Topic 13: Methods of Counting

The Pigeonhole Principle (1 / 2) (a.k.a. The Dirichlet Drawer Principle)

Example:

Definition: Pigeonhole Principle

Definition: Pigeonhole Principle (w/ functions)
The Pigeonhole Principle (2 / 2)

Example(s):

The Multiplication Principle (1 / 2)

Example(s):

Definition: Multiplication Principle (a.k.a. Product Rule)
The Multiplication Principle (2 / 2)

Example(s):

The Addition Principle (1 / 2)

**Definition:** *Addition Principle (a.k.a. Sum Rule)*

Example(s):
The Addition Principle (2 / 2)

Example(s):

The Principle of Inclusion-Exclusion (1 / 5)

A problem with the Addition Principle:

Example(s):
The Principle of Inclusion-Exclusion (2 / 5)

**Definition: Principle of Inclusion-Exclusion for Two Sets**

\[
|M \cup N| = |M| + |N| - |M \cap N|
\]

The Principle of Inclusion-Exclusion (3 / 5)

**Definition: Principle of Inclusion-Exclusion for Three Sets**

The cardinality of the union of sets \( M, N, \) and \( O \) is the sum of their individual cardinalities excluding the sum of the cardinalities of their pairwise intersections but including the cardinality of their intersection.

That is:

\[
|M \cup N \cup O| = |M| + |N| + |O| - (|M \cap N| + |M \cap O| + |N \cap O|) + |M \cap N \cap O|.
\]
The Principle of Inclusion-Exclusion (4 / 5)

Why so complex?

The Principle of Inclusion-Exclusion (5 / 5)

Example(s):
Permutations (1 / 2)

**Definition:** Permutation

**Example(s):**

Permutations (2 / 2)

**Conjecture:** There are \( n! \) possible permutations of \( n \) elements.
**Definition:** $r$-Permutation

Conjecture: The number of $r$-permutations of $n$ elements, denoted $P(n, r)$, is $n \cdot (n - 1) \cdot \ldots \cdot (n - r + 1)$, $r \leq n$.

**Observation:**

**Example(s):**
**r-Permutations (3 / 3)**

Example(s):

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**r-Combinations (1 / 3)**

**Definition:** *r-Combination*

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Other Notations:

Example(s):
The $r$-Permutation – $r$-Combination Connection:

Example(s):

Counting – CSc 245 v1.1 (McCann) – p. 19

Example(s):

Counting – CSc 245 v1.1 (McCann) – p. 20
Repetition and Permutations

We've already seen this!

Example(s):

In General: When object repetition is permitted, the number of $r$-permutations of a set of $n$ objects is $n^r$.

Repetition and Combinations (1 / 3)

Example(s): ‘Experienced’ Golf Balls

Red

Green

Blue
Repetition and Combinations (2 / 3)

Example(s):

In General: When repetition is allowed, the number of $r$–combinations of a set of $n$ elements is \( \binom{n+r-1}{r} = \binom{n+r-1}{n-1} \).

Repetition and Combinations (3 / 3)

A Small Extension:

Example(s):

In General: When repetition is allowed, the number of $r$–combinations of a set of $n$ elements when one of each element is included in $r$ is \( \binom{r-1}{r-n} = \binom{r-1}{n-1} \).
Another View of Repetition and Combinations (1 / 2)

Consider: An integer variable can represent the quantity of items selected with repetition.

Example(s):

Another View of Repetition and Combinations (2 / 2)

Example(s):
Generalized Permutations (1 / 3)

Idea: What if some elements are indistinguishable?

Example(s):

Generalized Permutations (2 / 3)

What if we have indistinguishable copies of multiple elements?

Example(s):

In General: If we have \( n \) objects of \( t \) different types, and there are \( i_k \) indistinguishable objects of type \( k \), then the number of distinct arrangements is

\[
P(n; i_1, i_2, \ldots, i_t) = \frac{n!}{i_1!i_2!\ldots i_t!}.
\]
Generalized Permutations (3 / 3)

We can view $P(n; i_1, i_2, \ldots, i_t)$ in terms of combinations:

Example(s):

In General:

$$P(n; i_1, i_2, \ldots, i_t) = \binom{n}{i_1} \binom{n-i_1}{i_2} \binom{n-i_1-i_2}{i_3} \cdots \binom{n-\ldots-i_{t-1}}{i_t}$$

More Fun with Combinations (1 / 2)

What if we created a table of $\binom{n}{k}$ values?

$$
\begin{array}{cccccc}
  & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & \phantom{0} & \phantom{1} & \phantom{2} & \phantom{3} & \phantom{4} & \phantom{5} \\
1 & \phantom{0} & \phantom{1} & \phantom{2} & \phantom{3} & \phantom{4} & \phantom{5} \\
2 & \phantom{0} & \phantom{1} & \phantom{2} & \phantom{3} & \phantom{4} & \phantom{5} \\
3 & \phantom{0} & \phantom{1} & \phantom{2} & \phantom{3} & \phantom{4} & \phantom{5} \\
4 & \phantom{0} & \phantom{1} & \phantom{2} & \phantom{3} & \phantom{4} & \phantom{5} \\
5 & \phantom{0} & \phantom{1} & \phantom{2} & \phantom{3} & \phantom{4} & \phantom{5} \\
\end{array}
$$
Pascal’s Triangle is the centered rows of the \( \binom{n}{k} \) table:

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\vdots
\end{array}
\]

Proving that Pascal’s Triangle is ‘Palindromic’

**Conjecture:** \( \binom{n}{k} = \binom{n}{n-k} \), where \( 0 \leq k \leq n \)
Pascal’s Identity (Combinatorial Argument Example)

Conjecture: \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \), where \( 1 \leq k \leq n \)

Pascal’s Identity [Combinatorial Proof (1 / 2)]

Definition: Combinatorial Proof

Conjecture: \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \), where \( 1 \leq k \leq n \)
The values of Pascal’s Triangle appear in numerous places. For instance:

\[(a + b)^0 = 1\]

\[(a + b)^1 = 1a + 1b\]

\[(a + b)^2 = 1a^2 + 2ab + 1b^2\]

\[(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3\]

Generalize this, and you’ve got the Binomial Theorem.
The Binomial Theorem (2 / 2)

**Theorem:** \((a + b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^{n-k} \cdot b^k\)

**Example(s):**