Propositions With Variables (1 / 2)

Propositions are static; variables are not allowed. But . . .

**Definition:** Predicate (a.k.a. Propositional Function)

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**Example(s):**
Propositions With Variables (2 / 2)

**Definition:** Domain (a.k.a. Universe) of Discourse

Example(s):

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Quantification

Idea: Establish truth of predicates over sets of values.

Two common generalizations:

**Note:** Do **not** use the book’s non-standard $\exists!x$ notation.
1. Universally Quantified Predicates

Example(s):

2. Existentially Quantified Predicates

Example(s):
Evaluating Mixed Quantifications (1 / 2)

First: Distinguishing \( \exists x \forall y S(x, y) \) from \( \forall i \exists k T(i, k) \):

Evaluating Mixed Quantifications (2 / 2)

Example(s):
Example: Universal Quantification (1 / 5)
Consider this conversational English statement:

All of McCann's students are geniuses.

How can we express that statement in logic notation?

Example: Universal Quantification (2 / 5)
Attempt #2: All of McCann's students are geniuses. → Logic
**Example: Universal Quantification (3 / 5)**

**Attempt #3:** All of McCann’s students are geniuses. → Logic

Let $P(x) : \text{Student } x \text{ is a genius, } x \in \text{People}$

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**Example: Universal Quantification (4 / 5)**

**Attempt #4:** All of McCann’s students are geniuses. → Logic

Let $P(x) : \text{Student } x \text{ is a genius, } x \in \text{People}$

Let $M(x) : x \text{ is enrolled in one of McCann’s classes, } x \in \text{People}$
**Example: Universal Quantification (5 / 5)**

Attempt #5: *All of McCann’s students are geniuses.* → Logic

Let $P(x)$: Student $x$ is a genius, $x \in \text{People}$

Let $M(x)$: $x$ is enrolled in one of McCann’s classes, $x \in \text{People}$

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**Implicit Quantification**

The “all” can be implicit in the English statement.

**Example(s):**
Example: Existential Quantification

Consider this conversational English statement:

At least one towel is dirty.

How can we express that statement in logic notation?

Another Example: Existential Quantification

Express this more specific statement in logic:

Some of the blue guest towels are dirty.

Let $D(x) : x$ is dirty, $x \in \text{Towels}$
Yet Another Example: Quantification

Now express this statement in logic:

Every last one of the blue guest towels are dirty!

Let \( B(x) : x \) is blue, \( x \in \text{Towels} \)

Let \( G(x) : x \) is used by guests, \( x \in \text{Towels} \)

Let \( D(x) : x \) is dirty, \( x \in \text{Towels} \)

Free vs. Bound Variables

Definition: Bound Variable

Other examples of ‘binding’ in CS:
Negations of Quantified Expressions

Remember De Morgan’s Laws for propositions? Well, . . .

**Definition: Generalized De Morgan’s Laws**

\[
\forall x \, P(x) \equiv \exists x \, \overline{P(x)} \quad (1 / 2)
\]

Let \( S(x) : x < 4, x \in \mathbb{Z} \)

The expression \( \forall x \, S(x), x \in \{1, 2, 3\} \) is true.

Equivalently, \( \forall x \, S(x) \) is false.

\[
\forall x \, S(x) \quad \equiv \quad S(1) \land S(2) \land S(3)
\]

so . . .

\[
\forall x \, S(x) \quad \equiv \quad S(1) \land S(2) \land S(3)
\]

\[
\equiv \quad \overline{S(1)} \lor \overline{S(2)} \lor \overline{S(3)} \quad \text{[De Morgan, 2x]}
\]

(Remember: \( \overline{S(1)} \lor \overline{S(2)} \lor \overline{S(3)} \) is still false.)

(Continues . . .)
Demonstration: $\forall x P(x) \equiv \exists x P(x)$ (2 / 2)

For $S(1) \lor S(2) \lor S(3)$ to be false, each term must be false; that is, no $S(x)$ is true (or all $S(x)$ are false).

It follows that the expression $\exists x S(x)$ must be false, completing the demonstration.

Example(s):

Expressing “Exactly one . . .” Statements (1 / 3)

Consider this conversational (& correct!) English statement:

Only one citizen of North Dakota is a member of the U.S. House of Representatives.

And consider this awkward but useful rewording:
Expressing “Exactly one . . .” Statements (2 / 3)

That rewording is useful because it can be directly expressed logically:

Expressing “Exactly one . . .” Statements (3 / 3)

A lingering problem:

The domain (“Citizens of North Dakota”) is too specific.

Solution: Add a predicate . . . but what, and where?
Expressing “Exactly two . . .” Statements (1 / 3)

Key observation:

**Question:** Can you say this in ‘awkward English’?

Exactly two citizens of North Dakota are U.S. Senators.

Expressing “Exactly two . . .” Statements (2 / 3)

Consider the two halves separately. Given:

\[ S(x) : x \text{ is a U.S. Senator, } x \in \text{People} \]

1. “At least two citizens of North Dakota are U.S. Senators”

2. “At most two citizens of North Dakota are U.S. Senators”
Finally, AND together

\[ \exists x \ \exists y \ (S(x) \land S(y) \land (x \neq y)) \]

and

\[ \forall x \ \forall y \ \forall z \ ((S(x) \land S(y) \land S(z)) \rightarrow (x = y \lor y = z \lor x = z)) \]