Topic 5:

Proofs of $p \rightarrow q$

Handful O’ Definitions (1 / 2)

Definition: Conjecture

Definition: Theorem

Definition: Proof
Handful O’ Definitions (2 / 2)

**Definition: Lemma**

**Definition: Corollary**

**Example(s):**

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Why do People Fear Proofs?

1. **Proofs don’t come from an assembly line.**
   - Need knowledge, persistence, and creativity

2. **Creating proofs seems like magic.**
   - But they are systematic in many ways

3. **Proofs are hard to read and understand.**
   - Only if the writer makes them so

4. **Institutionalized Fear.**
   - Many teachers avoid them in classes
Constructing a proof? Remember:

1. There are several proof techniques for a reason.
   ► One may be easier to use than the others

2. Knowledge of mathematics is important.
   ► Remember our Math Review?

3. There are “tricks” to know.
   ► Ex: Dividing an even # in half leaves no remainder

4. Practice helps . . . a lot!
   ► Just as it does for most everything else

5. Dead ends are expected.
   ► Proofs in books are the final, polished versions

Types Of Proof In This Class

1. Direct Proof
   ► The most common variety

2. Proof by Contraposition
   ► Like Direct, but with a twist

3. Proof by Contradiction
   ► A dark road on a foggy night

4. Proof by Mathematical Induction
   ► Wait for it . . .
Our First Conjecture

**Conjecture:** If \( n \) is even, then \( n^2 \) is also even, \( n \in \mathbb{Z} \).
Proof-Writing Miscellanea

• Remember: A conjecture isn’t a theorem until proven.

• Don’t lose sight of your destination.

• When writing proofs in this class:
  1. Always start with “Proof (style):”
  2. Stating your allowed assumptions can help.
  3. Define all introduced variables.
  4. End proofs with “Therefore,” and the conjecture.

[Outside of this class: “Q.E.D.” (*quod erat demonstrandum*,
Latin for “this was to be demonstrated.”)]

A Conjecture About Inequalities

**Conjecture:** If $0 < a < b$, then $a^2 < b^2$, $a, b \in \mathbb{R}$.
**Question:** How would you prove that $\forall x \, C(x)$ is true, where $x \in \{6, 28, 496\}$?

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**A Direct Proof Employing Cases**

**Conjecture:** $s \rightarrow r \equiv \lnot r \rightarrow \lnot s$

Proof (direct): Consider all possible combinations of values of $r$ and $s$:

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Case 1: $T \ T \ T \ T$
Case 2: $T \ F \ T \ T$
Case 3: $F \ T \ F \ F$
Case 4: $F \ F \ T \ T$

Therefore, $s \rightarrow r \equiv \lnot r \rightarrow \lnot s$.

(Yes, this truth table is a direct proof by cases.)
A More Interesting Direct Proof With Cases

**Conjecture:** \( x^2 \% 4 \in \{0, 1\}, \ x \in \mathbb{Z} \).

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**Poor Arguments \(\rightarrow\) Poor Proofs (1 / 2)**

**Conjecture:** \( 1 < 0 \).

**Proof or Goof?:**

Consider \( x \) such that \( 0 < x < 1 \). Take the base–10 logarithm of both sides of \( x < 1 \): \( \log_{10} x < \log_{10} 1 \). By definition, \( \log_{10} 1 = 0 \). Divide both sides by \( \log_{10} x \):

\[
\frac{\log_{10} x}{\log_{10} x} < \frac{0}{\log_{10} x},
\]

which reduces to \( 1 < 0 \).

Therefore, \( 1 < 0 \).
Poor Arguments —> Poor Proofs (2 / 2)

**Conjecture:** For all \( n \in \mathbb{Z}_{\text{odd}} \), \( (n^2 - 1) \mod 4 = 0 \).

**Proof or Goof?:**

Let \( x = 1 \). \( 1^2 - 1 = 0 \). \( 0 \mod 4 = 0 \). Let \( x = 3 \). \( 3^2 - 1 = 8 \). \( 8 \mod 4 = 0 \). Let \( x = 5 \). \( 5^2 - 1 = 24 \). \( 24 \mod 4 = 0 \). This shows no sign of failing to give a result of 0.

Therefore, for all \( n \in \mathbb{Z}_{\text{odd}} \), \( (n^2 - 1) \mod 4 = 0 \).

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Proof by Contraposition

(a.k.a. Proof of the Contrapositive)
Example #1: Proof by Contraposition

Conjecture: If $ac \leq bc$, then $c \leq 0$, when $a > b$.

Example #2: Proof by Contraposition

Conjecture: If $n^2$ is even, then $n$ is even.
Proof by Contradiction

(a.k.a. Reductio ad Absurdum)

Recall the Law of Implication: \( p \rightarrow q \equiv \neg p \lor q \)

Example #1: Proof by Contradiction

**Conjecture:** If \( 7n - 4 \) is odd, then \( n \) is odd.
Example #2: Proof by Contradiction (1 / 2)

**Conjecture:** The sum of the squares of two odd integers is never a perfect square. (Or: If $n = a^2 + b^2$, then $n$ is not a perfect square, where $a, b \in \mathbb{Z}^{\text{odd}}$.)
How To Prove Biconditional Expressions

(i.e., Conjectures Of The Form $p \iff q$)

Example(s):

Disproving Conjectures

Typical approaches:

1. Prove that the conjecture’s negation is true.
2. Find a counter-example. (Very commonly used!)

Example(s):