Propositions With Variables (1 / 2)

Propositions are static; variables are not allowed. But . . .

**Definition:** Predicate (a.k.a. Propositional Function)

Example(s):
Propositions With Variables (2 / 2)

Definition: Domain (a.k.a. Universe) of Discourse

Example(s):  

Quantification

Idea: Establish truth of predicates over sets of values.

Two common generalizations:

Note: Do not use the book’s non-standard $\exists!x$ notation.
1. Universally Quantified Predicates

Example(s):

2. Existentially Quantified Predicates

Example(s):
Evaluating Mixed Quantifications (1 / 2)

First: Distinguishing $\exists x \forall y S(x, y)$ from $\forall i \exists k T(i, k)$:

Evaluating Mixed Quantifications (2 / 2)

Example(s):
Example: Universal Quantification (1 / 5)
Consider this conversational English statement:

All of McCann’s students are geniuses.

How can we express that statement in logic notation?

Example: Universal Quantification (2 / 5)
Attempt #2: All of McCann's students are geniuses. → Logic
Example: Universal Quantification (3 / 5)

Attempt #3: All of McCann’s students are geniuses. → Logic

Let \( P(x) \) : Student \( x \) is a genius, \( x \in \text{People} \)

---

Example: Universal Quantification (4 / 5)

Attempt #4: All of McCann’s students are geniuses. → Logic

Let \( P(x) \) : Student \( x \) is a genius, \( x \in \text{People} \)

Let \( M(x) \) : \( x \) is enrolled in one of McCann’s classes, \( x \in \text{People} \)
Example: Universal Quantification (5 / 5)

Attempt #5: All of McCann’s students are geniuses. → Logic

Let $P(x) : \text{Student } x \text{ is a genius, } x \in \text{People}$

Let $M(x) : x \text{ is enrolled in one of McCann’s classes, } x \in \text{People}$

Implicit Quantification

The “all” can be implicit in the English statement.

Example(s):
Example: Existential Quantification

Consider this conversational English statement:

At least one towel is dirty.

How can we express that statement in logic notation?

Another Example: Existential Quantification

Express this more specific statement in logic:

Some of the blue guest towels are dirty.

Let $D(x) : x$ is dirty, $x \in$ Towels
Yet Another Example: Quantification

Now express this statement in logic:

Every last one of the blue guest towels are dirty!

Let $B(x) : x$ is blue, $x \in$ Towels
Let $G(x) : x$ is used by guests, $x \in$ Towels
Let $D(x) : x$ is dirty, $x \in$ Towels

Free vs. Bound Variables

**Definition: Bound Variable**

**Definition: Free (a.k.a. Unbound) Variable**

Other examples of ‘binding’ in CS:
Negations of Quantified Expressions

Remember De Morgan’s Laws for propositions? Well, . . .

**Definition: Generalized De Morgan’s Laws**

\[ \forall x P(x) \equiv \exists x \overline{P(x)} \quad \text{(1/2)} \]

Let \( S(x) : x < 4, x \in \mathbb{Z} \)

The expression \( \forall x S(x), x \in \{1, 2, 3\} \) is true.

Equivalently, \( \forall x S(x) \) is false.

\[
\begin{align*}
\forall x S(x) & \equiv S(1) \land S(2) \land S(3) \\
\forall x \overline{S(x)} & \equiv \overline{S(1)} \lor \overline{S(2)} \lor \overline{S(3)} \\
& \equiv \overline{S(1)} \lor \overline{S(2)} \lor \overline{S(3)} \quad \text{[De Morgan, 2x]} 
\end{align*}
\]

(Remember: \( \overline{S(1)} \lor \overline{S(2)} \lor \overline{S(3)} \) is still false.)

(Continues . . .)
Demonstration: \(\forall x P(x) \equiv \exists x P(x)\) (2 / 2)

For \(S(1) \lor S(2) \lor S(3)\) to be false, each term must be false; that is, no \(S(x)\) is true (or all \(S(x)\) are false).

It follows that the expression \(\exists x \overline{S(x)}\) must be false, completing the demonstration.

Example(s):

Expressing “Exactly one . . . ” Statements (1 / 3)

Consider this conversational (& correct!) English statement:

Only one citizen of North Dakota is a member of the U.S. House of Representatives.

And consider this awkward but useful rewording:
Expressing “Exactly one . . .” Statements (2 / 3)

That rewording is useful because it can be directly expressed logically:

Expressing “Exactly one . . .” Statements (3 / 3)

A lingering problem:

The domain (“Citizens of North Dakota”) is too specific.

Solution: Add a predicate . . . but what, and where?
Expressing “Exactly two . . .” Statements (1 / 3)

Key observation:

Question: Can you say this in ‘awkward English’?

Exactly two citizens of North Dakota are U.S. Senators.

Expressing “Exactly two . . .” Statements (2 / 3)

Consider the two halves separately:

1. “At least two citizens of North Dakota are U.S. Senators”

2. “At most two citizens of North Dakota are U.S. Senators”
Finally, AND together

\[ \exists x \exists y \left( S(x) \land S(y) \land (x \neq y) \right) \]

and

\[ \forall x \forall y \forall z \left( (S(x) \land S(y) \land S(z)) \rightarrow (x = y \lor y = z \lor x = z) \right) : \]