Topic 5:

Proofs of $p \rightarrow q$

Handful O’ Definitions (1 / 2)

**Definition:** Conjecture

**Definition:** Theorem

Definition: Proof
Handful O’ Definitions (2 / 2)

Definition: Lemma

Example(s):

Definition: Corollary

Example(s):

Why do People Fear Proofs?

1. Proofs don’t come from an assembly line.
   - Need knowledge, persistence, and creativity

2. Creating proofs seems like magic.
   - But they are systematic in many ways

3. Proofs are hard to read and understand.
   - Only if the writer makes them so

4. Institutionalized Fear.
   - Many teachers avoid them in classes
Constructing a proof? Remember:

1. There are several proof techniques for a reason.
   ▶ One may be easier to use than the others

2. Knowledge of mathematics is important.
   ▶ Remember our Math Review?

3. There are “tricks” to know.
   ▶ Ex: Dividing an even # in half leaves no remainder

4. Practice helps . . . a lot!
   ▶ Just as it does for most everything else

5. Dead ends are expected.
   ▶ Proofs in books are the final, polished versions

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Types Of Proof In This Class

1. Direct Proof
   ▶ The most common variety

2. Proof by Contraposition
   ▶ Like Direct, but with a twist

3. Proof by Contradiction
   ▶ A dark road on a foggy night

4. Proof by Mathematical Induction
   ▶ Wait for it . . .
Our First Conjecture

**Conjecture:** If \( n \) is even, then \( n^2 \) is also even, \( n \in \mathbb{Z} \).
Proof-Writing Miscellanea

- Remember: A conjecture isn’t a theorem until proven.
- Don’t lose sight of your destination.
- When writing proofs in this class:
  1. Always start with “Proof (style):”
  2. Stating your allowed assumptions can help.
  3. Define all introduced variables.
  4. End proofs with “Therefore, ” and the conjecture.

[Outside of this class: “Q.E.D.” (quod erat demonstrandum, Latin for “this was to be demonstrated.”)]

A Conjecture About Inequalities

**Conjecture:** If $0 < a < b$, then $a^2 < b^2$, $a, b \in \mathbb{R}$.
**Proof By Cases**

**Question:** How would you prove that $\forall x \ C(x)$ is true, where $x \in \{6, 28, 496\}$?

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**A Direct Proof Employing Cases**

**Conjecture:** $s \to r \equiv \neg r \to \neg s$

Proof (direct): Consider all possible combinations of values of $r$ and $s$:

<table>
<thead>
<tr>
<th>$r$</th>
<th>$s$</th>
<th>$s \to r$</th>
<th>$\neg r \to \neg s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</table>

Case 1: T T T T
Case 2: T F T T
Case 3: F T F F
Case 4: F F T T

Therefore, $s \to r \equiv \neg r \to \neg s$.

(Yes, this truth table is a direct proof by cases.)
A More Interesting Direct Proof With Cases

**Conjecture:** \( x^2 \equiv 4 \pmod{4}, x \in \mathbb{Z} \).

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Poor Arguments \( \rightarrow \) Poor Proofs (1 / 2)

**Conjecture:** \( 1 < 0 \).

**Proof or Goof?:**

Consider \( x \) such that \( 0 < x < 1 \). Take the base–10 logarithm of both sides of \( x < 1 \): \( \log_{10} x < \log_{10} 1 \). By definition, \( \log_{10} 1 = 0 \). Divide both sides by \( \log_{10} x \):

\[
\frac{\log_{10} x}{\log_{10} x} < \frac{0}{\log_{10} x},
\]

which reduces to \( 1 < 0 \).

Therefore, \( 1 < 0 \).
**Conjecture:** For all \( n \in \mathbb{Z}^{odd} \), \((n^2 - 1) \% 4 = 0\).

**Proof or Goof?:**
Let \( x = 1 \). \( 1^2 - 1 = 0 \). \( 0 \% 4 = 0 \). Let \( x = 3 \). \( 3^2 - 1 = 8 \). \( 8 \% 4 = 0 \). Let \( x = 5 \). \( 5^2 - 1 = 24 \). \( 24 \% 4 = 0 \). This shows no sign of failing to give a result of 0.

Therefore, for all \( n \in \mathbb{Z}^{odd} \), \((n^2 - 1) \% 4 = 0\).

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**Proof by Contraposition**

(a.k.a. **Proof of the Contrapositive**)

Proofs – CSci 245 v1.1 (McCann) – p. 15/24

Proofs – CSci 245 v1.1 (McCann) – p. 16/24
Example #1: Proof by Contraposition

**Conjecture:** If $ac \leq bc$, then $c \leq 0$, when $a > b$.

Example #2: Proof by Contraposition

**Conjecture:** If $n^2$ is even, then $n$ is even.
Proof by Contradiction

(a.k.a. Reductio ad Absurdum)

Recall the Law of Implication: \( p \rightarrow q \equiv \neg p \lor q \)

Example #1: Proof by Contradiction

Conjecture: If \( 3(n - 6) \) is odd, then \( n \) is odd.
Conjecture: The sum of the squares of two odd integers is never a perfect square. (Or: If $n = a^2 + b^2$, then $n$ is not a perfect square, where $a, b \in \mathbb{Z}^{\text{odd}}$.)
How To Prove Biconditional Expressions

(i.e., Conjectures Of The Form $p \leftrightarrow q$)

Example(s):

Disproving Conjectures

Typical approaches:

1. Prove that the conjecture’s negation is true.
2. Find a counter-example. (Very commonly used!)

Example(s):