity losses,*

$$Z_\alpha^*(\lambda \mathbf{X}^1 + (1 - \lambda) \mathbf{X}^2) \geq \lambda Z_\alpha^*(\mathbf{X}^1) + (1 - \lambda) Z_\alpha^*(\mathbf{X}^2), \quad 0 \leq \lambda \leq 1. \quad (20)$$

**Proof:** This statement is a straightforward extension of the well-known fact that the optimal value of, e.g., a linear programming problem with maximization objective is concave in the right hand side vector of its constraints, see, for instance, [13].  $\square$

**Proposition 2.7** *For any  $\mathbf{m} \in \mathbb{R}^{|E|}$  it holds that*

$$Z_\alpha^*(\mathbf{X} + \mathbf{m}) = -(1 - \alpha) \mathbf{1}^\top \mathbf{m} + Z_\alpha^*(\mathbf{X}) \quad (21)$$

**Proof:** Consider problem (12) without the nonnegativity constraint on  $\boldsymbol{\eta}$ :

$$\begin{aligned} Z_\alpha^*(\mathbf{X} + \mathbf{m}) = \max \quad & -(1 - \alpha) \mathbf{1}^\top \boldsymbol{\eta} + u^{-1} \mathbb{E} u(\mathbf{a}_s^\top \mathbf{x}(\omega)) \\ \text{s.t.} \quad & A\mathbf{x}(\omega) = \mathbf{0}, \quad \omega \in \Omega \\ & \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\omega) + \mathbf{m} - \boldsymbol{\eta})_+, \quad \omega \in \Omega \\ & \boldsymbol{\eta} \in \mathbb{R}^{|E|}. \end{aligned}$$

By substituting  $\boldsymbol{\eta} = \boldsymbol{\eta}' + \mathbf{m}$ , we obtain

$$\begin{aligned} Z_\alpha^*(\mathbf{X} + \mathbf{m}) = \max \quad & -(1 - \alpha) \mathbf{1}^\top (\boldsymbol{\eta}' + \mathbf{m}) + u^{-1} \mathbb{E} u(\mathbf{a}_s^\top \mathbf{x}(\omega)) \\ \text{s.t.} \quad & A\mathbf{x}(\omega) = \mathbf{0}, \quad \omega \in \Omega \\ & \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\omega) - \boldsymbol{\eta}')_+, \quad \omega \in \Omega \\ & \boldsymbol{\eta}' \in \mathbb{R}^{|E|}, \end{aligned}$$

which establishes (21).  $\square$

Propositions 2.5, 2.6, 2.7 imply that the quantity

$$\zeta(\mathbf{X}) = -\frac{1}{1 - \alpha} Z_\alpha^*(\mathbf{X}),$$

where  $Z_\alpha^*(\mathbf{X})$  is the optimal value of resilient maximum flow problem (12) without nonnegativity constraints on the resource preallocations  $\boldsymbol{\eta}$  is a convex measure of risk with respect to the vector of stochastic capacity losses, in that it satisfies the monotonicity, convexity, and translation invariance properties:

- (A1)  $\zeta(\mathbf{X}^1) \leq \zeta(\mathbf{X}^2)$  for any  $\mathbf{X}^1(\omega) \leq \mathbf{X}^2(\omega)$  a.s.;
- (A2)  $\zeta(\lambda \mathbf{X}^1 + (1 - \lambda) \mathbf{X}^2) \leq \lambda \zeta(\mathbf{X}^1) + (1 - \lambda) \zeta(\mathbf{X}^2)$  for any  $\mathbf{X}^1(\omega)$ ,  $\mathbf{X}^2(\omega)$  and  $0 \leq \lambda \leq 1$ ;
- (A3)  $\zeta(\mathbf{X} + \mathbf{m}) = \zeta(\mathbf{X}) + \mathbf{1}^\top \mathbf{m}$  for any  $\mathbf{X}(\omega)$  and  $\mathbf{m} \in \mathbb{R}^{|E|}$ .

In view of the above, the function  $\zeta(\mathbf{X})$ , defined through the optimal value  $Z_\alpha^*(\mathbf{X})$  of problem (12) without nonnegativity constraints on resource preallocations can be considered as a *network measure of risk* that quantifies the risk exposure of network's operation (namely, the maximum network flow) with respect to a set of stochastic factors that embody the unpredictable losses of arcs' carrying capacities. It is worth noting that, in contrast to various measures of risk proposed in the literature that are functions of a scalar stochastic factor  $X$ , the introduced network measure of risk  $\zeta(\cdot)$  is fundamentally a function of a multidimensional stochastic factor  $\mathbf{X}$ . Indeed, in the presence of multiple risk-inducing stochastic factors  $Z_1(\omega), \dots, Z_r(\omega)$ , the common and implicit assumption in the literature is that one can scalarize a multidimensional stochastic factor by introducing an explicitly and finitely defined *loss function*  $f(\mathbf{x}, \mathbf{Z})$  [57], where  $\mathbf{Z} = (Z_1, \dots, Z_r)^\top$  and  $\mathbf{x}$  is a vector of decision parameters, and consider a risk measure  $\rho(X)$  with  $X = f(\mathbf{x}, \mathbf{Z})$ . Conversely, in the case of the network measure of risk  $\zeta(\mathbf{X})$  the contributions of the elements of the stochastic vector  $\mathbf{X}$  to the overall risk are rather nontrivial and are determined via solution of the underlying maximum network flow problem.

Recall that risk measures such as (4)–(6) are *tail measures of risk*, meaning that for the values  $\alpha_1 < \alpha_2$  one has  $\eta_{\alpha_1}^*(X) \leq \eta_{\alpha_2}^*(X)$ , where

$$\eta_\alpha^*(X) = \min \{ \arg \min \{ \eta + (1 - \alpha)^{-1} v^{-1} \mathbb{E} v(X - \eta)_+ \} \}$$

represents the cutoff point of the right-hand tail  $(X - \eta)_+$  of the distribution of  $X$  that is used for computation of risk in (2). In view of this, the above characterization of tail measures of risk can be extended to the introduced network measures of risk:

**Corollary 2.8** *Proposition 2.2 implies that the network measures of risk  $\zeta(\mathbf{X})$  can be considered as a tail measures of risk, in the sense that  $\mathbf{1}^\top \boldsymbol{\eta}^* \leq \mathbf{1}^\top \boldsymbol{\eta}^{**}$ , where  $\boldsymbol{\eta}^*, \boldsymbol{\eta}^{**}$  represent optimal values of vector  $\boldsymbol{\eta}$  in problem (12) with values of parameters  $\alpha_1 < \alpha_2$ , respectively, and nonnegativity constraints on  $\boldsymbol{\eta}$  relaxed.*

Indeed, note that nonnegativity of  $\boldsymbol{\eta}$  was not used in the relevant part of proof of Proposition 2.2.

Continuing the analogy with the measures of risk discussed in Section 2.1, the network measures of risk can be seamlessly integrated in stochastic optimization problems with the purpose of minimization and/or control of network-related risks induced by stochastic losses. Namely, suppose that the stochastic vector  $\mathbf{X}$  depends parametrically on some decision variables  $\mathbf{y} \in \mathbb{R}^k$ ,  $\mathbf{X} = \mathbf{X}(\mathbf{y}, \omega)$ , and consider the following risk-constrained stochastic optimization problem

$$\min \{ g(\mathbf{y}) \mid \zeta(\mathbf{X}(\mathbf{y}, \omega)) \leq h(\mathbf{y}), \mathbf{y} \in C \}, \quad (22a)$$

which we assume to be convex. Taking into account the mathematical programming definition of  $\zeta$ , problem (22a) can be equivalently reformulated as

$$\begin{aligned} \min \quad & g(\mathbf{y}) \\ \text{s.t.} \quad & \mathbf{1}^\top \boldsymbol{\eta} - (1 - \alpha)^{-1} u^{-1} \mathbb{E} u(\mathbf{a}_s^\top \mathbf{x}(\omega)) \leq h(\mathbf{y}) \\ & A\mathbf{x}(\omega) = \mathbf{0}, \quad \omega \in \Omega \\ & \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\mathbf{y}, \omega) - \boldsymbol{\eta})_+, \quad \omega \in \Omega \\ & \mathbf{y} \in C, \end{aligned} \quad (22b)$$

where the exact meaning of equivalence between (22a) and (22b) is given next.

**Proposition 2.9** *Consider problem (22a), where  $C$  is a compact convex set, functions  $g$  and  $-h$  are convex on  $C$ , and each component of vector  $\mathbf{X}(\mathbf{y}, \omega)$  is also convex in  $\mathbf{y}$  on  $C$ . Then, problems (22a) and (22b) are equivalent, meaning that they achieve minima at the same values of  $\mathbf{y}$  and the corresponding optimal objective values coincide. In addition, if the risk constraint in (22a) is binding at optimality, then  $(\mathbf{y}^*, \mathbf{x}^*(\omega), \boldsymbol{\eta}^*)$  is an optimal solution of (22b) if and only if  $\mathbf{y}^*$  and  $(\mathbf{x}^*(\omega), \boldsymbol{\eta}^*)$  are, respectively, optimal solutions of (22a) and*

$$\min \{ \mathbf{1}^\top \boldsymbol{\eta} - (1 - \alpha)^{-1} u^{-1} \mathbb{E} u(\mathbf{a}_s^\top \mathbf{x}(\omega)) \mid A\mathbf{x}(\omega) = \mathbf{0}, \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\mathbf{y}^*, \omega) - \boldsymbol{\eta})_+, \omega \in \Omega \}. \quad (22c)$$

**Proof:** The proof is analogous to that of Theorem 3 in [43].  $\square$

In other words, Proposition 2.9 implies that the mathematical programming representation of the network measures of risk  $\zeta$  (based on problem (12) without nonnegativity constraints on  $\eta$ ) can be used to formulate stochastic optimization models with network measures of risk in the objective and/or constraints.

## 2.4 Resilient Minimum Cost Network Flow Model

In this section we illustrate the proposed idea of constructing metrics of network resilience on the example of the minimum cost flow problem. Using the notations introduced above, the standard minimum cost network flow model can be stated as

$$\begin{aligned} \min \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A^+ \mathbf{x} = \mathbf{q} \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{u}, \end{aligned} \tag{23}$$

where, as before,  $A^+$  is the node-arc incidence matrix,  $\mathbf{u}$  is the vector of capacities of arcs in  $E$ , and  $\mathbf{c}$  is the cost vector, whose elements represent the costs of sending one unit of flow along the arcs in  $E$ , and the vector  $\mathbf{q}$  of supplies/demands at network's nodes is such that  $q_i > 0$  (respectively,  $q_i < 0$ ) if  $i$  is a supply (respectively, demand) node, and  $q_i = 0$  otherwise. Obviously,  $\mathbf{q}$  has to satisfy

$$\mathbf{1}^\top \mathbf{q} = 0.$$

As before, assume that operational arc capacities  $\mathbf{u}$  are subject to stochastic capacity losses  $\mathbf{X}(\omega)$  such that

$$\mathbf{0} \leq \mathbf{X}(\omega) \leq \mathbf{u}, \quad \omega \in \Omega.$$

Following the same line of arguments as presented above, we formulate the resilient minimum cost network flow problem as follows:

$$\bar{Z}_\alpha^*(\mathbf{X}) = \min (1 - \alpha) \mathbf{c}^\top \boldsymbol{\eta} + v^{-1} \mathbb{E} v(\mathbf{c}^\top \mathbf{x}(\omega)) \tag{24a}$$

$$\text{s.t.} \quad A^+ \mathbf{x} = \mathbf{q}, \quad \omega \in \Omega \tag{24b}$$

$$\mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\omega) - \boldsymbol{\eta})_+, \quad \omega \in \Omega \tag{24c}$$

$$\boldsymbol{\eta} \geq \mathbf{0}. \tag{24d}$$

As before,  $\boldsymbol{\eta}$  denotes the amount of preallocated resources to offset the uncertain capacity losses  $\mathbf{X}(\omega)$ , such that the network flow  $\mathbf{x}(\omega)$  realized upon observing the outcome of capacity reductions  $\mathbf{X}(\omega)$  has to satisfy the bounds (24c) as well as the flow balance constraints (24b). Given that the smaller values of the stochastic cost  $\mathbf{c}^\top \mathbf{x}(\omega)$  of the network flow are preferred, its deterministic equivalent value is quantified using the deutility function based certainty equivalent  $v^{-1} \mathbb{E} v(\mathbf{c}^\top \mathbf{x}(\omega))$ , which is to be minimized together with the discounted cost of capacity preallocations  $(1 - \alpha) \mathbf{c}^\top \boldsymbol{\eta}$ . It is assumed that the deutility function is such that the certainty equivalent term in the objective (24) is convex.

The resilient minimum cost network flow model (24) possesses a number of properties analogous to those of the resilient network flow model discussed in Sections 2.2–2.3. In particular, it similarly possesses the integrality of solution, provided that the arc capacities, capacity losses, and supply/demand values at the nodes are integer, and the deutility function is linear or concave.

**Proposition 2.10 (Integrality)** *If the arc capacity vector  $\mathbf{u}$ , the supply/demand vector  $\mathbf{q}$ , and the stochastic capacity loss vector  $\mathbf{X}(\omega)$  are integral for any  $\omega \in \Omega$ , and the deutility function  $v$  is linear, then an optimal solution  $(\boldsymbol{\eta}^*, \mathbf{x}^*(\omega))$  of the resilient minimum network flow model (24) is also integral.*

Analogously to the resilient maximum network flow model, the optimal value of the resilient min-cost flow problem (24) is monotonic with respect to the “discount coefficient”  $\alpha$ , but since the objective function (24) represents the cost of the network flow, it is non-increasing in  $\alpha$ .

**Proposition 2.11 (Monotonicity with respect to  $\alpha$ )** *The optimal value of problem (24) is non-increasing as a function of  $\alpha \in [0, 1]$ :  $\bar{Z}_{\alpha_1}^*(\mathbf{X}) \geq \bar{Z}_{\alpha_2}^*(\mathbf{X})$ ,  $\alpha_1 < \alpha_2$ . In particular, both the costs of capacity preallocations  $\mathbf{c}^\top \boldsymbol{\eta}^*$  and the certainty equivalents of the network flow cost  $v^{-1} \mathbb{E} v(\mathbf{c}^\top \mathbf{x}^*(\omega))$  are non-increasing in  $\alpha$  at optimality.*

According to Proposition 2.11, the smallest possible objective value in problem (24) is attained when the resources  $\boldsymbol{\eta}$  for counteracting the capacity losses  $\mathbf{X}(\omega)$  can be preallocated for free,  $\alpha = 1$ . Similarly to above, in this case one can select  $\eta_i = \max_{\omega} X_i(\omega)$ , whereby the resilient min-cost flow problem (24) reduces to its deterministic version.

**Corollary 2.12** *If  $\alpha = 1$  and the deutility function is linear, there exists an optimal solution  $(\boldsymbol{\eta}^*, \mathbf{x}^*(\omega))$  of (24) such that*

$$\eta_{ij}^* = \max_{\omega} X_{ij}(\omega) \quad \forall (i, j) \in E \quad \text{and} \quad \mathbf{x}^*(\omega) = \mathbf{x}^* \in \arg \min \{ \mathbf{c}^\top \mathbf{x} \mid A\mathbf{x} = \mathbf{q}, \mathbf{0} \leq \mathbf{x} \leq \mathbf{u} \}$$

*i.e.,  $\mathbf{x}^*(\omega) = \mathbf{x}^*$ ,  $\omega \in \Omega$ , is a solution of the deterministic minimum cost network flow problem without capacity losses (23).*

In contrast to the resilient maximum flow model described in Section 2.2, in the case when there is no “discount” for preallocating the resources,  $\alpha = 0$ , forgoing the resource preallocation,  $\boldsymbol{\eta} = \mathbf{0}$ , is not optimal in general. Indeed, consider a simple network consisting of two nodes connected by an arc, such that the first node is a supply node with outflow value  $q_1 = q > 0$ , the second node is a demand node with inflow value  $q_2 = -q$ , and the capacity of the arc is  $u_{12} = q$ . In this situation, for any positive capacity loss  $X_{12}(\omega) > 0$ , zero resource preallocation  $\eta_{12} = 0$  cannot be feasible and therefore optimal.

In line with the discussion in Section 2.3, constraints (9) on the capacity loss vector  $\mathbf{X}(\omega)$  as well as the nonnegativity constraints on the vector  $\boldsymbol{\eta}$  of capacity preallocations can be relaxed, in the sense that problem (24) remains well defined and bounded after that. In such a case, the objective value of the resilient minimum cost network flow model (24) behaves as a convex measure of risk. More formally, the quantity

$$\bar{\zeta}(\mathbf{X}) = \frac{1}{1 - \alpha} \bar{Z}_{\alpha}^*(\mathbf{X}),$$

where  $\bar{Z}_{\alpha}^*(\mathbf{X})$  is the optimal value of resilient maximum flow problem (24) without nonnegativity constraints on the resource preallocations  $\boldsymbol{\eta}$ , is a convex measure of risk with respect to the vector of stochastic capacity losses by virtue of the fact that it satisfies the monotonicity, translation invariance, and convexity properties,

- (B1)  $\bar{\zeta}(\mathbf{X}^1) \leq \bar{\zeta}(\mathbf{X}^2)$  for any  $\mathbf{X}^1(\omega) \leq \mathbf{X}^2(\omega)$  a.s.;
- (B2)  $\bar{\zeta}(\lambda \mathbf{X}^1 + (1 - \lambda) \mathbf{X}^2) \leq \lambda \bar{\zeta}(\mathbf{X}^1) + (1 - \lambda) \bar{\zeta}(\mathbf{X}^2)$  for any  $\mathbf{X}^1(\omega)$ ,  $\mathbf{X}^2(\omega)$  and  $0 \leq \lambda \leq 1$ ;
- (B3)  $\bar{\zeta}(\mathbf{X} + \mathbf{m}) = \bar{\zeta}(\mathbf{X}) + \mathbf{c}^\top \mathbf{m}$  for any  $\mathbf{X}(\omega)$  and  $\mathbf{m} \in \mathbb{R}^{|E|}$ ,

as evidenced by Propositions 2.13–2.15 below, where it is assumed that the capacity losses  $\mathbf{X}(\omega)$  are unbounded per (17) and the preallocations  $\boldsymbol{\eta}$  in (24) are unrestricted.

**Proposition 2.13 (Monotonicity with respect to capacity losses)** *The optimal value  $\bar{Z}_{\alpha}^*(\mathbf{X})$  of problem (24) is non-decreasing as a function of stochastic capacity losses  $\mathbf{X} = \mathbf{X}(\omega)$ ,  $\omega \in \Omega$ . Namely, if two realizations  $\mathbf{X}^1(\omega)$ ,  $\mathbf{X}^2(\omega)$  of stochastic capacity losses are such that*

$$\mathbf{X}^1(\omega) \leq \mathbf{X}^2(\omega), \quad \omega \in \Omega, \tag{25}$$

*then the corresponding optimal values of (24) satisfy*

$$\bar{Z}_{\alpha}^*(\mathbf{X}^1) \leq \bar{Z}_{\alpha}^*(\mathbf{X}^2). \tag{26}$$

**Proposition 2.14 (Convexity with respect to capacity losses)** *The optimal value  $\bar{Z}_\alpha^*(\mathbf{X})$  of (24) is a convex function of stochastic capacity losses  $\mathbf{X} = \mathbf{X}(\omega)$ ,  $\omega \in \Omega$ , namely*

$$\bar{Z}_\alpha^*(\lambda \mathbf{X}^1 + (1 - \lambda) \mathbf{X}^2) \leq \lambda \bar{Z}_\alpha^*(\mathbf{X}^1) + (1 - \lambda) \bar{Z}_\alpha^*(\mathbf{X}^2), \quad 0 \leq \lambda \leq 1. \quad (27)$$

**Proposition 2.15** *For any  $\mathbf{m} \in \mathbb{R}^{|E|}$  it holds that*

$$\bar{Z}_\alpha^*(\mathbf{X} + \mathbf{m}) = (1 - \alpha) \mathbf{c}^\top \mathbf{m} + \bar{Z}_\alpha^*(\mathbf{X}) \quad (28)$$

Similarly to the analysis in Section 2.3, the introduced quantity  $\bar{\zeta}(\mathbf{X})$  can be viewed as a network measure of risk that quantifies the risk exposure of the minimum cost flow in a network with respect to a set of stochastic factors whose contributions to the overall risk picture are rather nontrivial and are dictated by the topological structure of the network. Also, by analogy with Proposition 2.9, stochastic programming problem involving  $\bar{\zeta}(\mathbf{X})$  of the form

$$\min \{g(\mathbf{y}) \mid \bar{\zeta}(\mathbf{X}(\mathbf{y}, \omega)) \leq h(\mathbf{y}), \mathbf{y} \in C\}, \quad (29a)$$

where  $C$  is convex and compact, functions  $g$ ,  $-h$ , and the components of  $\mathbf{X}$  are convex in  $\mathbf{y}$  on  $C$ , can be equivalently reformulated as

$$\begin{aligned} \min \quad & g(\mathbf{y}) \\ \text{s.t.} \quad & \mathbf{1}^\top \boldsymbol{\eta} + (1 - \alpha)^{-1} v^{-1} \mathbb{E} v(\mathbf{a}_s^\top \mathbf{x}(\omega)) \leq h(\mathbf{y}) \\ & A^+ \mathbf{x}(\omega) = \mathbf{q}, \quad \omega \in \Omega \\ & \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\mathbf{y}, \omega) - \boldsymbol{\eta})_+, \quad \omega \in \Omega \\ & \mathbf{y} \in C, \end{aligned} \quad (29b)$$

in the sense that they achieve minima at the same values of  $\mathbf{y}$  and the corresponding optimal objective values coincide. Likewise, if the risk constraint in (29a) is binding at optimality, then  $(\mathbf{y}^*, \mathbf{x}^*(\omega), \boldsymbol{\eta}^*)$  is an optimal solution of (29b) if and only if  $\mathbf{y}^*$  is an optimal solution of (29a) and  $(\mathbf{x}^*(\omega), \boldsymbol{\eta}^*)$  achieves the minimum of

$$\min \{ \mathbf{1}^\top \boldsymbol{\eta} + (1 - \alpha)^{-1} u^{-1} \mathbb{E} u(\mathbf{a}_s^\top \mathbf{x}(\omega)) \mid A^+ \mathbf{x}(\omega) = \mathbf{q}, \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\mathbf{y}^*, \omega) - \boldsymbol{\eta})_+, \omega \in \Omega \}.$$

This allows one to utilize the network measure of risk  $\bar{\zeta}(\mathbf{X})$  in stochastic optimization models where the stochastic loss vector  $\mathbf{X}$  can be controlled via decision parameters.

### 3 Decomposition Methods for Resilient Network Flow Problems

This section proposes efficient solution methods for the introduced above resilient maximum network flow problem and resilient minimum cost network flow problem in the case when the set  $\Omega$  of random events is finite, a setting that is common in the stochastic optimization literature. Namely, we assume  $\Omega = \{\omega_1, \dots, \omega_N\}$ , where  $\mathbb{P}(\omega_k) = \pi_k > 0$ ,  $k = 1, \dots, N$ ,  $\sum_{k=1}^N \pi_k = 1$ . As a general approach, we pursue the Benders decomposition techniques, due to the fact that the capacity preallocation vector  $\boldsymbol{\eta}$  can be considered as a complicating variable, or, in the terminology of stochastic programming, the first-stage variable, and the network flow vector  $\mathbf{x}(\omega)$  can be regarded as a second-stage variable. Below we discuss separate variants of Benders decomposition algorithm for resilient network problems with linear and nonlinear utility functions, respectively.

For brevity of exposition, below we present the decomposition algorithms for the resilient maximum network flow problem (12); the corresponding methods for the resilient minimum cost network flow problem (24) are constructed analogously.



### 3.1 Linear Resilient Maximum Flow Problem

First, we consider the resilient maximum network flow problem (12) with a linear utility function:

$$\begin{aligned}
\max \quad & -(1-\alpha)\mathbf{1}^\top \boldsymbol{\eta} + \mathbb{E}[\mathbf{a}_s^\top \mathbf{x}(\omega)] \\
\text{s.t.} \quad & A\mathbf{x}(\omega) = \mathbf{0}, \quad \omega \in \Omega \\
& \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\omega) - \boldsymbol{\eta})_+, \quad \omega \in \Omega \\
& \boldsymbol{\eta} \geq \mathbf{0}.
\end{aligned} \tag{30}$$

By regarding the capacity preallocations  $\boldsymbol{\eta}$  as the first-stage variables and the network flows  $\mathbf{x}(\omega)$  as second-stage variables in (12), for a fixed  $\omega \in \Omega$  we define

$$\Theta(\boldsymbol{\eta}, \omega) = \max \left\{ \mathbf{a}_s^\top \mathbf{x}(\omega) \mid A\mathbf{x}(\omega) = \mathbf{0}, \mathbf{0} \leq \mathbf{x}(\omega) \leq \mathbf{u} - (\mathbf{X}(\omega) - \boldsymbol{\eta})_+ \right\}. \tag{31}$$

Since (31) represents a standard maximum network flow problem, the value of  $\Theta(\boldsymbol{\eta}, \omega)$  can be computed efficiently for given values of  $\boldsymbol{\eta}$  and  $\omega$ ; namely, due to the maximum flow-minimum cut duality, we have that

$$\Theta(\boldsymbol{\eta}, \omega_k) = \sum_{(i,j) \in M(\boldsymbol{\eta}, \omega_k)} \left( u_{ij} - (X_{ij}(\omega_k) - \eta_{ij})_+ \right),$$

where  $M(\boldsymbol{\eta}, \omega_k) \subset E$  is the set of arcs in a minimum cut under scenario  $\omega_k$ , given the preallocated capacities  $\boldsymbol{\eta}$ . Then the original problem (30) can be written as

$$\max \left\{ -(1-\alpha)\mathbf{1}^\top \boldsymbol{\eta} + \mathbb{E}[\Theta(\boldsymbol{\eta}, \omega)] \mid \boldsymbol{\eta} \geq \mathbf{0} \right\},$$

which in a discrete probability space,  $\mathbb{P}(\omega_k) = \pi_k > 0$ , can be presented in the form

$$\max \left\{ -(1-\alpha)\mathbf{1}^\top \boldsymbol{\eta} + \theta \mid \theta \leq \sum_{k=1}^N \pi_k \Theta(\boldsymbol{\eta}, \omega_k), \boldsymbol{\eta} \geq \mathbf{0} \right\}. \tag{32}$$

Based on (32), the master problem can be written as

$$\max \quad -(1-\alpha)\mathbf{1}^\top \boldsymbol{\eta} + \theta \tag{33a}$$

$$\text{s.t.} \quad \theta \leq \sum_{k=1}^N \pi_k \sum_{(i,j) \in M(\boldsymbol{\eta}^v, \omega_k)} \left( u_{ij} - (X_{ij}(\omega_k) - \eta_{ij})_+ \right), \quad v = 1, \dots, r-1 \tag{33b}$$

$$\boldsymbol{\eta} \geq \mathbf{0} \tag{33c}$$

Alternatively, a multicut version of the master problem can be formulated as

$$\max \quad -(1-\alpha)\mathbf{1}^\top \boldsymbol{\eta} + \boldsymbol{\pi}^\top \boldsymbol{\theta} \tag{34a}$$

$$\text{s.t.} \quad \theta_k \leq \sum_{(i,j) \in M(\boldsymbol{\eta}^{v_k}, \omega_k)} \left( u_{ij} - (X_{ij}(\omega_k) - \eta_{ij})_+ \right), \quad v_k = 1, \dots, s_k, k = 1, \dots, N \tag{34b}$$

$$\boldsymbol{\eta} \geq \mathbf{0}, \tag{34c}$$

where  $\boldsymbol{\pi} = (\pi_1, \dots, \pi_N)^\top$ ,  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)^\top$ . Let  $(\theta^r, \boldsymbol{\eta}^r)$  be an optimal solution of the master problem (33) problem at iteration  $r$ . If it holds that

$$\theta^r > \sum_{k=1}^N \pi_k \sum_{(i,j) \in M(\boldsymbol{\eta}^r, \omega_k)} \left( u_{ij} - (X_{ij}(\omega_k) - \eta_{ij}^r)_+ \right), \tag{35}$$

then the cut

$$\theta \leq \sum_{k=1}^N \pi_k \sum_{(i,j) \in M(\boldsymbol{\eta}^r, \omega_k)} \left( u_{ij} - (X_{ij}(\omega_k) - \eta_{ij})_+ \right), \quad (36)$$

is sent to the master (33), the iteration counter is incremented,  $r := r + 1$ , and the master problem is resolved again. Once the condition (35) is violated, the corresponding solution  $(\theta^r, \boldsymbol{\eta}^r)$  is optimal for problem (32), and, correspondingly,  $\boldsymbol{\eta}^r$  is optimal for (30).

In the case of multi-cut master (34), let  $(\theta^r, \boldsymbol{\eta}^r)$  be its solution at iteration  $r$ . If there exists  $k \in \{1, \dots, N\}$  such that

$$\theta_k^r > \sum_{(i,j) \in M(\boldsymbol{\eta}^r, \omega_k)} \left( u_{ij} - (X_{ij}(\omega_k) - \eta_{ij}^r)_+ \right), \quad (37)$$

then one sets  $s_k := s_k + 1$ ,  $\boldsymbol{\eta}^{s_k} = \boldsymbol{\eta}^r$ , and the cut

$$\theta_k \leq \sum_{(i,j) \in M(\boldsymbol{\eta}^{s_k}, \omega_k)} \left( u_{ij} - (X_{ij}(\omega_k) - \eta_{ij})_+ \right), \quad (38)$$

is added to (34). If condition (37) is not satisfied for all  $k = 1, \dots, K$ , then  $\boldsymbol{\eta}^r$  is optimal for (32) and (30).

### 3.2 Nonlinear Resilient Maximum Network Flow Problem

Now let us consider the case when the utility function  $u$  in (12) is such that the certainty equivalent  $u^{-1} \mathbb{E}u(\cdot)$  is a concave function, whereby the resilient maximum network flow problem (12) is a convex programming problem. The described above Benders decomposition method does not readily extend to this nonlinear case, due to the fact that cuts of the form (36) explicitly rely on a linear aggregation of values  $\Theta(\boldsymbol{\eta}, \omega)$ . According to our preliminary computational experiments, the traditional nonlinear Benders decomposition approach for convex nonlinear programming problems (see, e.g., [16]) in application to problem (12) does not provide the expected computational advantages compared to solving the full formulation (12) using an off-the-shelf solver. Therefore, below we present a multi-cut variation of Benders decomposition method described in the previous section that relies on max flow-min cut duality for computational efficiency.

Similarly to above, by introducing the subproblem value function  $\Theta(\boldsymbol{\theta}, \omega)$  per (31), the original nonlinear problem (12) can be represented in the form

$$\max \quad -(1 - \alpha) \mathbf{1}^\top \boldsymbol{\eta} + \theta \quad (39a)$$

$$\text{s.t.} \quad \theta \leq u^{-1} \left( \sum_{k=1}^N \pi_k u(y_k) \right) \quad (39b)$$

$$y_k \leq \Theta(\boldsymbol{\eta}, \omega_k), \quad k = 1, \dots, N, \quad (39c)$$

$$\boldsymbol{\eta} \geq \mathbf{0}, \quad (39d)$$

where, as before,  $\pi_k = \mathbb{P}(\omega_k)$ ,  $k = 1, \dots, N$ . Note that, in accordance with our assumptions on the utility function  $u(\cdot)$ , constraint (39b) is convex. Then, master problem can be formulated as

$$\max \quad -(1 - \alpha) \mathbf{1}^\top \boldsymbol{\eta} + \theta \quad (40a)$$

$$\text{s.t.} \quad \theta \leq u^{-1} \left( \sum_{k=1}^N \pi_k u(y_k) \right) \quad (40b)$$

$$y_k \leq \sum_{(i,j) \in M(\boldsymbol{\eta}^v, \omega_k)} \left( u_{ij} - (X_{ij}(\omega_k) - \eta_{ij})_+ \right), \quad v = 1, \dots, (r-1), \quad k = 1, \dots, N \quad (40c)$$

$$\boldsymbol{\eta} \geq \mathbf{0}. \quad (40d)$$

Let  $(\theta^r, \eta^r)$  be an optimal solution of the master problem (40) problem at iteration  $r$ . If it holds that

$$\theta^r > u^{-1} \left( \sum_{k=1}^N \pi_k u \left( \sum_{(i,j) \in M(\eta^r, \omega_k)} (u_{ij} - (X_{ij}(\omega_k) - \eta_{ij}^r)_+) \right) \right), \quad (41)$$

then the cuts

$$y_k \leq \sum_{(i,j) \in M(\eta^r, \omega_k)} (u_{ij} - (X_{ij}(\omega_k) - \eta_{ij}^r)_+), \quad k = 1, \dots, N, \quad (42)$$

are sent to the master (40), the iteration counter is incremented,  $r := r + 1$ , and the master problem is resolved again. Once the condition (41) is failed, the corresponding solution  $(\theta^r, \eta^r)$  is optimal for problem (39), and, correspondingly,  $\eta^r$  is optimal for (12).

## 4 Numerical Experiments and Results

The performance of the proposed Benders decomposition algorithms was evaluated against CPLEX solver. All computations were conducted on a PC with 3.40GHz CPU and 32GB of RAM, running 64-bit Windows 10 environment. The algorithms presented in Section 3 have been coded in C++, and IBM CPLEX version 12.10 has been used to solve the complete mathematical programming formulations of the corresponding problems. Tables 1 through 4 report the computational running times of the corresponding solution methods for various network instances within a 2,000 second time limit. The best average running times are highlighted in bold, and symbol “—” indicates a failure to find an optimal solution within the prescribed time limit.

### 4.1 Resilient Maximum Network Flow Problem

The numerical studies of the solution methods for the resilient maximum network flow problem were conducted on network instances of order  $|V| = 96, 160, 500, 640, 960$ , and 1440 from the *MP-Testdata* repository [61], see [31] for details. The scenario sets of sizes  $|\Omega| = 3$  to 2000 for different problems were generated. In each scenario set, the stochastic capacity losses along the network’s arcs were generated as i.i.d. samples from the uniform  $U[0, 1]$  distribution multiplied by the arc’s original capacity.

It is also worthwhile to briefly comment on the choice of values of the parameter  $\alpha$  in our studies. During preliminary computational runs, we noticed that in some network instances the values of  $\alpha$  in the range of  $0 \leq \alpha \leq 0.3$  resulted in zero preallocations, meaning that the corresponding problem instances were less challenging and/or interesting comparing to those with nonzero preallocations at optimality. Hence, Tables 1 and 2 in this subsection report computational results for values  $\alpha \geq 0.4$ , which also applies to the results presented in the next subsection.

#### 4.1.1 Linear Resilient Maximum Network Flow Problem

In the case of the resilient maximum network flow problem with a linear utility function (30), both the single-cut and multi-cut Benders decomposition algorithms were implemented in C++, and their running times were compared against the runtimes of CPLEX solver used to solve the full mathematical programming formulations of the corresponding problem instances. Table 1 presents the mean computational times, in seconds, averaged over five randomly generated instances of a problem of the given size. It follows from the results in Table 1 that both the single-cut and multi-cut versions of the Benders decomposition method offer significant computational savings over the CPLEX solver. It is also worth noting that the multi-cut variant of the Benders decomposition algorithm mostly performs better than the single-cut version on instances with a small number of scenarios, and vice versa.

**Table 1:** Average running times (in seconds) obtained by using different solution methods for the resilient maximum network flow problem with a linear utility function (30). All running times are averaged over 5 instances, and symbol “—” indicates that an optimal solution could not be found within the time limit of 2000 seconds.

$ V $	$ \Omega $	$\alpha$	SINGLE-CUT	MULTI-CUT	CPLEX
98	3	0.4	<b>1</b>	<b>1</b>	14.2
	10	0.7	2	<b>1.8</b>	33.4
	50	0.95	9.4	<b>8.6</b>	143.2
	150	0.4	<b>25.8</b>	—	785.2
	1000	0.7	<b>172.8</b>	—	—
	2000	0.95	<b>381.4</b>	—	—
160	3	0.4	3	<b>2.8</b>	25.8
	10	0.7	9	<b>8.2</b>	47
	50	0.95	<b>26.6</b>	38.6	209
	150	0.4	<b>82.2</b>	82.4	813.2
	1000	0.7	<b>586</b>	—	—
500	3	0.4	42.8	<b>32.4</b>	162.6
	10	0.7	<b>64.8</b>	91.2	225.6
	50	0.95	<b>323.6</b>	505.6	902.8
	100	0.4	<b>703</b>	727.8	—
640	3	0.4	49.4	<b>47.8</b>	258.2
	10	0.7	151.2	<b>148.8</b>	339
	20	0.7	303.6	<b>299.8</b>	469
	100	0.95	<b>1575.2</b>	1599.6	—
960	3	0.4	88.2	<b>86.2</b>	628.6
	10	0.7	<b>270.6</b>	380.2	779.2
	20	0.95	1540	<b>906.8</b>	3210.4
	50	0.4	<b>1512.4</b>	—	—
1440	3	0.4	194.6	<b>177</b>	1840.8
	10	0.7	<b>658</b>	665.6	—
	20	0.95	1354	<b>1297.4</b>	—

#### 4.1.2 Nonlinear Resilient Maximum Network Flow Problem With a Power Utility Function

In order to illustrate the performance of the algorithm proposed in Section 3.2 for the nonlinear convex resilient maximum network flow problem, we consider formulation (12) with a power utility function  $u(t) = t^{1/p}$ ,  $t \geq 0$ , where  $p > 1$ . Clearly,  $u(t)$  in this case is an increasing concave utility function on  $t \geq 0$ , and the certainty equivalent  $u^{-1} \mathbb{E}u(\mathbf{a}_s \mathbf{x}(\omega))$  is then a concave function of  $\mathbf{x}$ . The resilient maximum flow model then becomes

$$\begin{aligned}
 Z_\alpha^*(\mathbf{X}) = \max \quad & -(1-\alpha)\mathbf{1}^\top \boldsymbol{\eta} + (\mathbb{E}[(\mathbf{a}_s^\top \mathbf{x}(\omega))^{1/p}])^p \\
 \text{s.t.} \quad & (12b), (12c), (12d).
 \end{aligned} \tag{43}$$

Under a finite scenario model,  $\Omega = \{\omega_1, \dots, \omega_N\}$ , where  $\pi_k = \mathbb{P}(\omega_k) > 0$ ,  $\sum_{k=1}^N \pi_k = 1$ , the above problem can be written in a mathematical programming form as follows

$$\begin{aligned}
Z_\alpha^*(\mathbf{X}) = \max \quad & -(1-\alpha)\mathbf{1}^\top \boldsymbol{\eta} + \theta \\
\text{s.t.} \quad & A\mathbf{x}_k = \mathbf{0}, \quad k = 1, \dots, N \\
& \theta \leq (y_1^{1/p} + \dots + y_N^{1/p})^p \\
& y_k = \pi_k^p \mathbf{a}_s^\top \mathbf{x}_k, \quad k = 1, \dots, N \\
& \mathbf{0} \leq \mathbf{x}_k \leq \mathbf{u} - (\mathbf{X}_j - \boldsymbol{\eta})_+, \quad k = 1, \dots, N \\
& \boldsymbol{\eta} \geq \mathbf{0}.
\end{aligned} \tag{44}$$

The corresponding formulation of the master problem then becomes

$$\begin{aligned}
\max \quad & -(1-\alpha)\mathbf{1}^\top \boldsymbol{\eta} + \theta \\
\text{s.t.} \quad & \theta \leq (y_1^{1/p} + \dots + y_N^{1/p})^p \\
& y_k \leq \pi_k^p \sum_{(i,j) \in M(\boldsymbol{\eta}^v, \omega_k)} (u_{ij} - (X_{ij}(\omega_k) - \eta_{ij})_+), \quad v = 1, \dots, (r-1), k = 1, \dots, N \\
& \boldsymbol{\eta} \geq \mathbf{0}.
\end{aligned} \tag{45}$$

Observe that the set

$$\left\{ (\theta, y_1, \dots, y_N) \in \mathbb{R}_+^{N+1} \mid \theta \leq (y_1^{1/p} + \dots + y_N^{1/p})^p \right\}$$

is a convex cone in  $\mathbb{R}_+^{N+1}$  for  $p \geq 1$ , and admits a lifted representation via  $N$  power cones:

$$w_k^p \leq y_k \theta^{p-1}, \quad k = 1, \dots, N, \quad \theta \leq w_1 + \dots + w_N.$$

Further, for each  $k = 1, \dots, N$  the power cones  $w_k^p \leq y_k \theta^{p-1}$  in the case of  $p = 2$  reduce to the so-called rotated second order cones

$$w_k^2 \leq y_k \theta, \quad k = 1, \dots, N,$$

and in the case of  $p = 3$  they can be represented by the rotated second order cones of the form

$$w_k^2 \leq v_k \theta, \quad v_k^2 \leq w_k y_k, \quad k = 1, \dots, N.$$

Since problems (44) and (45) can be formulated as second order cone programming (SOCP) problems for  $p = 2$  and  $p = 3$ , they can be solved using a number of off-the-shelf solvers, including CPLEX. In our numerical experiments, we compared the running times of solving the SOCP reformulation of problem (44) using CPLEX with the nonlinear decomposition algorithm proposed in Section 3.1, where the master problem (45) was likewise reformulated as a SOCP problem and solved by CPLEX. Table 2 presents the mean computational times, in seconds, averaged over five randomly generated instances of a problem of the given size. According to the results in Table 2, the nonlinear decomposition algorithm allows for substantial computational savings in comparison to solving the full mathematical programming formulation of the problem with the CPLEX solver.

## 4.2 Resilient Minimum Cost Network Flow Problem

Case studies of the computational approaches to the resilient minimum cost network flow problem involved network instances with  $|V| = 96, 200, 400, 500, 640, 960$  nodes from the same repository. Similarly to before, scenario sets for the arc capacity losses were generated with the number of scenarios  $|\Omega|$  ranging from 3 to 1000, where the arc capacity losses were obtained as  $U[0, 1]$  i.i.d. samples multiplied by the arc's original capacity value. The results of numerical experiments are reported in Table 3 for values of parameter  $\alpha = 0.4, 0.7, 0.95$ , where all running times are averaged over 5 randomly generated instances.

**Table 2:** Average running times (in seconds) obtained by using different solution methods for the nonlinear resilient maximum network flow problem with a power utility function,  $u(t) = t^{1/p}$ ,  $t \geq 0$ . All running times are averaged over 5 instances, and symbol “—” indicates that an optimal solution could not be found within the time limit of 2,000 seconds.

			$p = 2$		$p = 3$	
$ V $	$ \Omega $	$\alpha$	MULTI-CUT	CPLEX	MULTI-CUT	CPLEX
96	3	0.4	<b>0.8</b>	16.4	<b>0.8</b>	16.6
	10	0.7	<b>4.0</b>	41.2	<b>4.0</b>	40.0
	50	0.95	<b>13.4</b>	215.0	<b>14.2</b>	222.6
	150	0.4	<b>32.2</b>	966.4	<b>30.8</b>	872.2
	1000	0.7	<b>264.6</b>	—	<b>462.4</b>	—
	2000	0.95	<b>507.4</b>	—	<b>578.0</b>	—
160	3	0.4	<b>5.2</b>	36.8	<b>5.0</b>	36.0
	10	0.7	<b>17.2</b>	74.6	<b>15.0</b>	72.8
	50	0.95	<b>43.6</b>	375.8	<b>38.6</b>	399.2
	150	0.4	<b>144.6</b>	—	<b>90.2</b>	—
	1000	0.7	<b>602.2</b>	—	<b>595.6</b>	—
500	3	0.4	<b>36.6</b>	173.2	<b>34.8</b>	183.0
	10	0.7	<b>106.8</b>	237.6	<b>105.0</b>	235.4
	50	0.95	<b>344.0</b>	975.4	<b>355.2</b>	938.8
	100	0.4	<b>726.2</b>	—	<b>711.4</b>	—
640	3	0.4	<b>57.4</b>	304.2	<b>56.8</b>	281.4
	10	0.7	<b>161.2</b>	341.0	<b>159.4</b>	343.6
	20	0.7	<b>339.8</b>	487.6	<b>337.6</b>	514.2
	100	0.95	<b>1679.6</b>	—	<b>1775.0</b>	—
960	3	0.4	<b>502.8</b>	854.2	<b>493.0</b>	839.4
	10	0.7	<b>797.4</b>	1507.6	<b>775.8</b>	1497.2
	20	0.95	<b>1081.2</b>	—	<b>1035.2</b>	—
	50	0.4	<b>1746.8</b>	—	<b>1663.6</b>	—
1440	3	0.4	<b>380.0</b>	—	<b>379.4</b>	—
	10	0.7	<b>817.6</b>	—	<b>787.0</b>	—
	20	0.95	<b>1510.2</b>	—	<b>1393.8</b>	—

As it follows from the data in Table 3, the single-cut Benders algorithm still affords substantial computational advantage over the off-the-shelf solver CPLEX. But, unlike in the case of the resilient maximum network flow problem, the multi-cut version of the Benders method performs significantly worse than the single-cut variant, and arguably worse than CPLEX as it fails to reach an optimal solution except for the problems with smallest scenario sets.

Computational experiments on the nonlinear resilient minimum cost flow problem were conducted analogously to those involving the nonlinear resilient maximum network flow problem reported in the previous subsection. Namely, a power utility function was used,  $u(t) = t^p$ ,  $t \geq 0$ , and the corresponding problem was reformulated as a second-order conic programming problem in the cases of  $p = 2$  and  $p = 3$ , similarly to that of Section 4.1.2. A multi-cut Benders algorithm for the nonlinear resilient minimum cost network flow problem was constructed by analogy with the corresponding method described in Section 3.2, and tested on five randomly generated instances

**Table 3:** Average running times (in seconds) obtained by using different solution methods for the linear resilient minimum cost network flow problem. All running times are averaged over 5 instances, and symbol “—” indicates that an optimal solution could not be found within the time limit of 2000 seconds.

$ V $	$ \Omega $	$\alpha$	SINGLE-CUT	MULTI-CUT	CPLEX
96	3	0.4	<b>4.4</b>	<b>4.4</b>	21.2
	10	0.7	<b>15.2</b>	350.4	45.8
	50	0.95	<b>176.6</b>	—	232.6
	150	0.4	<b>611.8</b>	—	942
	500	0.7	<b>1328.6</b>	—	—
200	3	0.4	7	<b>5.4</b>	46.4
	10	0.7	13.2	<b>12.4</b>	119.8
	50	0.95	<b>80.8</b>	—	934.4
	150	0.4	<b>130.2</b>	—	—
	500	0.7	<b>534.4</b>	—	—
	1000	0.95	<b>1582</b>	—	—
400	3	0.4	<b>38</b>	99	199.6
	10	0.7	<b>148</b>	—	1475.4
500	3	0.4	<b>43.4</b>	55	548.4
	10	0.7	<b>91.2</b>	—	—
	50	0.95	<b>590</b>	—	—
	100	0.4	<b>1974.2</b>	—	—
640	3	0.4	<b>152.4</b>	207	—
	10	0.7	<b>372.2</b>	—	—
	40	0.95	<b>1985.2</b>	—	—
960	3	0.4	<b>177.2</b>	897.6	—
	10	0.7	<b>796.4</b>	—	—

for each combination of network size and size of scenario set. The computational results are presented in Table 4, and confirm that the proposed nonlinear Benders algorithm provides substantial advantages over using the CPLEX solver.

In general, it follows from the conducted numerical studies that the single-cut Benders decomposition approach outperforms its traditional multi-cut variant on linear resilient maximum network flow and minimum cost network flow problems, where the multiple cuts are scenario-based. In contrast, in nonlinear problems, the proposed multi-cut method is substantially more efficient than the traditional (single-cut) Benders decomposition method for general nonlinear programming problems.

### 4.3 Analysis of the Effects of Network’s Structure on Its Resilience

In this subsection we elucidate the effects of a network’s structure, or topology, on its resilience; in other words, we attempt to answer the question “are specific types of networks more resilient than others?” For concreteness, we employ the definition of resilience as given by the resilient maximum network flow formulation with a linear utility function (12). In order to obtain the most clear-cut answer on whether a specific network structure is more conducive to resilient operations, we conduct numerical experiments on several “idealized”, or “pure” versions of some known network topologies, including those of a uniform random graph, also known as Erdős-Renyi random

**Table 4:** Average running times (in seconds) obtained by using different solution methods for the nonlinear resilient minimum cost network flow problem with a power utility function,  $u(t) = t^p$ ,  $t \geq 0$ . All running times are averaged over 5 instances, and symbol “—” indicates that an optimal solution could not be found within the time limit of 2000 seconds.

$ V $	$ \Omega $	$\alpha$	$p = 2$		$p = 3$	
			MULTI-CUT	CPLEX	MULTI-CUT	CPLEX
96	3	0.4	<b>7.2</b>	22.8	<b>7.2</b>	23.4
	10	0.7	<b>21.4</b>	48.2	<b>22.6</b>	49.4
	50	0.95	<b>203.0</b>	251.8	<b>209.8</b>	256.0
	150	0.4	<b>828.6</b>	1023.6	<b>835.0</b>	1047.8
	500	0.7	<b>1612.4</b>	—	<b>1635.2</b>	—
200	3	0.4	<b>11.0</b>	50.4	<b>11.6</b>	50.8
	10	0.7	<b>25.8</b>	127.6	<b>27.0</b>	130.4
	50	0.95	<b>117.6</b>	984.0	<b>126.8</b>	996.2
	150	0.4	<b>268.2</b>	—	<b>283.4</b>	—
	500	0.7	<b>727.4</b>	—	<b>759.4</b>	—
	1000	0.95	<b>1853.6</b>	—	<b>1874.8</b>	—
400	3	0.4	<b>64.2</b>	209.8	<b>65.6</b>	214.6
	10	0.7	<b>196.4</b>	1629.4	<b>200.6</b>	1641.8
500	3	0.4	<b>68.8</b>	596.2	<b>70.0</b>	605.0
	10	0.7	<b>165.4</b>	—	<b>168.4</b>	—
	50	0.7	<b>943.6</b>	—	<b>1027.2</b>	—
640	3	0.4	<b>186.0</b>	—	<b>193.2</b>	—
	10	0.7	<b>430.6</b>	—	<b>439.8</b>	—
960	3	0.4	<b>316.8</b>	—	<b>320.4</b>	—
	10	0.7	<b>1294.2</b>	—	<b>1325.0</b>	—

graph, a scale-free graph topology, and a grid network structure. Real-life networks may combine features of all these topologies to a varying degree.

The Erdős-Rényi uniform random graph on the node set  $V$  is obtained by connecting each pair of nodes with an arc at a constant probability  $p \in (0, 1)$ , independently of other arcs in the graph [24]. The uniform random graphs, despite their theoretical significance, possess a binomial degree distribution, and as such do not provide good models for many real-life networked systems, which are known to contain several nodes of very high degree with the remaining nodes having relatively few neighbors. Such networks fit well the model of scale-free, or power law networks, i.e., networks whose degree distribution follows a power law. The well-known Barabasi-Albert preferential attachment algorithm [6] represents one of the primary methods for generating scale-free graphs. Grid networks are characterized by high clustering and a small average distance length between nodes. A grid network is typically constructed by partitioning nodes into a number of subsets, such that each node is only independently connected to nodes in the same subset or the neighboring subset.

In order to achieve the most “apples-to-apples” comparison of network flows in these types of networks, a special care was taken during generation of the network instances and selection of the pairs of source-sink nodes as described below.

The numerical studies of the resilience of the maximum network flow were conducted on randomly generated uniform (Erdős-Rényi), scale-free (Barabasi-Albert), and grid networks with 100 nodes and the same density of



about  $d = 0.1$ . To make sure that the instances of the resilient maximum network flow problem shown above that were solved on these network topologies were otherwise as similar as possible, the following procedure had been followed. First, a Barabasi-Albert random network was generated, its density computed, and a uniform random Erdős-Renyi network and a grid network were then generated with the same density. Arc capacities for each of the generated networks were sampled at random from the set  $\{1, \dots, 20\}$ .

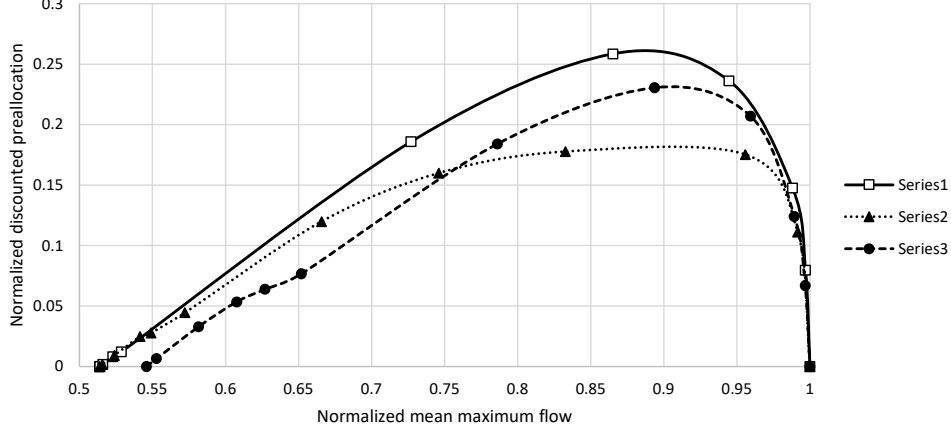
In order to choose the source-sink node pairs in each of the constructed networks that would provide for “most comparable” maximum network flow problem instances differing primarily in the underlying network structure, the following procedure was implemented. A maximum network flow problem was solved for every pair of nodes in a network, and among the pairs with highest flow throughput in all three networks we selected the source-sink pairs that had identical maximum network flow and the same distance between them, as given by the number of arcs in the shortest path. In other words, as a result of this selection procedure, the constructed problem instances had an identical value of the deterministic maximum network flow, and the same distance between the source and sink nodes (typically, 4 to 5), such that the differences in resilience of network flow among the three networks could be attributed primarily to the underlying network structure. The described procedure of constructing a triple of network instances was repeated five times to create a sample of 5 tuples, each containing randomly generated network instances with uniform, scale-free, and grid topologies.

Next, for each tuple of network instances we generated three scenario sets of capacity losses  $\mathbf{X}(\omega)$ . Namely, in the first scenario set, which we call “*independent*”, the capacity losses were generated as i.i.d. samples from the uniform  $U[0, 1]$  distribution multiplied by the deterministic capacity of the arc. The second scenario set models the circumstances in which the capacity losses in arcs increase proportionally with the degree of the arc’s origin node. In other words, if an arc leaves a high-degree node, it is likely to experience higher capacity losses than an arc leaving a low-degree node. We refer to this scenario set as “*increasing*”. Lastly, the third scenario set models an opposite situation, when an arc leaving a high-degree node is likely to sustain smaller capacity losses than an arc leaving a low-degree node; this scenario set is called “*decreasing*”. Each instance of problem (30), for every network topology and scenario of capacity losses, was solved with values of the discount factor  $\alpha \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.95, 1\}$ , and the obtained optimal values of the “discounted” capacity preallocations  $(1 - \alpha)\mathbf{1}^\top \boldsymbol{\eta}^*$  and the mean network flow  $\mathbb{E}[\mathbf{a}_s^\top \mathbf{x}^*(\omega)]$  were normalized by the value of deterministic maximum flow in the network corresponding to the case of  $\alpha = 1$  (see Corollary 2.3). Figures 1–3 present the obtained results, averaged over five randomly generated instances.

Namely, Figure 1 displays resilience characteristics of networks through the relationship between the normalized optimal discounted amount of preallocated capacity and the normalized mean network flow in the case when the capacity losses are i.i.d. and do not depend on the node degree (scenario set “*independent*”). In accordance to Proposition 2.2, the maximum mean flow in all three networks increases with  $\alpha$ ; therefore the data points in the graphs correspond to the values of  $\alpha = 0, 0.1, \dots, 0.95, 1$  in an ascending order when viewed from left to right. In such a way, for instance, the leftmost datapoints correspond to  $\alpha = 0$  and, per Corollary 2.4, represent networks with no flow resilience. Recall that  $\alpha = 0$  means there is no incentive for preallocating resources in advance, in which case zero preallocation  $\boldsymbol{\eta}^* = \mathbf{0}$  is optimal, and the value of the maximum mean network flow is smallest, by virtue of Proposition 2.2. Respectively, the rightmost datapoints correspond to the value of  $\alpha = 1$ , a 100% discount on capacity preallocations, and represent ideally resilient networks, since, by Corollary 2.3, the mean optimal flow is restored to its deterministic pre-disruption value, at zero cost to the decision maker (while the optimal preallocation  $\boldsymbol{\eta}^*$  is nonzero, the discounted preallocation  $(1 - \alpha)\mathbf{1}^\top \boldsymbol{\eta}^* = 0$  for  $\alpha = 1$ ).

The graphs in Figure 1 show that for values of discount factor from  $\alpha = 0$  to about  $\alpha = 0.6$  the mean maximum flow in the networks with uniform, scale-free, and grid topologies grows approximately proportionately to discounted preallocation amounts, and at approximately the same rate for all three types of network structure. The grid network structure appears to be the most resilient within this range of discount factor values, as it allows for higher values of expected maximum flow at lower preallocation amounts. For values of discount factor  $\alpha$  greater than 0.7, the scale-free network structure exhibits most resilience; in particular, it can be seen that for a certain range of discount factor values, a significant increase in the expected maximum flow can be achieved with very little increase in discounted preallocated capacity amount.

Figure 2 presents the dependency between the discounted preallocation capacity amounts and expected maximum



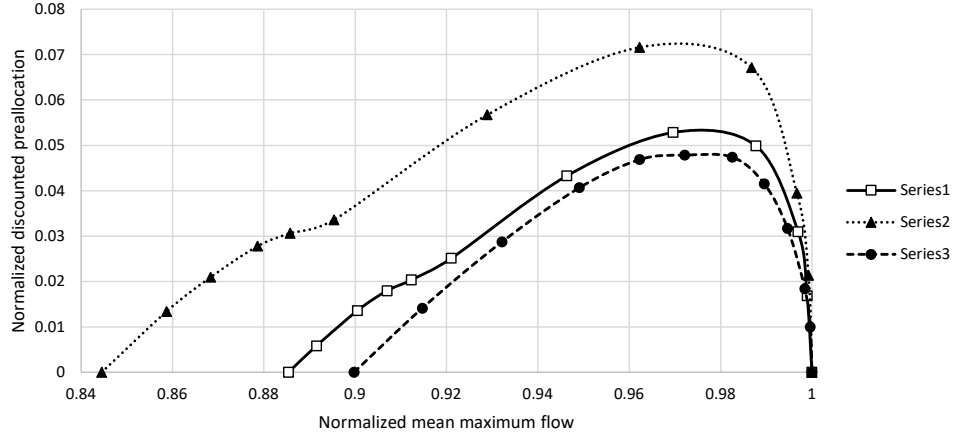
**Figure 1:** The normalized mean maximum flow and discounted capacity preallocation amounts in networks of different topologies under the assumption that capacity losses do not depend on the degrees of nodes. The data points, indicated by the squares, triangles, and circles, correspond to values of the discount factor  $\alpha = 0, 0.1, \dots, 0.95, 1$  in the ascending order from left to right.

network flow in the case when capacity losses are likely to be larger in the arcs that leave high-degree nodes, i.e., under scenario set “*increasing*”. Under such circumstances, clearly, the scale-free, or power-law topology of the network is the least resilient, in the sense that larger discounted preallocated capacity amounts are required for achieving a partial restoration of the mean network flow, as graphs in Figure 2 show. This outcome is not unexpected, since, in accordance with the power-law degree distribution, the scale-free network sustains higher cumulative capacity losses comparing to networks of other topologies. The network topology that is most conducive to high resilience of the maximum network flow in this case is the grid topology.

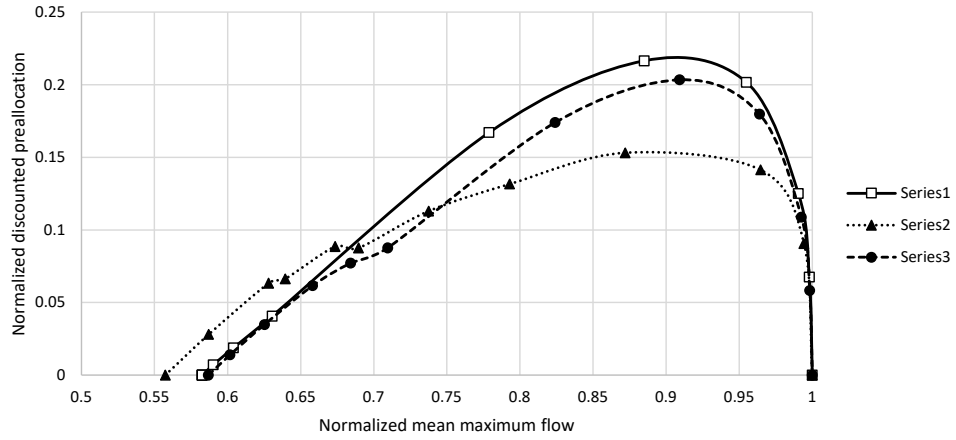
Finally, the case when random capacity losses are likely to be lower in arcs leaving high-degree nodes (scenario set “*decreasing*”) is presented in Figure 3. Under this scenario set, the patterns are similar to those observed under “independent” scenario set in Figure 1. Namely, scale-free network topology appears to be more conducive to achieving high resiliency when the value of the discount factor  $\alpha$  is around 0.7 or higher. For values of  $\alpha$  between approximately 0.3 and 0.7, the grid network topology is the most conducive to resilience of network flows; for values of the discount factor between 0 and 0.3, both the uniform and grid network structures appear to result in the same degree of resiliency.

## 5 Conclusions

In this work we have proposed an approach to constructing metrics of network resilience, and illustrated it on the examples of the resilient maximum network flow problem and the resilient minimum cost network flow problem. Specifically, the network flows in these problems are designed for resilience against unpredictable losses of network arc capacities, and the mechanism of attaining a degree of resilience is through preallocation of resources to (at least partially) restoring the capacities of the arcs. It is shown that the resulting formulations of resilient network flow problems possess a number of useful properties; for instance, similarly to the standard network flow problems, the solution is integral if the arc capacities, costs, etc., are integral. It is also shown that the proposed formulations of resilient network flow problems can be viewed as “network measures of risk”, similar in properties and behavior to convex measures of risk. Computationally efficient decomposition algorithms have been proposed for both the resilient maximum network flow problem and the resilient minimum cost network flow problem, and a study of the effects of network structure on resiliency of maximum network flow has been conducted on networks with three types of topology: that of uniform random graphs, scale-free graphs, and grid graphs.



**Figure 2:** The normalized mean maximum flow and discounted capacity preallocation amounts in networks of different topologies under the assumption that capacity losses are likely to be higher in arcs that leave high-degree nodes. The data points, indicated by the squares, triangles, and circles, correspond to values of the discount factor  $\alpha = 0, 0.1, \dots, 0.95, 1$  in the ascending order from left to right.



**Figure 3:** The normalized mean maximum flow and discounted capacity preallocation amounts in networks of different topologies under the assumption that capacity losses are likely to be lower in arcs that leave high-degree nodes. The data points, indicated by the squares, triangles, and circles, correspond to values of the discount factor  $\alpha = 0, 0.1, \dots, 0.95, 1$  in the ascending order from left to right.

## 6 Acknowledgements

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