Economics 519 Midterm Exam
University of Arizona
Fall 2017

Closed-book part: Don't consult books or notes until you've handed in your solutions to Problems #1 - #3. After you do consult books or notes, your solutions to Problems #1 - #3 will not be accepted.

1. A sequence $\{x_k\}$ of real numbers is said to be **bounded** if there is a number M such that $|x_k| \leqq M$ for every $k = 1, 2, \ldots$. Verify that the set B of bounded sequences is a subspace of the space $\mathbb{R}^\infty$ of all real sequences. I suggest writing sequences $\mathbf{x} \in \mathbb{R}^\infty$ as $\mathbf{x} = (x_1, x_2, x_3, \ldots)$.

2. Let $u : \mathbb{R}^2_{++} \to \mathbb{R}$ be defined by $u(x_1, x_2) = x_1 x_2$.

(a) Determine the gradient and the Hessian matrix of u as functions of (x_1, x_2).

(b) Determine any critical points of u.

For any strictly positive values of p_1, p_2, and M, define the constrained maximization problem

$$(\mathbf{P}) \qquad \max_{\mathbf{x} \in \mathbb{R}^2_{++}} u(x_1, x_2) = x_1 x_2 \qquad \text{s.t.} \;\; x_1, x_2 \geqq 0 \;\text{ and }\; p_1 x_1 + p_2 x_2 \leqq M.$$

(c) The **solution function** for this problem, $\widehat{\mathbf{x}} : \mathbb{R}^3_{++} \to \mathbb{R}^3_{++}$, gives us a vector $(\widehat{x}_1, \widehat{x}_2, \lambda) \in \mathbb{R}^3_{++}$ as a function of the parameter vector (p_1, p_2, M), where $(\widehat{x}_1, \widehat{x}_2)$ is the unique solution of $(\mathbf{P})$ and λ is the value of the Lagrange multiplier at the solution. Apply the Implicit Function Theorem to verify that the solution function exists on all of $\mathbb{R}^3_{++}$.

(d) Solve the three equations in the problem $(\mathbf{P})$'s Kuhn-Tucker first-order conditions (FOC) to obtain the consumer's demand functions $\widehat{\mathbf{x}}_i(p_1, p_2, M), i = 1, 2$ and the function $\lambda(p_1, p_2, M)$ that gives the value of the Lagrange multiplier λ. [Once you actually have these functions, (c) becomes redundant.]

(e) Use the second-order conditions to verify that any $\widehat{\mathbf{x}}$ that satisfies the (FOC) is a maximum.

(f) Use the Lagrange multiplier to determine the derivative of the value function. Then determine the value function explicitly, as the utility level at the optimal solution, and check that its derivative is as you had just determined.

3. Provide the statement and a proof of the Sum-of-Sets Maximization Theorem for just two sets — *i.e.*, the sum of two sets A and B in a vector space V.

The open-book part of the exam is on the following page.

Open-book part: Be sure to turn in your solutions to Problems #1 - #3 before using notes.

4. In any vector space V there is a natural definition of scalar multiplication of subsets: for any $X \subseteq V$ and any $\lambda \in \mathbb{R}$, λX is the set $\{\lambda\mathbf{x} \in V \mid \mathbf{x} \in X\}$. Verify that the set $\mathcal{S}$ of all subsets of V, with the sum of sets as defined in our lecture notes and scalar multiplication of sets, is *not* a vector space.

5. Let M denote the set of all 2×2 matrices. You've shown in an exercise that M is a vector space under the usual (*i.e.*, component-wise) definitions of addition and scalar multiplication of matrices. Let S denote the set of all *symmetric* 2×2 matrices, and let D_+, D_-, and D be the following subsets of S:

$$\begin{aligned} D_+ &= \{A \in S \mid A \text{ is positive semidefinite } \}, \\ D_- &= \{A \in S \mid A \text{ is negative semidefinite } \}, \\ D &= D_+ \cup D_-. \end{aligned}$$

(a) Determine which, if any, of the sets S, D_+, D_-, and D are vector subspaces of M (and verify your answers).

(b) The set $\mathcal{F}$ of all functions $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is a vector space under the usual definitions of addition and scalar multiplication of real-valued functions. What, if anything, do the results in (a) tell us about whether any of the following sets is a subspace of $\mathcal{F}$: the set C_- of all concave functions on $\mathbb{R}^n_{++}$; the set C_+ of all convex functions on $\mathbb{R}^n_{++}$; and the set C of all functions that are either concave or convex.