ECON 519 MIDTERM EXAM SOLUTIONS
FALL 2017

① B IS NONEMPTY: $\underline{0} = (0,0,\ldots)$ IS BOUNDED, $\therefore$ IN B.

B SATISFIES (VS1): SUPPOSE $x, x' \in B$; THEN $\exists M, M' > 0$ s.t. $|x_k| \leq M$ AND $|x'_k| \leq M'$ FOR $k = 1, 2, \ldots$. THEREFORE $|x_k + x'_k| \leq |x_k| + |x'_k| \leq M + M'$, $k = 1, 2, \ldots$, WHERE THE FIRST INEQUALITY IS THE TRIANGLE INEQUALITY. THEREFORE $x + x'$ IS BOUNDED, i.e., $x + x' \in B$.

B SATISFIES (VS2): SUPPOSE $x \in B$ AND $\lambda \in \mathbb{R}$. THEN $\exists M > 0$ s.t. $|x_k| \leq M$, $k = 1, 2, \ldots$ LET $M' \doteq |\lambda| M$. THEN $|\lambda x_k| \leq |\lambda| |x_k| \leq |\lambda| M = M'$ FOR $k = 1, 2, \ldots$ THEREFORE $\lambda x \in B$.

② $u(x_1, x_2) = x_1 x_2$

(a) $\frac{\partial u}{\partial x_1} = x_2$ AND $\frac{\partial u}{\partial x_2} = x_1$, SO $\nabla u = (x_2, x_1)$.

(b) $H = D^2 u(x) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, WHERE $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}(x)$.

(c) WE APPLY THE IFT TO THE FOC EQUATIONS:

$$\left.\begin{aligned} F_1(x_1,x_2,\lambda,p_1,p_2,m) &= u_1-\lambda p_1 = x_2-\lambda p_1 &= 0\\ F_2(x_1,x_2,\lambda,p_1,p_2,m) &= u_2-\lambda p_2 = x_1-\lambda p_2 &= 0\\ F_3(x_1,x_2,\lambda,p_1,p_2,m) &= p_1x_1+p_2x_2-m &= 0\end{aligned}\right\}\text{FOC}$$

THE IFT ENSURES THAT A (SOLUTION) FUNCTION $f:(p_1,p_2,m)\mapsto(x_1,x_2,\lambda)$ EXISTS (AND IS DIFFERENTIABLE) IF

$$\left[\frac{\partial F_i}{\partial x_j}\right] \text{ IS NONSINGULAR, i.e., IF } \left|\left[\frac{\partial F_i}{\partial x_j}\right]\right| \neq 0.$$

WE HAVE

$$\left|\left[\frac{\partial F_i}{\partial x_j}\right]\right| = \begin{vmatrix} u_{11} & u_{12} & -p_1\\ u_{21} & u_{22} & -p_2\\ p_1 & p_2 & 0\end{vmatrix} = -u_{12}p_1p_2-u_{21}p_1p_2+u_{11}p_2^2+u_{22}p_1^2$$

$$= \begin{vmatrix} 0 & 1 & -p_1\\ 1 & 0 & -p_2\\ p_1 & p_2 & 0\end{vmatrix} = -p_1p_2-p_1p_2 = -2p_1p_2<0.$$

SINCE $\left|\left[\frac{\partial F_i}{\partial x_j}\right]\right| \neq 0$ AT ALL $x\in\mathbb{R}^2_{++}$, THERE IS A FUNCTION $f:(p_1,p_2,m)\mapsto(x_1,x_2,\lambda)$ — i.e., A FUNCTION $\hat{x}(p_1,p_2,m)$, THE CONSUMER'S DEMAND FUNCTION.

(d) $x_2 = \lambda p_1$ AND $x_1 = \lambda p_2$,

~~SO WE HAVE~~ $\cancel{p_1 x_1 + p_2 x_2 = \lambda p_1 p_2 + \lambda p_1 p_2 = 2\lambda p_1 p_2}$.

SINCE $p_1 x_1 + p_2 x_2 = M$, WE HAVE $2\lambda p_1 p_2 = M$, i.e.,

$$\lambda = \frac{M}{2p_1p_2} \text{ AND } \therefore \hat{x}_1 = \frac{M}{2p_1p_2} p_2 = \frac{M}{2p_1}$$

$$\text{AND } \hat{x}_2 = \frac{M}{2p_1p_2} p_1 = \frac{M}{2p_2}.$$

(e) A SUFFICIENT CONDITION TO ENSURE THAT $\hat{x}$ IS A (GLOBAL) MAXIMUM OF u SUBJECT TO THE LINEAR BUDGET CONSTRAINT IS THAT $\hat{x}$ SATISFIES THE FOC AND THAT u IS QUASICONCAVE, FOR WHICH IN TURN A SUFFICIENT CONDITION IS THAT THE BORDERED HESSIAN MATRIX

$$\mathbb{B} = \begin{bmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{21} & u_{22} \end{bmatrix}$$ HAS A POSITIVE DETERMINANT.

WE HAVE $|\mathbb{B}| = \begin{vmatrix} 0 & x_2 & x_1 \\ x_2 & 0 & 1 \\ x_1 & 1 & 0 \end{vmatrix} = x_1x_2 + x_1x_2 = 2x_1x_2 > 0$,

THEREFORE $\hat{x}$ IS A GLOBAL MAXIMUM OF $u(\cdot)$ SUBJECT TO THE BUDGET CONSTRAINT.

(f) WE KNOW THAT $\frac{\partial v}{\partial m} = \lambda = \frac{M}{2p_1p_2}$, FROM ABOVE, IN (d).

DIRECTLY, WE HAVE

$$v(p_1, p_2, M) = \hat{x}_1(p_1, p_2, M)\, \hat{x}_2(p_1, p_2, M)$$

$$= \left(\frac{M}{2p_1}\right)\left(\frac{M}{2p_2}\right), \text{ FROM (d)}$$

$$= \frac{M^2}{4p_1p_2}. \quad \therefore \frac{\partial v}{\partial M} = 2\frac{M}{4p_1p_2} = \frac{M}{2p_1p_2}.$$

(3) THEOREM: LET $f: V \to \mathbb{R}$ BE A REAL-VALUED LINEAR FUNCTION ON A VECTOR SPACE V; LET A AND B BE SUBSETS OF V; AND LET $\bar{x}_a \in A$ AND $\bar{x}_b \in B$. THEN $\bar{x}_a + \bar{x}_b$ MAXIMIZES f ON $A+B$ IF AND ONLY IF $\bar{x}_a$ MAXIMIZES f ON A AND $\bar{x}_b$ MAXIMIZES f ON B.

PROOF:

FIRST ASSUME THAT $\bar{x}_a$ MAXIMIZES f ON A AND $\bar{x}_b$ MAXIMIZES f ON B, AND LET $\bar{x} = \bar{x}_a + \bar{x}_b$. WE SHOW THAT $\bar{x}$ MAXIMIZES f ON $A+B$. LET $x \in A+B$, AND WE SHOW THAT $f(x) \leqq f(\bar{x})$. LET $x_a \in A$ AND $x_b \in B$ BE SUCH THAT $x_a + x_b = x$. WE HAVE

SUCH x_a, x_b EXIST, BECAUSE $x \in A+B$

$$\begin{aligned} f(x) &= f(x_a + x_b) = f(x_a) + f(x_b), \text{ BECAUSE } f \text{ IS LINEAR} \\ &\leqq f(\bar{x}_a) + f(\bar{x}_b), \text{ BECAUSE } \bar{x}_a \text{ AND } \bar{x}_b \text{ MAXIMIZE } f \\ &= f(\bar{x}_a + \bar{x}_b), \text{ BECAUSE } f \text{ IS LINEAR} \\ &= f(\bar{x}). \end{aligned}$$

TO ESTABLISH THE CONVERSE, ASSUME THAT $\bar{x}$ MAXIMIZES f ON $A+B$, AND LET $\bar{x}_a \in A$ AND $\bar{x}_b \in B$ BE SUCH THAT $\bar{x}_a + \bar{x}_b = \bar{x}$. SUPPOSE (WLOG) THAT $\bar{x}_a$ DOES NOT MAXIMIZE f ON A — i.e., THERE IS AN $x_a \in A$ s.t. $f(x_a) > f(\bar{x}_a)$. THEN WE HAVE

SUCH $\bar{x}_a, \bar{x}_b$ EXIST BECAUSE $\bar{x} \in A+B$

$$\begin{aligned} f(x_a + \bar{x}_b) &= f(x_a) + f(\bar{x}_b), \text{ BECAUSE } f \text{ IS LINEAR} \\ &> f(\bar{x}_a) + f(\bar{x}_b), \text{ BECAUSE } f(x_a) > f(\bar{x}_a) \\ &= f(\bar{x}_a + \bar{x}_b), \text{ BECAUSE } f \text{ IS LINEAR} \\ &= f(\bar{x}). \end{aligned}$$

THEREFORE $\bar{x}$ DOES NOT MAXIMIZE f ON $A+B$, A CONTRADICTION. ||

(4) IT'S OBVIOUS THAT $\mathcal{S}$ IS NONEMPTY (FOR EXAMPLE, ONE OF ITS ELEMENTS IS THE SET V ITSELF). AND IT'S OBVIOUS THAT THE GIVEN OPERATIONS SATISFY (VS1) AND (VS2). SO, SINCE WE'RE TOLD THAT $\mathcal{S}$ IS <u>NOT</u> A VECTOR SPACE, $\mathcal{S}$ MUST NOT BE A SUBSET OF V. AND INDEED THAT'S THE CASE: THE ELEMENTS OF $\mathcal{S}$ ARE NOT <u>ELEMENTS</u> OF V, BUT <u>SUBSETS</u> OF V.

NEVERTHELESS, IF THE OPERATIONS SATISFY (VS3) – (VS8), THEN $\mathcal{S}$ WOULD BE A VECTOR SPACE (BUT NOT A SUBSPACE OF V). SO IT MUST BE THAT ONE OF THESE SIX PROPERTIES IS VIOLATED. (VS3), (VS4), AND (VS8) ARE OBVIOUSLY SATISFIED, AND IT SEEMS AS IF (VS5) PROBABLY IS TOO (AND THAT CAN BE EASILY SHOWN). ALSO (VS6) IS SATISFIED: THE SINGLETON SET $\{\underline{0}\}$ CONSISTING OF JUST THE ORIGIN OF V SATISFIES $\{\underline{0}\}+A=A$ FOR ALL SETS $A \subseteq V$ — i.e., $\{\underline{0}\}$ IS THE ADDITIVE IDENTITY IN $\mathcal{S}$.

THIS LEAVES (VS7), AND INDEED, FOR ANY SUBSET $A \subseteq V$ CONSISTING OF MORE THAN ONE VECTOR, THERE IS NO SET B FOR WHICH $A+B=\{\underline{0}\}$. (FOR EXAMPLE, LET $A=\{\underline{0}, a\}$ FOR SOME $a \neq \underline{0}$; THEN $A+B=A \cup (a+B) \neq \{\underline{0}\}$.)

THE ABOVE SOLUTION HAS WAY MORE DETAIL THAN NECESSARY. A PERFECTLY GOOD ANSWER IS TO POINT OUT THAT $\{\underline{0}\}$ IS THE ADDITIVE IDENTITY AND THAT NO SET B SATISFIES $A+B=\{\underline{0}\}$ IF A HAS TWO OR MORE ELEMENTS. NOTHING MORE IS NEEDED.

(5) (a) S IS OBVIOUSLY NONEMPTY, AND IT SATISFIES (VS1) AND (VS2): THE SUM OF SYMMETRIC MATRICES IS CLEARLY SYMMETRIC, AND A SCALAR MULTIPLE OF A SYMMETRIC MATRIX IS CLEARLY SYMMETRIC. THEREFORE S IS A VECTOR SUBSPACE OF M.

D_+ AND D_- ARE NOT SUBSPACES, AS THEY VIOLATE (VS2): IF A IS A NONZERO POSITIVE SEMIDEFINITE MATRIX, THEN $(-1)A$ IS NEGATIVE SEMIDEFINITE, AND THE ONLY MATRIX THAT'S BOTH POSITIVE AND NEGATIVE SEMIDEFINITE IS THE ZERO MATRIX. SIMILARLY FOR A NEGATIVE SEMIDEFINITE MATRIX A.

D IS NOT A SUBSPACE BECAUSE IT VIOLATES (VS1): LET $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ AND $B=\begin{bmatrix}-2&0\\0&0\end{bmatrix}$. THEN A IS POSITIVE SEMIDEFINITE AND B IS NEGATIVE SEMIDEFINITE, SO $A\in D$ AND $B\in D$. BUT $A+B=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$, WHICH IS INDEFINITE, SO $A+B\notin D$.

(b) IF C_+ WERE A SUBSPACE, THEN D_+, THE SET OF ALL HESSIAN MATRICES OF THE TWICE-DIFFERENTIABLE FUNCTIONS IN C_+, WOULD BE A SUBSPACE OF M, BUT WE'VE SHOWN IT'S NOT. SIMILARLY, C_- IS NOT A SUBPACE.

C IS NOT A SUBSPACE BECAUSE D IS THE SET OF ALL HESSIAN OF THE TWICE-DIFFERENTIABLE FUNCTIONS IN C, AND D IS NOT A SUBSPACE.