Propositions With Variables (1 / 2)

Propositions are static; variables are not allowed. But . . .

**Definition:** Predicate (a.k.a. Propositional Function)

Example(s):
Propositions With Variables (2 / 2)

Definition: Domain (a.k.a. Universe) of Discourse

Example(s):

Quantification

Idea: Establish truth of predicates over sets of values.

Two common generalizations:

Note: Do not use the book's non-standard $\exists!x$ notation.
1. Universally Quantified Predicates

Example(s):

2. Existentially Quantified Predicates

Example(s):
Evaluating Mixed Quantifications (1 / 2)

First: Distinguishing $\exists x \forall y S(x, y)$ from $\forall i \exists k T(i, k)$:

Evaluating Mixed Quantifications (2 / 2)

Example(s):
Example: Universal Quantification (1 / 5)

Consider this conversational English statement:

All of McCann’s students are geniuses.

How can we express that statement in logic notation?

Example: Universal Quantification (2 / 5)

Attempt #2: All of McCann's students are geniuses. → Logic
Example: Universal Quantification (3 / 5)

Attempt #3: All of McCann’s students are geniuses. → Logic

Let $P(x) : \text{Student } x \text{ is a genius, } x \in \text{People}$

Example: Universal Quantification (4 / 5)

Attempt #4: All of McCann’s students are geniuses. → Logic

Let $P(x) : \text{Student } x \text{ is a genius, } x \in \text{People}$

Let $M(x) : x \text{ is enrolled in one of McCann’s classes, } x \in \text{People}$
Example: Universal Quantification (5 / 5)

Attempt #5: All of McCann’s students are geniuses. → Logic

Let \( P(x) \) : Student \( x \) is a genius, \( x \in \text{People} \)

Let \( M(x) \) : \( x \) is enrolled in one of McCann’s classes, \( x \in \text{People} \)

Implicit Quantification

The “all” can be implicit in the English statement.

Example(s):
Example: Existential Quantification

Consider this conversational English statement:

At least one towel is dirty.

How can we express that statement in logic notation?

Another Example: Existential Quantification

Express this more specific statement in logic:

Some of the blue guest towels are dirty.

Let $D(x) : x$ is dirty, $x \in$ Towels
Yet Another Example: Quantification

Now express this statement in logic:

Every last one of the blue guest towels are dirty!

Let $B(x) : x$ is blue, $x \in$ Towels
Let $G(x) : x$ is used by guests, $x \in$ Towels
Let $D(x) : x$ is dirty, $x \in$ Towels

Free vs. Bound Variables

Definition: Bound Variable

Definition: Free (a.k.a. Unbound) Variable

Other examples of ‘binding’ in CS:
Demonstration: $\forall x P(x) \equiv \exists x \overline{P(x)}$ (1 / 2)

Let $S(x) : x < 4, x \in \mathbb{Z}$

The expression $\forall x S(x), x \in \{1, 2, 3\}$ is true.

Equivalently, $\forall x S(x)$ is false.

$\forall x S(x) \equiv S(1) \land S(2) \land S(3)$  so …

$\forall x S(x) \equiv \overline{S(1) \land S(2) \land S(3)}$

$\equiv \overline{S(1)} \lor \overline{S(2)} \lor \overline{S(3)}$  [De Morgan, 2x]

(Remember: $\overline{S(1)} \lor \overline{S(2)} \lor \overline{S(3)}$ is still false.)

(Continues …)
Demonstration: $\forall x P(x) \equiv \exists x P(x)$

For $S(1) \lor S(2) \lor S(3)$ to be false, each term must be false; that is, no $S(x)$ is true (or all $S(x)$ are false).

It follows that the expression $\exists x S(x)$ must be false, completing the demonstration.

Example(s):

Expressing “Exactly one . . .” Statements (1 / 3)

Consider this conversational (& correct!) English statement:

Only one citizen of North Dakota is a member of the U.S. House of Representatives.

And consider this awkward but useful rewording:
Expressing “Exactly one . . .” Statements (2 / 3)

That rewording is useful because it can be directly expressed logically:

Expressing “Exactly one . . .” Statements (3 / 3)

A lingering problem:

The domain ("Citizens of North Dakota") is too specific.

Solution: Add a predicate . . . but what, and where?
Expressing “Exactly two . . .” Statements (1 / 3)

Key observation:

**Question:** Can you say this in ‘awkward English’?

Exactly two citizens of North Dakota are U.S. Senators.

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Expressing “Exactly two . . .” Statements (2 / 3)

Consider the two halves separately:

1. “At least two citizens of North Dakota are U.S. Senators”

2. “At most two citizens of North Dakota are U.S. Senators”
Finally, AND together

\[ \exists x \exists y (S(x) \wedge S(y) \wedge (x \neq y)) \]

and

\[ \forall x \forall y \forall z ((S(x) \wedge S(y) \wedge S(z)) \rightarrow (x = y \lor y = z \lor x = z)) : \]