Handful O’ Definitions (1 / 2)

**Definition: Conjecture**

**Definition: Theorem**

**Definition: Proof**
Why do People Fear Proofs?

1. **Proofs don’t come from an assembly line.**
   - Need knowledge, persistence, and creativity

2. **Creating proofs seems like magic.**
   - But they are systematic in many ways

3. **Proofs are hard to read and understand.**
   - Only if the writer makes them so

4. **Institutionalized Fear.**
   - Many teachers avoid them in classes
Constructing a proof? Remember:

1. There are several proof techniques for a reason.
   ▶ One may be easier to use than the others

2. Knowledge of mathematics is important.
   ▶ Remember our Math Review?

3. There are “tricks” to know.
   ▶ Ex: Dividing an even # in half leaves no remainder

4. Practice helps . . . a lot!
   ▶ Just as it does for most everything else

5. Dead ends are expected.
   ▶ Proofs in books are the final, polished versions

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Types Of Proof In This Class

1. Direct Proof
   ▶ The most common variety

2. Proof by Contraposition
   ▶ Like Direct, but with a twist

3. Proof by Contradiction
   ▶ A dark road on a foggy night

4. Proof by Mathematical Induction
   ▶ Wait for it . . .
Our First Conjecture

**Conjecture**: If \( n \) is even, then \( n^2 \) is also even, \( n \in \mathbb{Z} \).
Proof-Writing Miscellanea

- Remember: A conjecture isn’t a theorem until proven.
- Don’t lose sight of your destination.
- When writing proofs in this class:
  1. Always start with “Proof (style):”
  2. Stating your allowed assumptions can help.
  3. Define all introduced variables.
  4. End proofs with “Therefore,” and the conjecture.

[Outside of this class: “Q.E.D.” (quod erat demonstrandum, Latin for “this was to be demonstrated.”)]

A Conjecture About Inequalities

**Conjecture:** If $0 < a < b$, then $a^2 < b^2$, $a, b \in \mathbb{R}$.
“Proof By Cases”

**Question:** How would you prove that \( \forall x \ C(x) \) is true, where \( x \in \{6, 28, 496\} \)?

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**A Direct Proof Employing Cases**

**Conjecture:** \( s \rightarrow r \equiv \neg r \rightarrow \neg s \).

**Proof (direct):** Consider all possible combinations of values of \( r \) and \( s \):

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<tr>
<th>( r )</th>
<th>( s )</th>
<th>( s \rightarrow r )</th>
<th>( \neg r \rightarrow \neg s )</th>
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Therefore, \( s \rightarrow r \equiv \neg r \rightarrow \neg s \).

(Yes, this truth table is a direct proof by cases.)
A More Interesting Direct Proof With Cases

**Conjecture:** $x^2 \% 4 \in \{0, 1\}, \ x \in \mathbb{Z}$.

Poor Arguments $\rightarrow$ Poor Proofs (1 / 2)

**Conjecture:** $1 < 0$.

**Proof or Goof?:**

Consider $x$ such that $0 < x < 1$. Take the base–10 logarithm of both sides of $x < 1$: $\log_{10} x < \log_{10} 1$. By definition, $\log_{10} 1 = 0$. Divide both sides by $\log_{10} x$:

$\frac{\log_{10} x}{\log_{10} 1} < \frac{0}{\log_{10} x}$, which reduces to $1 < 0$.

Therefore, $1 < 0$. 
Conjecture: For all $n \in \mathbb{Z}^{odd}$, $(n^2 - 1) \mod 4 = 0$.

Proof or Goof?:
Let $x = 1$. $1^2 - 1 = 0$. $0 \mod 4 = 0$. Let $x = 3$. $3^2 - 1 = 8$. $8 \mod 4 = 0$. Let $x = 5$. $5^2 - 1 = 24$. $24 \mod 4 = 0$. This shows no sign of failing to give a result of 0.

Therefore, for all $n \in \mathbb{Z}^{odd}$, $(n^2 - 1) \mod 4 = 0$.

Proof by Contraposition

(a.k.a. Proof of the Contrapositive)
Example #1: Proof by Contraposition

**Conjecture:** If \( ac \leq bc \), then \( c \leq 0 \), when \( a > b \).

Example #2: Proof by Contraposition

**Conjecture:** If \( n^2 \) is even, then \( n \) is even.
Proof by Contradiction

(a.k.a. Reductio ad Absurdum)

Recall the Law of Implication: \( p \rightarrow q \equiv \neg p \lor q \)

Example #1: Proof by Contradiction

Conjecture: If \( 3n + 2 \) is odd, then \( n \) is odd.
Example #2: Proof by Contradiction (1 / 2)

Conjecture: The sum of the squares of two odd integers is never a perfect square. (Or: If $n = a^2 + b^2$, then $n$ is not a perfect square, where $a, b \in \mathbb{Z}_{odd}$.)
How To Prove Biconditional Expressions

(i.e., Conjectures Of The Form \( p \leftrightarrow q \))

Example(s):

Disproving Conjectures

Typical approaches:

1. Prove that the conjecture’s negation is true.
2. Find a counter-example. (Very commonly used!)

Example(s):