A friend who’s in liquor production, has a still of astounding construction. The alcohol boils, through old magnet coils; he says that it’s . . .

**Proof by Induction**

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**Review: Inductive and Deductive Arguments**

**Definition: Deductive Argument**

An argument that uses accepted general principles to explain a specific situation is a *deductive* argument.

**Definition: Inductive Argument**

An argument that moves from specific observations to a general conclusion is an *inductive* argument.

**Old Example:** 3, 5, and and 7 are prime numbers. Therefore, all positive odd integers above 1 are prime numbers.
The First Principle of Mathematical Induction

(Often called “Weak Induction”)

Definition: First Principle of Mathematical Induction

In the $p \rightarrow q$ form:
The Second Principle of Mathematical Induction

(Often called “Strong Induction”)

Just one (big!) difference from the 1st Principle:

**Definition: Second Principle of Mathematical Induction**

If: (i) $P(a)$ is true for the starting point $a \in \mathbb{Z}^+$, and

Then: $P(n)$ is true, for all $n \in \mathbb{Z}^+$, $n \geq a$.

But . . . Which One Should I Use?
Example #1: A Summation (1 / 2)

Conjecture: \[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

Example #1: A Summation (2 / 2)
Example #2: An Inequality (1 / 3)

**Conjecture:** $n^2 > n + 1, \forall n \geq 2, n \in \mathbb{Z}^+$

Example #2: An Inequality (2 / 3)
Example #2: An Inequality (3 / 3)

Here’s an alternative inductive step, starting with the I.H.:

<table>
<thead>
<tr>
<th>Inductive: If $n^2 &gt; n + 1$, then $(n + 1)^2 &gt; n + 2$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $n^2 &gt; n + 1$ [Given (Inductive Hypothesis)]</td>
</tr>
<tr>
<td>(2) $n^2 + 2n + 1 &gt; n + 1 + 2n + 1$ [Add $2n + 1$ to both sides]</td>
</tr>
<tr>
<td>(3) $(n + 1)^2 &gt; 3n + 2$ [Algebra]</td>
</tr>
<tr>
<td>(4) $n \geq 2$ [Given]</td>
</tr>
<tr>
<td>(5) $n \geq 0$ [Follows from (4)]</td>
</tr>
<tr>
<td>(6) $2n \geq 0$ [Multiply both sides by 2]</td>
</tr>
<tr>
<td>(7) $3n + 2 \geq n + 2$ [Add $n + 2$ to both sides]</td>
</tr>
<tr>
<td>(8) $(n + 1)^2 &gt; n + 2$ [(3),(7),Transitivity]</td>
</tr>
</tbody>
</table>

The Basis Step Only Seems Pointless

Conjecture: $\sum_{i=0}^{n-1} 2^i = 2^n$, $\forall n \geq 1$
Some “Obvious” Patterns That Aren’t (1 / 2)

1. Circle Division by Chords

- 2 points → 2 regions
- 3 points → 4 regions
- 4 points → 8 regions

Some “Obvious” Patterns That Aren’t (2 / 2)

2. Is $n^2 - n + 41$ prime $\forall n \geq 1$?

3. Is $991n^2 + 1$ never a perfect square $\forall n \geq 1$?
Strong Induction

Review:

In *Strong Induction*, $P(a)$ must be true and, for any $k \geq a$, if $P(a) \land P(a + 1) \land \ldots \land P(k - 1) \land P(k)$ is true, then $P(k + 1)$ is true.

Example: Strong Induction (1 / 2)

**Conjecture:** Let $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$. Also, assume $a_k = a_{k-1} + a_{k-2} + a_{k-3}$. Is $a_n \leq 2^n$, $\forall n \geq 3$?
Example: Strong Induction (2 / 2)

Structural Induction