The Pigeonhole Principle (1 / 2)  (a.k.a. The Dirichlet Drawer Principle)

Example:

Definition: Pigeonhole Principle

Definition: Pigeonhole Principle (w/ functions)
The Pigeonhole Principle (2 / 2)

Example(s):

The Multiplication Principle (1 / 2)

Example(s):

Definition: Multiplication Principle (a.k.a. Product Rule)
The Multiplication Principle (2 / 2)

Example(s):

The Addition Principle (1 / 2)

**Definition:** Addition Principle (a.k.a. Sum Rule)

Example(s):
The Addition Principle (2 / 2)

Example(s):

The Principle of Inclusion-Exclusion (1 / 5)

A problem with the Addition Principle:

Example(s):
The Principle of Inclusion-Exclusion (2 / 5)

**Definition:** Principle of Inclusion-Exclusion for Two Sets

The cardinality of the union of sets $M$, $N$, and $O$ is the sum of their individual cardinalities excluding the sum of the cardinalities of their pairwise intersections but including the cardinality of their intersection.

That is: $|M \cup N \cup O| = |M| + |N| + |O| - (|M \cap N| + |M \cap O| + |N \cap O|) + |M \cap N \cap O|.$
The Principle of Inclusion-Exclusion (4 / 5)

Why so complex?

The Principle of Inclusion-Exclusion (5 / 5)

Example(s):
Definition: Permutation

Example(s):

Conjecture: There are $n!$ possible permutations of $n$ elements.
**Definition: r-Permutation**

**Conjecture:** The number of $r$-permutations of $n$ elements, denoted $P(n, r)$, is $n \cdot (n - 1) \cdot \ldots \cdot (n - r + 1)$, $r \leq n$.  

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**r-Permutations (2 / 3)**

**Observation:**

**Example(s):**
\(r\)-Permutations (3 / 3)

Example(s):

\(r\)-Combinations (1 / 3)

**Definition:** \(r\)-Combination

\[ \text{}\]

Other Notations:

Example(s):
The $r$-Permutation – $r$-Combination Connection:

Example(s):

\[ \text{Example(s):} \]

\[ \text{Example(s):} \]
Repetition and Permutations

We’ve already seen this!

Example(s):

In General: When object repetition is permitted, the number of \( r \)-permutations of a set of \( n \) objects is \( n^r \).

Repetition and Combinations (1 / 3)

Example(s): ‘Experienced’ Golf Balls

[Diagram of red, green, and blue golf balls]
Repetition and Combinations (2 / 3)

Example(s):

In General: When repetition is allowed, the number of \( r \)-combinations of a set of \( n \) elements is
\[
\binom{n+r-1}{r} = \binom{n+r-1}{n-1}.
\]

Repetition and Combinations (3 / 3)

A Small Extension:

Example(s):

In General: When repetition is allowed, the number of \( r \)-combinations of a set of \( n \) elements when one of each element is included in \( r \) is
\[
\binom{r-1}{r-n} = \binom{r-1}{n-1}.
\]
Another View of Repetition and Combinations (1 / 2)

Consider: An integer variable can represent the quantity of items selected with repetition.

Example(s):

Another View of Repetition and Combinations (2 / 2)

Example(s):
Generalized Permutations (1 / 3)

Idea: What if some elements are indistinguishable?

Example(s):

Generalized Permutations (2 / 3)

What if we have indistinguishable copies of multiple elements?

Example(s):

In General: If we have $n$ objects of $t$ different types, and there are $i_k$ indistinguishable objects of type $k$, then the number of distinct arrangements is

$$P(n; i_1, i_2, \ldots, i_t) = \frac{n!}{i_1!i_2!\ldots i_t!}.$$
Generalized Permutations (3 / 3)

We can view \( P(n; i_1, i_2, \ldots, i_t) \) in terms of combinations:

**Example(s):**

In General:

\[
P(n; i_1, i_2, \ldots, i_t) = \binom{n}{i_1} \binom{n-i_1}{i_2} \binom{n-i_1-i_2}{i_3} \cdots \binom{n-\cdots-i_{t-1}}{i_t}
\]

More Fun with Combinations (1 / 2)

What if we created a table of \( \binom{n}{k} \) values?

\[
\begin{array}{cccccc}
  & 0 & 1 & 2 & 3 & 4 & 5 \\
 0 & 0 & 1 & 2 & 3 & 4 & 5 \\
 1 & 0 & 1 & 2 & 3 & 4 & 5 \\
 2 & 0 & 1 & 2 & 3 & 4 & 5 \\
 3 & 0 & 1 & 2 & 3 & 4 & 5 \\
 4 & 0 & 1 & 2 & 3 & 4 & 5 \\
 5 & 0 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]
Pascal’s Triangle is the centered rows of the \( \binom{n}{k} \) table:

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\vdots
\end{array}
\]

Proving that Pascal’s Triangle is ‘Palindromic’

Conjecture: \( \binom{n}{k} = \binom{n}{n-k} \), where \( 0 \leq k \leq n \)
Pascal’s Identity [Combinatorial Proof (1/2)]

Definition: Combinatorial Proof

Conjecture: \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \), where \( 1 \leq k \leq n \)
The Binomial Theorem (1 / 2)

The values of Pascal’s Triangle appear in numerous places.

For instance:

\[
(a + b)^0 = 1 \\
(a + b)^1 = 1a + 1b \\
(a + b)^2 = 1a^2 + 2ab + 1b^2 \\
(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3
\]

Generalize this, and you’ve got the Binomial Theorem.
The Binomial Theorem (2 / 2)

**Theorem:** \((a + b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^{n-k} \cdot b^k\)

**Example(s):**