Math Camp 2020: Exercise Set #1

1. For each of the following statements, determine whether the statement is true or false, and if false indicate *why* it's false.

(a) $\{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^3 \times \mathbb{R}^2$.

(b) $\{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^3 \cup \mathbb{R}^2$.

(c) $\{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$.

(d) $\big((2,1,1),(2,1),(1,1)\big) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$.

(e) $\big\{\big((2,1,1),(2,1),(1,1)\big)\big\} \subseteq \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$.

(f) $\exists n \in \mathbb{N} : \{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^n$.

(g) $(1,2,3) \subseteq \mathbb{N}$.

(h) $(1,2,3) \in \mathbb{N}$.

(i) $(1,2,3) \in \mathbb{R}^3$.

(j) $\{(2,1,1),(2,1),(1,1)\} = \{(2,1),(2,1,1),(1,1)\}$.

(k) $\mathbb{R}^2 \cup \mathbb{R}^3 = \mathbb{R}^3 \cup \mathbb{R}^2$.

(l) $\mathbb{R}^2 \times \mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^2$.

2. In our first Logic lecture (Lecture #5) we listed compound statements that covered 11 of the 16 possible "profiles" of truth values. Using only negation, conjunction, and disjunction, for each of the remaining five truth-value profiles construct a compound statement that has that truth-value profile.

3. Let A be an $m \times n$ matrix and let f be the function $f(x) = Ax$. Sections 7.3 and 7.4 of Simon & Blume might be helpful for this exercise. Drawing diagrams will also be helpful.

(a) What is the domain of f and what is the target space of f?

(b) What conditions on m, n, and A are both necessary and sufficient for f to be one-to-one and onto its target space?

(c) If $m = 2$ and $n = 1$, under what conditions (if any) will f be one-to-one? Onto?

(d) If $m = 2$ and $n = 3$, under what conditions (if any) will f be one-to-one? Onto?

4. "You can fool some of the people all the time, and you can fool all the people some of the time, but you can't fool all the people all the time." (Attributed by some to Abraham Lincoln.) Hammack's Exercise #2.9.12 asks you to translate this sentence into symbolic logic. But the sentence is ambiguous — *i.e.*, it can be translated into different formal statements that have different truth values.

Let's define the following three sentences, or statements:

SA: You can fool some of the people all the time.

AS: You can fool all the people some of the time.

AA: You can fool all the people all the time.

It's clear that the original sentence is the conjunction SA $\wedge$ AS $\wedge \sim$ AA. However, the sentences SA and AS, as written here in English, are ambiguous — as you're going to demonstrate.

Let's say there are three people (Abby, Beth, and Carl) and three times (Monday, Wednesday, and Friday). Let $X = \{a, b, c\}$ and $Y = \{m, w, f\}$. Let's write the (open) statement "x can be fooled at time y" as $\varphi(x, y)$. Note that you can think of φ as a function, $\varphi : X \times Y \to \{T, F\}$, where T or F is the truth value of $\varphi(x, y)$. You can also write $\Phi = \{(x, y) \,|\, \varphi(x, y) = T\}$, the set of pairs (x, y) for which x can be fooled at time y. Figure 1 depicts the statement AA; Figure 2 depicts the statement "Abby can be fooled on Monday and Wednesday (only), Beth can be fooled on Monday (only), and Carl can be fooled on Monday and Friday (only)."

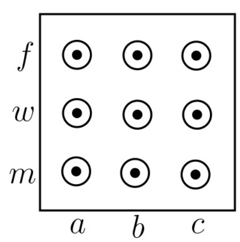


Figure 1

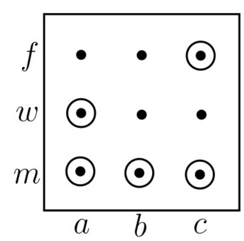


Figure 2

Now note that the statements SA and AS can each have two different meanings:

SA could mean $\exists x \in X \colon \forall y \in Y \colon \varphi(x, y) = T$, which we'll denote as SA_1, or

SA could mean $\forall y \in Y \colon \exists x \in X \colon \varphi(x, y) = T$, which we'll denote as SA_2.

AS could mean $\exists y \in Y \colon \forall x \in X \colon \varphi(x, y) = T$, which we'll denote as AS_1, or

AS could mean $\forall x \in X \colon \exists y \in Y \colon \varphi(x, y) = T$, which we'll denote as AS_2.

As in Figures 1 and 2, for each of the following four statements L_i $(i = 1, 2, 3, 4)$ draw a diagram of the Cartesian product $X \times Y$, and use the diagram to represent a function φ (*i.e.*, a set Φ) for which the statement is True and as many of the other three statements as possible are False. (For each statement L_i there are multiple correct solutions — *i.e.*, multiple correct sets $\Phi \subseteq X \times Y$. You only need to draw one correct solution for each L_i.) In each of the four cases indicate which statements are True and which are False.

$$\mathrm{L}_1 = \mathrm{SA}_1 \wedge \mathrm{AS}_1 \wedge \sim \mathrm{AA} \qquad \mathrm{L}_3 = \mathrm{SA}_2 \wedge \mathrm{AS}_1 \wedge \sim \mathrm{AA}$$
$$\mathrm{L}_2 = \mathrm{SA}_1 \wedge \mathrm{AS}_2 \wedge \sim \mathrm{AA} \qquad \mathrm{L}_4 = \mathrm{SA}_2 \wedge \mathrm{AS}_2 \wedge \sim \mathrm{AA}.$$

5. Now let X and Y be arbitrary sets and let $\varphi(x, y)$ be an arbitrary open statement (or Φ an arbitrary subset of $X \times Y$). Prove that SA_1 implies SA_2 or that AS_1 implies AS_2. (These two implications are clearly equivalent to one another, so you only need to prove one of them.) This proof should require no more than a sentence or two.

Note: Understanding Exercises #4 and #5 will pretty much guarantee that you will have no trouble understanding uniform continuity and uniform convergence. Failing to understand these two exercises will likely mean you'll have a hard time with uniform continuity and uniform convergence.

6. Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$, and define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = ax^n$.

(a) Use your knowledge of proof by induction to verify that if n is even, then f is neither one-to-one nor onto $\mathbb{R}$.

(b) Assuming that the function f is continuous (it is, but don't prove it here), can you determine whether f is one-to-one and/or onto if n is odd instead of even? You might want to use the Intermediate Value Theorem here. (Oddly, the Intermediate Value Theorem doesn't appear in Simon & Blume. It's on page 60 of Sundaram, and a Google search will also work fine; for example, *https://en.wikipedia.org/wiki/Intermediate_value_theorem*.

7. (Hammack, Chapter 10, Exercise #2)
Prove that for every $n \in \mathbb{N}$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

8. For each of the following vectors, determine the vector's length and determine a vector of length 1 that points in the opposite direction.

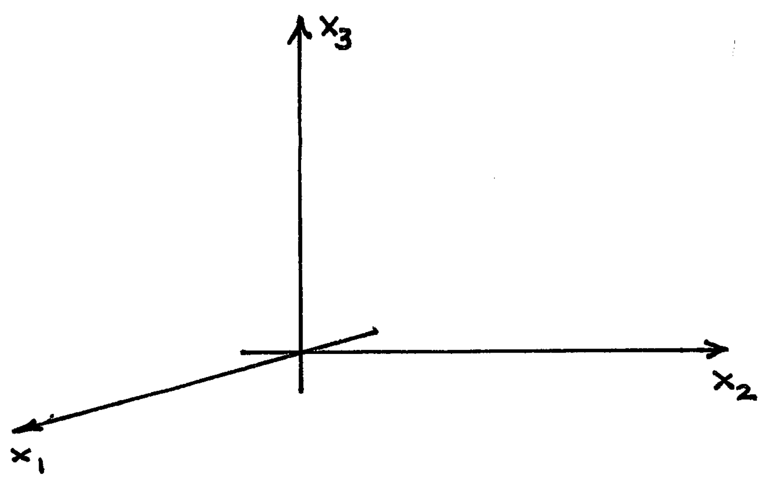
(a) $(12, 5)$ (b) $(1, 1)$ (c) $(3, 3)$ (d) $(3, -3)$ (e) $(-1, 1, -1)$
(f) $(1, 1, 1, 1)$ (g) $(12, 0, 0, 5)$ (h) $(12, -1, 1, 5)$.

9. For each of the following vectors $\mathbf{v}$, find two vectors $\mathbf{u}$ and $\mathbf{w}$ that are orthogonal to $\mathbf{v}$ and have the same length as $\mathbf{v}$:

(a) $\mathbf{v} = (1, 0)$ (b) $\mathbf{v} = (1, 1)$ (c) $\mathbf{v} = (1, 0, 0)$ (d) $\mathbf{v} = (1, 1, 1)$.

10. Draw a diagram depicting the following, in $\mathbb{R}^2$: the vector $\mathbf{a} = (1, -1)$; the line $H = \{\mathbf{x} \in \mathbb{R}^2 \,|\, \mathbf{a} \cdot \mathbf{x} = 0\}$; and the vectors $(1, 1), (1, 0), (2, 1), (0, -1), (0, 1), (-1, 2), (-1, 0), (-1, 1)$. Denoting each of these vectors in turn by $\mathbf{v}$, determine $\mathbf{a} \cdot \mathbf{v}$; which side of the line H the vector $\mathbf{v}$ lies on (lower-right (SE) or upper-left (NW)); and, both visually (by looking at the diagram) and analytically (via the dot product) whether the angle between $\mathbf{a}$ and $\mathbf{v}$ is acute, obtuse, or a right angle (they're orthogonal).

Here's something I find helpful for visualizing things in $\mathbb{R}^3$: Assuming you're in a room with walls that are perpendicular to one another and to the floor, look at a corner where two of the walls meet the floor; imagine the positive x_1-axis to be the line where the wall to your left meets the floor; the positive x_2-axis to be the line where the wall to your right meets the floor; and the positive x_3-axis to be the line where the two walls meet.



11. Let $\mathbf{a}$ be the vector $\mathbf{a} = (1, 1, 1)$ and let H be the plane $H = \{\mathbf{x} \in \mathbb{R}^3 \,|\, \mathbf{a} \cdot \mathbf{x} = 0\}$ in $\mathbb{R}^3$. For each of the following vectors $\mathbf{v}$ determine analytically (via the dot product) whether the angle between $\mathbf{a}$ and $\mathbf{v}$ is acute, obtuse, or is a right angle (they're orthogonal); and whether $\mathbf{v}$ lies on the same side of H as the vector $\mathbf{a}$, on the opposite side of H, or *on* the plane H:

(a) $\mathbf{v} = (1, 0, 0)$ (b) $\mathbf{v} = (0, 0, 1)$ (c) $\mathbf{v} = (0, -1, 0)$ (d) $\mathbf{v} = (0, 1, 1)$ (e) $\mathbf{v} = (1, -1, 0)$.

12. Prove that if $n > 1$ then the x_1-axis in $\mathbb{R}^n$ is a closed set (or equivalently, prove that its complement is an open set).

13. Prove that every finite subset of $\mathbb{R}^n$ is a closed set.

Math Camp 2020: Exercise Set #2

1. The set $\mathbb{R}^\infty$ of sequences of real numbers is a vector space under the usual (component-wise) definitions of vector addition and scalar multiplication:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) := (x_1 + y_1, x_2 + y_2, \ldots) \quad \text{and} \quad \alpha(x_1, x_2, \ldots) := (\alpha x_1, \alpha x_2, \ldots).$$

Verify that the set ℓ^∞ of bounded sequences of real numbers, with the operations of vector addition and scalar multiplication it inherits from $\mathbb{R}^\infty$, is a vector subspace of $\mathbb{R}^\infty$. [A sequence $(x_n)_1^\infty \in \mathbb{R}^\infty$ is **bounded** if $\exists M \in \mathbb{R} : \forall n \in \mathbb{N} : |x_n| \leqq M$. See this link for an application to economics.]

2. Let S be the set of all real sequences that have only a finite number of non-zero terms — *i.e.*, $S = \left\{(x_n)_1^\infty \in \mathbb{R}^\infty \mid \{n \in \mathbb{N} \mid x_n \neq 0\} \text{ is a finite set } \right\}$. Is S a vector subspace of $\mathbb{R}^\infty$? Is S a vector subspace of ℓ^∞? Verify your answers.

3. Provide a proof of the following proposition: If $\dim V = n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span V, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V.

4. At about 32:00 of Lecture 14(B) I say"You might guess that $\mathbb{R}^\infty$ is infinite-dimensional. Well, let's see." Then I go on to instead check whether the *infinite* set consisting of the unit vectors is a basis. In the end I point out that we *didn't* check whether $\mathbb{R}^\infty$ has a finite basis, and that that would be a good exercise. Here's that exercise. It's a little bit challenging, but what we're going to do is a good example of how to prove something when it's not clear how to do it. The original problem — to show that no finite set can be a basis of $\mathbb{R}^\infty$ — is part (d), and that was originally going to be this exercise. When I tried to do it myself I came up with basically the right idea pretty quickly, but I couldn't see exactly how to work out the details. So what I did is what one typically has to do in this situation: you cook up a simpler version of the problem and solve that, hoping that that will give you the insight how to solve the real problem. What we're going to do here is go *several steps* simpler — working backward from (d), getting simpler at each step — and work our way back up to the actual problem. (It's still going to be kind of challenging.)

(a) Prove that the set $\{a^1, a^2\}$, in which $a^1 = (1, 2, 1)$ and $a^2 = (2, 1, 1)$, is not a basis of $\mathbb{R}^3$.

(b) Prove that no subset of $\mathbb{R}^3$ with only two vectors can be a basis of $\mathbb{R}^3$.

(c) Prove that the set $\{a^1, a^2\}$, in which $a^1 = (1, 2, 1, 1, 1, \ldots)$ and $a^2 = (2, 1, 1, 1, 1, \ldots)$ is not a basis of $\mathbb{R}^\infty$.

(d) Prove that no subset of $\mathbb{R}^\infty$ with only a finite number of vectors can be a basis of $\mathbb{R}^\infty$.

5. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear function for which $f(1,0,0) = (1,1,1)$, $f(0,1,0) = (0,1,0)$, and $f(0,0,1) = (2,3,2)$.

(a) Write down the matrix A that defines f — *i.e.*, the matrix for which $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

The column space of A is the space spanned by the columns of A; the row space of A is the space spanned by the rows of A; and the nullspace of A is the set of solutions of the equation system $A\mathbf{x} = \mathbf{0}$ — denoted $\mathrm{Col}(A)$, $\mathrm{Row}(A)$, and $\mathrm{Null}(A)$, respectively.

(b) Determine bases for each of the spaces $\mathrm{Col}(A)$, $\mathrm{Row}(A)$, and $\mathrm{Null}(A)$.

(c) Is f one-to-one? Is it onto $\mathbb{R}^3$? Explain how you determined the answers to these two questions.

6. Provide a proof by induction that if S is a convex set in a vector space V, then every convex combination of vectors in S is also in S. Note that this, together with its converse, is a theorem in Lecture 15. As we said there, the converse is a trivial immediate consequence of the definition of a convex set. The initial step of the induction proof here is equally trivial: that every convex combination of *two* vectors in S is also in S, which is obviously true, as the definition of a convex set.

7. Exercise #21.19 in Simon & Blume. Note that the definition of quasiconcave function they're using is the one that says a quasiconcave function is one for which all upper-contour sets are convex.

8. Here's a useful theorem: If f_1 and f_2 are strictly concave functions defined on a convex subset of a vector space V, then the sum $f_1 + f_2$ is also strictly concave.

(a) Provide a proof of the theorem.

(b) Let $u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$ be a utility function defined on $\mathbb{R}^2_+$. The real function $f(z) = \sqrt{z}$ is strictly concave. Explain why the theorem doesn't apply to this situation — the theorem can't be invoked to establish that u is strictly concave.

Here's another extremely useful theorem: If X_1 and X_2 are intervals in $\mathbb{R}$ and if $f_1 : X_1 \to \mathbb{R}$ and $f_2 : X_2 \to \mathbb{R}$ are strictly concave functions, then the function $f : X_1 \times X_2 \to \mathbb{R}$ defined by $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ is strictly concave. Note that this theorem *does* imply that the function $u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$ is strictly concave (because the real function $f(z) = \sqrt{z}$ is strictly concave).

(c) Provide a proof of this second theorem.

Math Camp 2020: Exercise Set #3

1. Exercise #10.16 in Simon & Blume. Note how much easier it is to prove the Triangle Inequality for these norms than it is for the Euclidean norm, where we needed to first prove the Cauchy-Schwarz Inequality. This is one more example of how one or the other of these two norms is often easier to work with than the Euclidean norm. [In the S&B exercise the norms are written with *three* vertical lines on each side. That's a typo; they should be just $\|(u_1, u_2)\|$.]

2. Prove that if a sequence $\{\mathbf{x}(n)\}$ converges to $\overline{\mathbf{x}} \in \mathbb{R}^\ell$, then for each $k \in \{1, \ldots, \ell\}$ the component sequence $\{x_k(n)\}$ converges to $\overline{x}_k$. [In Lecture 18 we said that if a set in $\mathbb{R}^\ell$ is open in one norm on $\mathbb{R}^\ell$ then it's open in all other norms on $\mathbb{R}^\ell$, so you can use whichever one of the norms $\|\cdot\|_1$, $\|\cdot\|_2$, or $\|\cdot\|_\infty$ that makes your proof easiest.]

3. Our definition of a compact set is a set in which every sequence has a subsequence that converges to a point in the set. Prove directly from this definition that every finite set of points in $\mathbb{R}^n$ is compact. [As in #1, you can use any norm on $\mathbb{R}^n$ here.]

4. Determine whether the utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ is continuous if u is defined by

$$u(x_1, x_2) = \begin{cases} x_1 x_2, & \text{if } x_1 x_2 \leq 4 \\ 4, & \text{if } 4 < x_1 x_2 < 9 \\ x_1 x_2 - 5, & \text{if } x_1 x_2 \geq 9 \end{cases}$$

5. There are two elements of "doing proofs". One element is being able to *create* a proof of something. The second element, which is necessary for the first one, is being able to tell whether a proof is valid — *i.e.*, whether a given proof actually succeeds at establishing the proposition we're trying to prove. Each of the following exercises consists of a conjecture and a proposed proof. In each case the conjecture may be true or false, and the proof may be correct (valid) or not. Of course, we can't have a conjecture that's false and a proof that's valid. What you're to do, for each exercise, is to determine which of the three remaining categories the exercise falls into: (a) the conjecture is false (in this case, provide a counterexample to verify that it's false *and* point out why the proof is not valid); (b) the conjecture is true and the proof is valid (here you only need to say that's the case); or (c) the conjecture is true but the proof is not valid (in this case, point out why the proof is invalid and fix it or otherwise give a correct proof).

Conjecture 1: Let $X' = [0, 1]$, the closed unit interval in $\mathbb{R}$, with the usual norm (the absolute value). The set $S = [0, \frac{1}{2}) = \{x \in X' \mid 0 \leqq x < \frac{1}{2}\}$ is open in X' (*i.e.*, open relative to X').

Proof?: Let $V = (-\frac{1}{2}, \frac{1}{2})$, the open interval in $\mathbb{R}$ between $-\frac{1}{2}$ and $\frac{1}{2}$. Then $S = V \cap X'$ and V is open in $\mathbb{R}$, so S is open in X'. ■

Conjecture 2: Let $\{x(n)\}$ be a sequence in $\mathbb{R}^\ell$. If for each $k \in \{1, \ldots, \ell\}$ the component sequence $\{x_k(n)\}$ converges to $\overline{x}_k \in \mathbb{R}$, then $\{x(n)\}$ converges to $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_\ell) \in \mathbb{R}^\ell$. [Note that this is the converse of a proposition you proved in a previous exercise.]

Proof?: Let $\epsilon > 0$. Then for every $k \in \{1, \ldots, \ell\}$ there is an $\overline{n}_k \in \mathbb{N}$ such that $n > \overline{n}_k \Rightarrow |x_k(n) - \overline{x}_k| < \frac{1}{\ell}\epsilon$. Therefore $\sum_{k=1}^{\ell} |x_k(n) - \overline{x}_k| < \epsilon$; *i.e.*, $\|x(n) - \overline{x}\|_1 < \epsilon$. Since this is true for every $\epsilon > 0$, $\{x(n)\}$ converges to $\overline{x}$. ■ [Do you think $\| \cdot \|_1$ was the best norm to use?]

Conjecture 3: The set $S = \{\, x \in \mathbb{Q} \mid 0 \leqq x < \sqrt{2}\,\}$ — *i.e.*, the set of nonnegative rational numbers less than $\sqrt{2}$ — is not closed in $\mathbb{Q}$.

Proof?: Let $\{x_n\}$ be a sequence of rational numbers in S (each is therefore less than $\sqrt{2}$) that converges to $\sqrt{2}$. Since $\sqrt{2} \notin S$, S is not a closed set. ■

Conjecture 4: If $\{\mathbf{x}_n\}$ converges to $\overline{\mathbf{x}}$ in $\mathbb{R}^\ell$, then every subsequence of $\{\mathbf{x}_n\}$ converges to $\overline{\mathbf{x}}$.

Proof?: Let $\{\mathbf{x}_{n_m}\}$ be a subsequence of $\{\mathbf{x}_n\}$ and suppose $\{\mathbf{x}_{n_m}\}$ converges to a point $\widetilde{\mathbf{x}}$. We'll show that $\widetilde{\mathbf{x}} = \overline{\mathbf{x}}$. Suppose instead that $\widetilde{\mathbf{x}} \neq \overline{\mathbf{x}}$, and we'll obtain a contradiction. Let $\epsilon = \frac{1}{2}\|\widetilde{\mathbf{x}} - \overline{\mathbf{x}}\|$. Then

$$\exists\, \overline{m} \in \mathbb{N} : m > \overline{m} \Rightarrow \|\mathbf{x}_{n_m} - \widetilde{\mathbf{x}}\| < \epsilon \qquad \text{and} \qquad \exists\, \overline{n} \in \mathbb{N} : n > \overline{n} \Rightarrow \|\mathbf{x}_n - \overline{\mathbf{x}}\| < \epsilon.$$

Let m satisfy $m > \overline{m}$ and $n_m > \overline{n}$. Then $\|\mathbf{x}_{n_m} - \widetilde{\mathbf{x}}\| < \epsilon$ and $\|x_{n_m} - \overline{\mathbf{x}}\| < \epsilon$. Therefore the Triangle Inequality yields $\|\widetilde{\mathbf{x}} - \overline{\mathbf{x}}\| \leqq \|\mathbf{x}_{n_m} - \widetilde{\mathbf{x}}\| + \|\mathbf{x}_{n_m} - \overline{\mathbf{x}}\| < 2\epsilon$, a contradiction, since $\|\widetilde{\mathbf{x}} - \overline{\mathbf{x}}\| = 2\epsilon$. ■

Math Camp 2020: Exercise Set #4

1. Determine whether each of the following matrices is positive definite or semidefinite, or is indefinite. If the matrix is indefinite, provide a vector at which the associated quadratic form is positive and one at which the quadratic form is negative. If the matrix is semidefinite, provide a vector $\mathbf{x} \neq \mathbf{0}$ at which the quadratic form has the value zero.

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 2 & 4 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 2 & 3 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -3 \end{bmatrix}$

2. Use the differential function (also called the total differential, or the total derivative, as Simon & Blume call it) to give an informal argument why the gradient of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at $\overline{x} \in \mathbb{R}^2$ is orthogonal to the function's level curve at $\overline{x}$. Then recast the argument so that it applies to functions $f : \mathbb{R}^n \to \mathbb{R}$.

3. The function

$$f(x) = \begin{cases} x^2, & x \leqq 1 \\ 3x - 2, & x \geqq 1 \end{cases}$$

is continuous everywhere in $\mathbb{R}$. (Note that $f(1) = 1$.) By direct appeal to the definition of the derivative, show that f is not differentiable at $x = 1$.

4. A C^2 function $f : X \to \mathbb{R}$, defined on an open convex set $X \subseteq \mathbb{R}^n$, is concave if and only if its 2^{nd}-degree Taylor polynomial is concave at every $\mathbf{x} \in X$, which is the case if and only if the Hessian matrix of second partial derivatives, $D^2 f(\mathbf{x})$, or $H(\mathbf{x})$, is negative semidefinite at every $\mathbf{x} \in X$. Moreover, a sufficient condition for f to be *strictly* concave is that the Hessian matrix be negative definite at every $\mathbf{x} \in X$.

(a) Provide a counterexample to show that a negative definite Hessian matrix is not necessary for a C^2 function to be strictly concave.

(b) For parameter values a_1, a_2, and a_3 define the function $f : \mathbb{R}^2_{++} \to \mathbb{R}$ by $f(x_1, x_2) = a_1 log x_1 + a_2 e^{x_2} + a_3 x_1 x_2$. Determine values of the parameters (if any) for which f is concave, for which f is convex, and for which f is neither concave nor convex.

5. A **critical point** of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is a point $\mathbf{x} \in \mathbb{R}^n$ at which $\nabla f(\mathbf{x}) = \mathbf{0}$ — *i.e.*, at which all partial derivatives have the value zero. For each of the following exercises in Simon & Blume determine the critical points of the function and determine at each critical point whether the Hessian matrix (the matrix of second partial derivatives) is positive or negative definite or semidefinite or indefinite:

#17.1(a), 17.1(b), and 17.2(a).

The solutions in the back of the book for these three problems are not all correct.

Math Camp 2020: Exercise Set #5

1. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = x_1 x_2(3 - x_1 - x_2)$. Note that f is twice differentiable everywhere in $\mathbb{R}^2$. (Actually, as a polynomial, derivatives of f of all orders exist on all of $\mathbb{R}^2$.)

(a) Find all the critical points of f — *i.e.*, all points $\mathbf{x} \in \mathbb{R}^2$ at which $\nabla f(\mathbf{x}) = \mathbf{0}$.

(b) Determine the Hessian matrix $D^2 f(x_1, x_2)$ as a function of (x_1, x_2).

(c) Determine all local maxima or local minima of f, if any.

(d) Determine all global maxima or minima of f, if any.

(e) At the point $(\overline{x}_1, \overline{x}_2) = (1, 1)$ write the 2^{nd}-degree Taylor polynomial in Δx_1 and Δx_2 — *i.e.*, write all the polynomial's terms.

2. Suppose that we have n data points $(x_1, y_1), \ldots, (x_n, y_n)$ and we want to determine the least-squares line $y = mx + b$ for the given data. This is the line that minimizes the sum of the squares of the n residuals $r_i = mx_i + b - y_i$, *i.e.*, that minimizes the function

$$S(m, b) = \sum_{i=1}^{n} (mx_i + b - y_i)^2.$$

Note that the x_is and y_is are not variables in this problem, they're the data, which we're given. The variables are m and b, the slope and intercept of the line that minimizes the sum of squares.

(a) Draw a diagram depicting this problem with six data points. Indicate the residuals in the diagram.

(b) Prove that the least-squares line has slope and intercept

$$m = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2} \quad \text{and} \quad b = \frac{(\sum x_i^2)(\sum y_i) - (\sum x_i)(\sum x_i y_i)}{n \sum x_i^2 - (\sum x_i)^2}.$$

3. (See Exercise #21.12 in S&B) Assume that a one-product firm faces an inverse demand function $p = F(q)$ for its product and that the firm's cost function is $C(q)$. Assume that $F(\cdot)$ and $C(\cdot)$ are both twice differentiable.

(a) Obtain expressions for the firm's revenue function and its marginal revenue function.

(b) Determine some condition(s) that are sufficient to ensure that the firm's profit function $\pi(q)$ is concave, and some condition(s) that guarantee $\pi(\cdot)$ is *strictly* concave, What are the conditions for concavity and strict concavity of π when the demand function is linear? What would be the significance of $\pi(\cdot)$ being strictly concave?

For the remainder of this exercise assume that F is the linear function $F(q) = a - bq$ on the domain $0 \leqq q \leqq a/b$; that $C(q) = cq$; and that $a, b, c > 0$ and $a > c$.

(c) Write down the firm's revenue function and its profit function, and obtain its marginal revenue and marginal cost functions. All of these functions should have one decision variable and three parameters — for example, $\pi(q; a, b, c)$.

(d) Obtain the solution function of the firm's profit-maximization problem.

(e) Obtain the value function for the firm's profit-maximization problem.

(f) Determine the derivative of the solution function with respect to the parameter c.

(g) Determine the derivative of the value function with respect to the parameter c. Compare this to the derivative of the profit function with respect to c. How do you account for this comparison, in light of the fact that the optimal level of output changes in response to changes in c?

4. Assume that $a > 0$ and $b > 0$. On the domain $\mathbb{R}^2_+$ define the functions

$$f(\mathbf{x}; a) = ax_1 + x_2, \qquad G^1(\mathbf{x}) = 2x_1 + x_2, \qquad G^2(\mathbf{x}) = 2x_1 + 4x_2, \qquad G^3(\mathbf{x}) = x_1 + x_2$$

and the maximization problem

$$(\mathbf{P}) \quad \max f(\mathbf{x}; a) \quad s.t. \;\; G^1(\mathbf{x}) \leqq 9, \;\; G^2(\mathbf{x}) \leqq 18, \;\; G^3(\mathbf{x}) \leqq b.$$

For parts (a), (b), and (c) assume that $a = 1$, so the problem's only parameter is b.

(a) For each of the following values of b, draw the constraint set (the feasible set): $b = 3$, $b = 5$, and $b = 8$.

(b) For values of b greater than 8, determine the solution of the problem $(\mathbf{P})$; draw the gradients of the functions f, G^1, and G^2 at the solution; and determine the values of the Lagrange multipliers at the solution — *i.e.*, determine ∇f as a non-negative linear combination of the three G-gradients. Which constraints are binding at the solution and which are non-binding?

(c) Determine the solution function for the problem $(\mathbf{P})$. Note that for some values of b the solution function is not really a function: there are multiple solutions. Which values of b are those, and what are the solutions (the maximizers of f) in those cases? Determine the value function for the problem $(\mathbf{P})$.

For parts (d), (e), and (f) assume that $b = 5$, so the problem's only parameter is a.

(d) Draw the feasible set. Including the non-negativity constraints, there are 10 intersections of pairs of constraints. Identify which intersections are feasible points (vectors) and label them A, B, C, etc. At each of these feasible, labeled intersections, identify which constraints are binding and draw the gradients of the G^i functions for the binding "structural" constraints (*i.e.*, the binding G^i-constraints).

(e) At each intersection in part (d), determine the values of a for which that intersection is the optimal solution. Determine the solution function for the problem (**P**). Note that for some values of a the solution function is not really a function: there are multiple solutions. Which values of a are those, and what are the solutions (the maximizers of f) in those cases? Determine the value function for the problem (**P**) — remember, a is the only parameter in this part.

(f) For each intersection in part (d), determine the values of the Lagrange multipliers when that intersection is an optimal solution — *i.e.*, determine ∇f as a non-negative linear combination of the three G-gradients. The Lagrange multiplier values will depend upon a. (If you find it difficult to do this part for arbitrary values of a, then assume successively that $a = 1/4, a = 3/4, a = 3/2$, and $a = 3$.) At each intersection indicate the cone in which ∇f must lie.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and let $\overline{x} \in \mathbb{R}$ be a real number at which the first, second, and third derivatives of f all exist. Verify that the third-degree Taylor polynomial

$$P_3(\Delta x) := f(\overline{x}) + f'(\overline{x})\Delta x + \frac{1}{2}f''(\overline{x})(\Delta x)^2 + \frac{1}{6}f'''(\overline{x})(\Delta x)^3$$

that we use to approximate the function f near $\overline{x}$ has

(a) the same value as f at $\overline{x}$ (*i.e.*, at $\Delta x = 0$),

(b) the same slope (the same derivative) as f at $\overline{x}$,

(c) the same curvature (the same second derivative) as f at $\overline{x}$, and

(d) the same third derivative as f at $\overline{x}$.

Math Camp 2020: Exercise Set #6

1. The Kuhn-Tucker sufficient-condition theorem includes the following second-order conditions:

- each G^i is quasiconvex, and either
- f is concave, or
- f is quasiconcave and $\nabla f \neq \mathbf{0}$ at $\widehat{\mathbf{x}}$.

What we would like is a counterexample demonstrating that we can't dispense with the condition that $\nabla f \neq \mathbf{0}$ at $\widehat{\mathbf{x}}$ when f is merely quasiconcave but not concave. Provide such a counterexample using the following function from Exercise Set #3:

$$f(x_1, x_2) = \begin{cases} x_1 x_2, & \text{if } x_1 x_2 \leq 4 \\ 4, & \text{if } 4 < x_1 x_2 < 9 \\ x_1 x_2 - 5, & \text{if } x_1 x_2 \geq 9 \end{cases}$$

2. Define $f : \mathbb{R}^2_+ \to \mathbb{R}$ by $f(x_1, x_2) = x_1 x_2$. We wish to maximize f subject to the constraint $G(x_1, x_2) \leqq c$, where $G(x_1, x_2) = x_1^2 + x_2^2$ and $c > 0$.

(a) Determine the gradients of f and G as functions of $\mathbf{x} = (x_1, x_2)$.

(b) Determine the solution function of the maximization problem.

(c) Draw a diagram depicting the constraint, the solution, the objective-function contour through the solution, and the gradients of the objective and constraint functions at the solution.

(d) Determine the value function of the maximization problem.

(e) Verify that for every value of c the Kuhn-Tucker conditions are satisfied at the solution.

(f) Determine the value of the Lagrange multiplier at the solution.

(g) Assume that $c = 2$. What is the solution of the maximization problem? Apply the theorem on the last page of the lecture notes "Second-Order Conditions and Quadratic Forms with Constraints" to verify that the sufficient conditions for a constrained maximum are satisfied at your solution.

(h) On your diagram in (c), draw the line tangent to the constraint at the solution. Determine whether the objective function is positive or negative definite or semidefinite on this line.

3. It's trivial to show that the function $f(x_1, x_2) = x_1 x_2$, defined on the domain $\mathbb{R}^2_{++}$, is strictly convex if restricted to a line $x_2 = ax_1$ in $\mathbb{R}^2_{++}$ for any $a > 0$. Use the theorem on page 2 of the "Differentiable Quasiconcave Functions" lecture notes to verify that f is nevertheless quasiconcave on the entire domain $\mathbb{R}^2_{++}$.

4. In order to apply the same theorem as in #3 to the function $f(x_1, x_2, x_3) = x_1 x_2 x_3$ defined on the domain $\mathbb{R}^3_{++}$, you will need to check the signs of certain determinants.

(a) Write down in detail each of those determinants.

(b) In lieu of you evaluating the determinant of the 4×4 bordered matrix, I will tell you that the determinant is $-3(x_1 x_2 x_3)^2$. Any other determinants you will need for this question you should be able to evaluate. Use the theorem to determine whether f is quasiconcave, quasiconvex, or neither, by determining the signs of the required determinants.

5. In an example we determined the definiteness of the quadratic form $f(x_1, x_2) = x_1^2 - x_2^2$ on any line through the zero vector. Instead, use the theorem on page 8 of the lecture notes "Second-Order Conditions and Quadratic Forms with Constraints" to obtain the same result, about whether f is positive or negative definite or semidefinite, or indefinite, on any given line $x_2 = ax_1$.