CSc 144 — Discrete Mathematics for Computer Science I

# Sample First Midterm Exam — The Answers

## Background and Advice

Discrete Math students seem to appreciate seeing what one of my exams looks like before they take the first one. I think that reducing anxiety is a good thing, so here's a sample first midterm for you to use to help prepare for the first midterm. I suggest that you treat this as if it were the actual exam. That is, **do not** look at it until you have studied all of the topics, and then do it in the same setting as we use for midterms: A time period of 50 minutes with no book, no notes, and no calculator. Only when you are finished (or out of time!) should you download and look at the sample answers, and compare them to your own. (Need a timer? Try typing `set timer 50 minutes` into Google.)

The actual exam will have a dedicated cover page with directions, rather than all of this background, and we'll let you know when you can open the exam and get started. On both the actual exam and this practice exam, the Page O' Logical Equivalences is provided on the last page. And, yes, you can tear it off of the exam to make it easier to use.

Please remember: What you see here is just a sample set of questions. On the actual exam, you will see different questions, some on different topics, phrased in different ways, and presented in different formats. If you are expecting the actual exam to be only a slight variation of this sample exam, you will be disappointed. Study *everything* we've covered!

Finally: This is the only sample exam I will provide this semester. After seeing this (and the real first midterm exam), you'll have a very good idea of what to expect on future exams.

1. (11 points) Math Review

   (a) What is the symbol for the set of positive integers?

   $\mathbb{Z}^+$. $\mathbb{N}^*$ is also acceptable.

   (b) True or False: 14 is a rational number.

   **True.** Because we can write 14 as a ratio of integers ($\frac{14}{1}$), it is rational.

   (c) Evaluate: $\sum_{w=-4}^{2} \frac{w}{2}$

   Observe that the values of $w$ straddle 0. Thus, $\sum_{w=-2}^{2} \frac{w}{2} = 0$, and only the terms $w = -4$ and $w = -3$ matter. $\frac{-4}{2} + \frac{-3}{2} = \frac{-7}{2}$.

   (d) Create an example that shows subtraction does not distribute over multiplication.

   Here's one example: $4 - (6 * 2) = -8$, but $(4 - 6) * (4 - 2) = -4$.

   (e) Simplify: $\log_2((\frac{8}{32})^2)$

   $\log_2((\frac{8}{32})^2) = 2 * \log_2(\frac{8}{32}) = 2 * \log_2(\frac{1}{4}) = 2 * \log_2(2^{-2}) = 2 * -2 = -4$

   Or: $\log_2((\frac{8}{32})^2) = 2 * \log_2(\frac{8}{32}) = 2 * (\log_2(2^3) - \log_2(2^5)) = 2 * (3 - 5) = -4$

2. (21 points) More Math Review!

(a) What is $443_{10}$ in both binary and hexadecimal?

> $110111011_2$ and $1BB_{16}$

(b) What is $12A_{11}$ in Base 7?

> $12A_{11}$ is equal to $153_{10}$ $(1 * (11^2) + 2 * (11^1) + 10 * (11^0) = 121 + 22 + 10 = 153)$. In Base 7, the last three position values are 49, 7 and 1. Three 49s is 147, which is 6 shy of 153. So, we need 3 49s, 0 7s, and 6 1s to total the same amount; $12A_{11} = 306_7$.

(c) Simplify to a fraction of the form $\frac{a}{b}$: $1 + \frac{2}{3+\frac{4}{5}}$.

> $1 + \frac{2}{3+\frac{4}{5}} = 1 + \frac{2}{\frac{19}{5}} = 1 + \frac{10}{19} = \frac{29}{19}$

(d) Let $A = \{b, e, f, g\}$, $B = \{a, c, d, f, g, h\}$, and $C = \{b, d, g\}$, over the universe $\mathcal{U} = \{a, b, c, d, e, f, g, h\}$. What are the results of the evaluations of the expressions $C - (A \cap B)$ and $(\overline{A} \cap C) \cup B$?

> $C - (A \cap B) = \{b, d\}$, and $(\overline{A} \cap C) \cup B = \{a, c, d, f, g, h\} = B$.

(e) Evalute: $\prod_{a=3}^{3} \sum_{b=10}^{12} (2a + b)$

> $\prod_{a=3}^{3} \sum_{b=10}^{12} (2a+b) = \prod_{a=3}^{3} ((2a+10)+(2a+11)+(2a+12)) = \prod_{a=3}^{3} (6a+33) = 6(3)+33 = 51.$

(f) Evaluate all three expressions: $7 \mid 15$, $7 \setminus 15$, and $7 \,\%\, 15$.

> $7 \mid 15$ is false; 7 does not divide 15 evenly. '\' is integer division; $7 \setminus 15 = 0$, because we can't get any 15s out of 7. $7 \,\%\, 15 = 7$; we can't get any 15s out of 7, so the entire 7 is left over.

3. (3 points) Define: **Compound Proposition**

> "A statement that is a logical combination of multiple simple propositions."

4. (6 points) Is $\overline{p \to q} \equiv p \wedge \overline{q}$? Construct a truth table to determine your answer (and don't forget to tell us what your answer is!).

| $p$ | $q$ | $p \to q$ | $\overline{p \to q}$ | $\overline{q}$ | $p \wedge \overline{q}$ |
|---|---|---|---|---|---|
| T | T | T | $\boxed{\text{F}}$ | F | $\boxed{\text{F}}$ |
| T | F | F | $\boxed{\text{T}}$ | T | $\boxed{\text{T}}$ |
| F | T | T | $\boxed{\text{F}}$ | F | $\boxed{\text{F}}$ |
| F | F | T | $\boxed{\text{F}}$ | T | $\boxed{\text{F}}$ |

> Because $\overline{p \to q}$ and $p \wedge \overline{q}$ evaluate to the same outcomes for all possible inputs, yes, they are equivalent.

5. (6 points) Without using a truth table, show that $\overline{p \vee q} \equiv \overline{p \vee (\overline{p} \wedge q)}$ Remember to justify each step.

Starting with the most complex side, working toward the other side:

$$\begin{aligned} \overline{p \vee (\overline{p} \wedge q)} &\equiv \overline{(p \vee \overline{p}) \wedge (p \vee q)} && \text{[Distribution Law]} \\ &\equiv \overline{\mathbf{T} \wedge (p \vee q)} && \text{[Negation Law]} \\ &\equiv \overline{p \vee q} && \text{[Identity Law]} \end{aligned}$$

You could also answer this using reasoning, but that would be more difficult, given the complexity of the expressions. (But not bad; try it! Show that both sides are the same when $p$ is true, and again when $p$ is false.)

6. (2 points) What must be done to prove $\forall x P(x), x \in D$ is true, for a given propositional function $P(x)$ and domain $D$?

It must be shown that $P(x)$ is true for each $x \in D$.

7. (12 points) Convert the following English statements into logical notation in terms of the operators, symbols, and quantifiers used in class. Be sure to identify your propositions and predicates; domains are provided.

(a) Someone is taller than somebody. (domain: people)

Let $T(x, y) : x$ is taller than $y$, where $x, y \in$ People.
Answer: $\exists x\ \exists y\ T(x, y)$, where $x, y \in$ People.

(b) There is no free lunch. (domain: meals)

Let $F(x) : x$ is free, and let $L(x) : x$ is lunch, where $x \in$ Meals.
One Answer: $\neg\exists x\ (F(x) \wedge L(x))$, where $x \in$ Meals. You could apply various forms of De Morgan's Laws to create others.

(c) "The only consistent people are the dead." (domain: Creatures)

Let $D(x) : x$ is dead, $C(x) : x$ is consistent, and $P(x) : x$ is a person, all with $x \in$ Creatures.
Answer: $\forall x\ [(C(x) \wedge P(x)) \rightarrow D(x)]$, where $x \in$ Creatures.

(The quote is from Aldous Huxley.)

8. (14 points) Convert the following logical expressions to <u>conversational</u> English sentences. Assume that:

$B(\alpha)$ : $\alpha$ is a hummingbird
$L(\alpha)$ : $\alpha$ is large
$H(\alpha)$ : $\alpha$ eats honey
$C(\alpha)$ : $\alpha$ is brightly colored

(a) $\overline{L(Tweety)} \oplus C(Tweety)$

> Tweety is either not large or is brightly colored, but not both.

(b) $\exists y(B(y) \wedge \overline{L(y)})$, $y \in$ animals

> Some hummingbirds are small.

(c) $\forall z(\overline{H(z)} \rightarrow \overline{C(z)})$, $z \in$ animals

> Animals that do not eat honey are not brightly colored.

(d) $\forall w(\overline{L(w)} \vee \overline{H(w)})$, $w \in$ birds

> No large birds eat honey. (It helps to apply various forms of De Morgan's.)

9. (8 points) Consider this English sentence, and assume a domain of "Birds" : THERE IS A BIRD THAT IS LARGE, AND IS A HUMMINGBIRD, BUT IS DRAB.

(a) Using the same set of predicates from the previous question, express that sentence as an existentially quantified logical statement.

> Remembering that "but" is logically the same as "and", and knowing that "drab" means "not colorful", we can produce: $\exists x\,(L(x) \wedge B(x) \wedge \neg C(x))$, where $x \in$ Birds.

(b) Express the same sentence in logic, but this time use a quantified logical statement that starts with " $\neg\forall x$".

> This is a challenge. You can try to construct the answer in English directly, but starting with a "not" often confuses people. Another way to do it is to remember that, by Generalized De Morgan's Laws, $\neg\forall x P(x) \equiv \exists x\,\neg P(x)$. If we can express the $L(x) \wedge B(x) \wedge \neg C(x)$ section of the answer to part (a) in a negated form (that is, something of the form "$\neg P(x)$"), we can apply that version of Generalized De Morgan's and get our desired answer form.
>
> Remembering that implication goes well with $\forall$, we look to convert from a group of ANDs to an implication. Table 3 Line h says that $p \wedge \neg q \equiv \neg(p \rightarrow q)$. Letting $p$ represent $L(x) \wedge B(x)$, we see that $(L(x) \wedge B(x)) \wedge \neg C(x) \equiv \neg((L(x) \wedge B(x)) \rightarrow C(x))$. That gives us the "$\neg P(x)$" form we want for $\exists x\,\neg P(x)$. Applying Generalized De Morgan's, we get our final version: $\neg\forall x((L(x) \wedge B(x)) \rightarrow C(x))$, where $x \in$ Birds. In 'stilted' English: It isn't true that, considering all birds, if a bird is a large hummingbird, then it is brightly colored. Because that statement isn't true, there must be a drab large hummingbird.

10. (3 points) Fill in the Blanks! We know that there are at least three pairs of terms for $p$ and $q$ in the implication $p \to q$. Write in the missing name of each pair below.

(a) $p$ is the antecedent and $q$ is the consequent.

(b) $p$ is the hypothesis and $q$ is the conclusion.

(c) $p$ is sufficient and $q$ is necessary.

11. (3 points) Define: **Logical Equivalence** (We covered two definitions in class; either is acceptable.)

Our first definition: "Two propositions $p$ and $q$ are *(Logically) Equivalent* (written $p \equiv q$) when both evaluate to the same result when presented with the same input.

Our second defintion: "$p$ and $q$ are *(Logically) Equivalent* (written $p \equiv q$) if $p \leftrightarrow q$ is a tautology."

12. (6 points) Expression the statement "Exactly two prime numbers are less than five" in logic, using appropriate predicate(s) and domain(s).

We need to express "at least two" and add to it the limitation that there are no more than two.

Let $P(x)$ : $x$ is a prime number, and $L(x)$ : $x$ is less than five, where $x \in \mathbb{Z}^+$ for both (because primes are positive integers).

$\exists x \exists y (P(x) \wedge L(x) \wedge P(y) \wedge L(y) \wedge (x \neq y) \wedge \forall z[(P(z) \wedge L(z)) \to (z = x \vee z = y)])$.

CSc 144 — Discrete Mathematics for Computer Science I
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# The Page O' Logical Equivalences ("POLE")

## Table I: Some Equivalences using AND ($\wedge$) and OR ($\vee$):

| | | |
|---|---|---|
| (a) | $p \wedge p \equiv p$<br>$p \vee p \equiv p$ | Idempotent Laws |
| (b) | $p \wedge \mathbf{F} \equiv \mathbf{F}$<br>$p \vee \mathbf{T} \equiv \mathbf{T}$ | Domination Laws |
| (c) | $p \wedge \mathbf{T} \equiv p$<br>$p \vee \mathbf{F} \equiv p$ | Identity Laws |
| (d) | $p \wedge q \equiv q \wedge p$<br>$p \vee q \equiv q \vee p$ | Commutative Laws |
| (e) | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$<br>$(p \vee q) \vee r \equiv p \vee (q \vee r)$ | Associative Laws |
| (f) | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$<br>$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | Distributive Laws |
| (g) | $p \wedge (p \vee q) \equiv p$<br>$p \vee (p \wedge q) \equiv p$ | Absorption Laws |

## Table II: Some More Equivalences (adding Negation ($\neg$)):

| | | |
|---|---|---|
| (a) | $\neg(\neg p) \equiv p$ | Double Negation |
| (b) | $p \wedge \neg p \equiv \mathbf{F}$<br>$p \vee \neg p \equiv \mathbf{T}$ | Negation Laws |
| (c) | $\neg(p \wedge q) \equiv \neg p \vee \neg q$<br>$\neg(p \vee q) \equiv \neg p \wedge \neg q$ | De Morgan's Laws |

## Table III: Still More Equivalences (adding Implication ($\rightarrow$)):

| | | |
|---|---|---|
| (a) | $p \rightarrow q \equiv \neg p \vee q$ | Law of Implication |
| (b) | $p \rightarrow q \equiv \neg q \rightarrow \neg p$ | Law of the Contrapositive |
| (c) | $\mathbf{T} \rightarrow p \equiv p$ | "Law of the True Antecedent" |
| (d) | $p \rightarrow \mathbf{F} \equiv \neg p$ | "Law of the False Consequent" |
| (e) | $p \rightarrow p \equiv \mathbf{T}$ | Self-implication (a.k.a. Reflexivity) |
| (f) | $p \rightarrow q \equiv (p \wedge \neg q) \rightarrow \mathbf{F}$ | Reductio Ad Absurdum |
| (g) | $\neg p \rightarrow q \equiv p \vee q$ | |
| (h) | $\neg(p \rightarrow q) \equiv p \wedge \neg q$ | |
| (i) | $\neg(p \rightarrow \neg q) \equiv p \wedge q$ | |
| (j) | $(p \rightarrow q) \vee (q \rightarrow p) \equiv \mathbf{T}$ | Totality |
| (k) | $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$ | Exportation Law (a.k.a. Currying) |
| (l) | $(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r)$ | |
| (m) | $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$ | |
| (n) | $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$ | |
| (o) | $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$ | |
| (p) | $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \rightarrow r)$ | Commutativity of Antecedents |

## Table IV: Yet More Equivalences (adding Exclusive OR ($\oplus$) and Biimplication ($\leftrightarrow$)):

| | | |
|---|---|---|
| (a) | $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ | Definition of Biimplication |
| (b) | $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ | |
| (c) | $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ | |
| (d) | $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$ | Definition of Exclusive Or |
| (e) | $p \oplus q \equiv \neg(p \leftrightarrow q)$ | |
| (f) | $p \oplus q \equiv p \leftrightarrow \neg q \equiv \neg p \leftrightarrow q$ | |

**You may tear off this page if you wish!**

Notes:

1. $p$, $q$, and $r$ represent arbitrary logical expressions. They may represent equivalent expressions (e.g., if $p \equiv q$, then by Absorption $p \wedge (p \vee p) \equiv p$).
2. $\mathbf{T}$ and $\mathbf{F}$ represent the logical values True and False, respectively.
3. These tables show many of the common and/or useful logical equivalences; this is not an exhaustive collection!