Collected Definitions for Exam #3

This is the ‘official’ collection of need-to-know definitions for Exam #3. My pledge to you: If I ask you for a definition on the exam, the term will come from this list. Note that this is not a complete list of the definitions given in class. You should know the others, too, but I won’t specifically ask you for their definitions on the exam.

**Topic 8: Relations**

- The **inverse** of a relation \( R \), denoted \( R^{-1} \), contains all of the ordered pairs of \( R \) with their components exchanged. (That is, \( R^{-1} = \{(b, a) \mid (a, b) \in R\} \).
- Let \( G \) be a relation from set \( A \) to set \( B \), and let \( F \) be a relation from \( B \) to set \( C \). The **composite** of \( F \) and \( G \), denoted \( F \circ G \), is the relation of ordered pairs \((a, c), a \in A, c \in C \), such that \( b \in B, (a, b) \in G \), and \((b, c) \in F \).
- A relation \( R \) on set \( A \) is an **equivalence relation** if it is reflexive, symmetric, and transitive.
- A relation \( R \) on set \( A \) is a (reflexive/weak) **partial order** if it is reflexive, antisymmetric, and transitive.
- A relation \( R \) on set \( A \) is **irreflexive** if, for all members of \( A \), \((a, a) \notin R \).
- A relation \( R \) on set \( A \) is an (irreflexive/strict) **partial order** if it is irreflexive, antisymmetric, and transitive.
- Let \( R \) be a weak partial order on set \( A \). \( a \) and \( b \) are said to be **comparable** if \( a, b \in A \) and either \( a \preceq b \) or \( b \preceq a \) (that is, \((a, b) \in R \) or \((b, a) \in R \)).
- A weak partially-ordered relation \( R \) on set \( A \) is a **total order** if every pair of elements \( a, b \in A \) are comparable.

**Topic 9: Functions**

- A **function** from set \( X \) to set \( Y \), denoted \( f : X \to Y \), is a relation from \( X \) to \( Y \) such that \( f(x) \) is defined \( \forall x \in X \) and, for each \( x \in X \), there is exactly one \((x, y) \in f \).
- For each of the following, let \( f : X \to Y \) be a function, and assume \( f(n) = p \).
  - \( X \) is the **domain** of \( f \); \( Y \) is the **codomain** of \( f \).
  - \( f \) maps \( X \) to \( Y \).
  - \( p \) is the **image** of \( n \); \( n \) is the **pre-image** of \( p \).
  - The **range** of \( f \) is the set of all images of elements of \( X \). (Note that the range need not equal the codomain.)
- The **floor** of \( n \), denoted \( \lfloor n \rfloor \), is the largest integer \( \leq n \).
- The **ceiling** of a value \( m \), denoted \( \lceil m \rceil \), is the smallest integer \( \geq m \).
- A function \( f : X \to Y \) is **injective** (a.k.a. **one-to-one** if, for each \( y \in Y \), \( f(x) = y \) for at most one member of \( X \).
- A function \( f : X \to Y \) is **surjective** (a.k.a. **onto**) if \( f \)’s range is \( Y \) (the range = the codomain).
- A **bijective** function (a.k.a. a **one-to-one correspondence**) is both injective and surjective.
- The **inverse** of a bijective function \( f \), denoted \( f^{-1} \), is the relation \( \{(y, x) \mid (x, y) \in f\} \).
- Let \( f : Y \to Z \) and \( g : X \to Y \). The **composition** of \( f \) and \( g \), denoted \( f \circ g \), is the function \( h = f(g(x)) \), where \( h : X \to Z \).
- A function \( f : X \times Y \to Z \) (or \( f(x, y) = z \)) is a **binary** function.

(Continued . . .)
Topic 10: Properties of Integers

- Let \( i \) and \( j \) be positive integers. \( j \) is a factor of \( i \) when \( i \% j = 0 \).
- A positive integer \( p \) is prime if \( p \geq 2 \) and the only factors of \( p \) are 1 and \( p \).
- A positive integer \( p \) is composite if \( p \geq 2 \) and \( p \) is not prime.
- Let \( x \) and \( y \) be integers such that \( x \neq 0 \) and \( y \neq 0 \). The Greatest Common Divisor (GCD) of \( x \) and \( y \) is the largest integer \( i \) such that \( i \mid x \) and \( i \mid y \). That is, \( \gcd(x,y) = i \).
- If the GCD of \( a \) and \( b \) is 1, then \( a \) and \( b \) are relatively prime.
- When the members of a set of integers are all relatively prime to one another, they are pairwise relatively prime.
- Let \( x \) and \( y \) be positive integers. The Least Common Multiple (LCM) of \( x \) and \( y \) is the smallest integer \( s \) such that \( x \mid s \) and \( y \mid s \). That is, \( \text{lcm}(x,y) = s \).

Topic 11: Sequences and Strings

- A sequence is the ordered range of a function from a set of integers to a set \( S \).
- In an arithmetic sequence (a.k.a. arithmetic progression) \( a, a_n+1-a_n \) is constant. This constant is called the common difference of the sequence.
- In a geometric sequence (a.k.a. geometric progression) \( g, \frac{g_{n+1}}{g_n} \) is constant. This constant is called the common ratio of the sequence.
- An increasing (a.k.a. non-decreasing) sequence \( i \) is ordered such that \( i_n \leq i_{n+1} \).
- A strictly increasing sequence \( i \) is ordered such that \( i_n < i_{n+1} \).
- A non-increasing (a.k.a. decreasing) sequence \( i \) is ordered such that \( i_n \geq i_{n+1} \).
- A strictly decreasing sequence \( i \) is ordered such that \( i_n > i_{n+1} \).
- Sequence \( x \) is a subsequence of sequence \( y \) when the elements of \( x \) are found within \( y \) in the same relative order.
- A string is a contiguous finite sequence of zero or more elements drawn from a set called the alphabet.
- A set is finite if there exists a bijective mapping between it and a set of cardinality \( n, n \in \mathbb{Z}^* \).
- A set is countably infinite (a.k.a. denumerably infinite) if there exists a bijective mapping between the set and either \( \mathbb{Z}^* \) or \( \mathbb{Z}^+ \).
- A set is countable if it is either finite or countably infinite. If neither, the set is uncountable.

Topic 12: Induction

- The First Principle of Mathematical Induction: if (i) \( P(a) \) is true for the starting point \( a \in \mathbb{Z}^+ \), and (ii) if \( P(k) \) is true for any \( k \in \mathbb{Z}^+ \), then \( P(k+1) \) is true, then \( P(n) \) is true for all \( n \in \mathbb{Z}^+, n \geq a \).
- The Second Principle of Mathematical Induction: if (i) \( P(a) \) is true for the starting point \( a \in \mathbb{Z}^+ \), and (ii) (for any \( k \in \mathbb{Z}^+ \)) if \( P(j) \) is true for any \( j \in \mathbb{Z}^+ \) such that \( a \leq j \leq k \), then \( P(k+1) \) is true, then \( P(n) \) is true for all \( n \in \mathbb{Z}^+, n \geq a \).