Rosen, Section 5.2:

1. (12 points) Consider the infinite sequence \( \{s_n\} \) such that \( s_n = 2 \cdot s_{n-1} - s_{n-2} \), where \( s_0 = 1 \) and \( s_1 = 3 \).
   
   (a) Compute the terms \( s_2 \) through \( s_5 \).
   
   (b) Find a simple function, in terms of \( n \) rather than in terms of preceding elements of \( s \), that also seems to generate \( \{s_n\} \).
   
   (c) Prove, using strong induction, that your function does, in fact, generate exactly the sequence \( \{s_n\} \).

(Continued …)
2. (16 points) Imagine that famed archaeologist Iowa Kraving has discovered evidence that an ancient civilization used two sizes of rocks to weigh their crops. The reddish rocks each weighed 4 ugh, and the bluish rocks each weighed 7 ugh (where an ‘ugh’ is a grunt of exertion expressed by those who had to carry the rocks, as in, “Kraving said ‘ugh’ seven times while lifting that bluish rock onto the scales.”). The civilization could weigh 4 ugh of grain, or 7, or 8 (two 4-ugh rocks), or 11 (4+7), but not 9 or 10, because there’s no sum of 4s and/or 7s that equal 9 or 10.

(a) It is possible to weigh exactly 15-ugh of grain. What is the combination of 4–ugh and 7–ugh rocks that sums to 15?

(b) If Kraving can find four consecutive integers (e.g., 2, 3, 4, and 5 are four consecutive integers) worth of ‘ugh’ that can be formed from the two sizes of rocks, she’ll know that the civilization could weigh any integer ‘ugh’ amount they wanted to weigh from the smallest of those four integers on up. How does Kraving know that a sequence of four is needed, rather than, say, two or three?

(c) Kraving’s summer intern, Immah Thekhed, somehow manages to discover that one consecutive sequence of four amounts — 40, 41, 42, and 43 — can all be formed from the stones. Kraving suspects that a smaller such sequence of four amounts exists, and tasks Thekhed with finding the smallest such sequence. What is the smallest such sequence of four consecutive amounts, and how can each of the four be formed from 4–ugh and 7–ugh stones?

(d) Use your answers to the previous two questions to help you construct a proof by strong induction that any integer amount of grain could be weighed by this civilization, starting from the smallest amount in the smallest sequence.

Rosen, Section 6.1:

3. (6 points) Imagine a home alarm system that allows the homeowner to set a password that consists of four lower–case letters.

(a) How many such four–letter sequences can exist? Provide your answer both as an expression and as an integer evaluation of that expression.

(b) How many can exist if repeated letters are not allowed within a single password? Again, provide an expression and its evaluation.

(c) Now assume that the alarm’s password can be as short as a single letter, as long as four letters, or any length in–between. How many passwords can be created, if no repeated letters are allowed within a single password? Once again, provide an expression and its evaluation.

4. (4 points) Messenger ribonucleic acid (mRNA) is now used for the production of vaccines. A strand of mRNA is built of a sequence of the molecules adenine, cytosine, guanine, and uracil (A, C, G, and U, for short).

(a) How many 5–molecule strands of mRNA start with A?

(b) How many 5–molecule strands of mRNA do not contain A at all?

5. (4 points) You remember (we hope!) that a function maps domain elements to codomain elements.

(a) Assume a domain of 4 elements and a codomain of 8 elements. How many unique functions exist from the domain to the codomain?

(b) Now assume domain of 4 elements and a codomain of 8 elements. How many unique injective functions exist from the domain to the codomain?

(Continued . . .)
6. (6 points) An city council has a president, a city manager, and eight city councilors. A photographer is taking pictures of subsets of these people lined up in front of the city seal.

(a) In how many ways can eight of these 10 people be lined up?
(b) In how many ways can eight of these 10 people be lined up if the council president and the city manager both must be in the picture?
(c) In how many ways can eight of these 10 people be lined up if the council president or the city manager (but not both) must be in the picture?

7. (2 points) A UArizona fan has a drawer containing 7 shirts that are all or partly red, and 5 shirts that are all or partly blue. If three shirts have some combination of red and blue, how many total shirts are in the drawer?

8. (4 points) Consider three sets (D, E, and F) of items in a universe of 35 items. \(|D| = 25\), \(|E| = 21\), \(|D \cap E| = 14\), \(|D \cap F| = 15\), \(|E \cap F| = 12\), \(|D \cap E \cap F| = 8\), and \(|D \cup E \cup F| = 33\).

(a) Evaluate: \(|D \cup E \cup F|\).
(b) Evaluate: \(|F|\).

Rosen, Section 6.2:

9. (2 points) Use the Pigeonhole Principle to show that in any group of 21 cellphone–owning people, at least three of those people have phone numbers that end with the same digit.

10. (2 points) Consider a function from a domain \(D\) to a codomain \(C\) where \(|D| > |C|\). Use the Pigeonhole Principle to show that such a function cannot be an injective function.

Rosen, Section 6.3:

For all questions in this section, provide your answer both as an expression in terms of \(n!\), \(P(n,r)\), \(C(n,r)\), etc., as appropriate, and also as an integer answer. Why? We want to have some idea of how you arrived at your answers!

11. (2 points) How many different finishing orders exist for a race of six sailboats if all six finish and no ties occur? (Reminder: See the directions at the top of this section!)

12. (4 points) Assume that the act of flipping a coin always ends with the coin landing flat, with either ‘heads’ or ‘tails’ facing up.

(a) If a coin is flipped 10 times, how many sequences of outcomes are possible? (E.g., THHTHTHTTTT is one possible outcome sequence.)
(b) If a coin is flipped 10 times, how many sequences of outcomes contain exactly three heads?

13. (4 points) A bit string is just that — a string (sequence) of zeroes and/or ones.

(a) How many bit strings of length 10 contain at most three 1–bits?
(b) How many bit strings of length 10 contain at least three 1–bits?

14. (4 points) At a small grade school, at the end of recess, eight third–graders and four fourth–graders line up to go back into the school. In how many ways can the students line up so that none of the fourth–graders are standing next to each other? (Hint: First consider the positions for the third–graders.)
15. (6 points) A neighborhood community center organizes indoor soccer games in which each team has six players on the field at a time. One of the teams has nine available players.

(a) In how many ways can the team choose six players to start a game? Assume that positions (like goalkeeper) are not considered.
(b) In how many ways can the team choose six players to start a game, assuming that positions are considered. Each position is distinct.
(c) Of the nine players, three are children. If the league rules require that at least one of the six players be a child, and positions are not considered, in how many ways can six players be chosen to start a game?

Rosen, Section 6.5:

For all questions in this section, provide your answer both as an expression in terms of n!, P(n,r), C(n,r), etc., as appropriate, and also as an integer answer. Why? We want to have some idea of how you arrived at your answers!

16. (2 points) Your local grocery store has 10 varieties of individually–packaged ice cream bars for convenient impulse purchases. They always have plenty in stock, and all bars of any one variety are indistinguishable. How many ways are there to choose four bars?

17. (6 points) Consider the equation \( x_1 + x_2 + x_3 + x_4 = 17. \)

(a) If \( x_i \in \mathbb{Z}^* \), how many solutions are there to the equation?
(b) If \( x_i \in \mathbb{Z}^+ \), how many solutions are there to the equation?

18. (4 points) Consider the name of Winnie–the–Pooh’s friend EEYORE (yes, all upper–case).

(a) How many distinguishable arrangements of the letters in EEYORE are possible?
(b) How many distinguishable arrangements of the letters in EEYORE are possible, if all of the E’s are adjacent to one another?