Level: undergraduate math methods
Background: basic power series (Boas ch.1 or equivalent)
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Taylor Series: three kinds

A function whose derivatives all exist (and are continuous) is called C^∞ or *smooth*. If a function is smooth at a point a, then we can write out its Taylor series,

$$f(x) \approx f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + \tfrac{1}{3!}f^{(3)}(x-a)^3 + \dots \tag{1}$$

I'm being purposely vague here by writing $\approx$, because we have not yet established the sense in which the series approximates the function. You are probably thinking that the series converges to the function (at least sufficiently near $x = a$):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} x^n. \qquad \text{(analytic functions)} \tag{2}$$

But this doesn't always happen. When it does (i.e. when there is a finite radius of convergence R such that Eq. (2) holds in $|x - a| < R$), we say the function is *analytic* at $x = a$. But sometimes the radius of converge is *zero*—the series doesn't converge anywhere except the point $x = a$, where it is useless. And there is actually another, more sinister pathology: Sometimes the Taylor series converges to something that isn't the original function! We therefore divide smooth functions into three types:[1]

- Analytic functions: the Taylor series converges to the function.
- Smooth, non-analytic type 1: The Taylor series diverges (zero radius of convergence).
- Smooth, non-analytic type 2: The Taylor series converges to something else.

You have every right to be surprised, but please don't be discouraged: the Taylor series is actually quite useful even for non-analytic functions. The point is to stop thinking about convergence, but instead think about something called *asymptotic* equality. To explain, let's imagine truncating the series at some order, writing

$$f(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + O((x-a)^3). \tag{3}$$

You are probably used to thinking of $O((x-a)^3)$ as a shorthand for "some terms involving $(x-a)^3$ and all higher powers." With this meaning, Eq. (3) is correct only for analytic functions. However, it turns out that Eq. (3) remains correct for all smooth functions if we instead say that $O((x-a)^3)$ stands for "a function that scales to zero like $(x-a)^3$ or faster as $x \to a$". To be completely precise, we say[2]

$$g(x) = O((x-a)^n) \;\; \text{means:} \;\; \lim_{x\to a} \frac{g(x)}{(x-a)^n} < \infty. \tag{4}$$

[1]Note that while analytic and non-analytic are standard names, the types 1 and 2 are my own terminology.
[2]This definition is adequate when we deal with smooth functions $g(x)$. More generally you need the notion of a limit supremum, or an alternative notion of the boundedness of the ratio.

You can think of $O((x-a)^n)$ as "anything that goes to zero like $(x-a)^n$ or faster, including higher powers and/or really nasty stuff like $e^{-1/x}$." (See below for a physical discussion of this nasty stuff.) For example, $3x^2 = O(x^2)$, $3x^2 = O(x)$, $x^3 \sin x = O(x^3)$, $15x^{21.22} = O(x^{13})$, $e^{-1/x} = O(x^n)$ for all n. Now we can clarify (3) just by rearranging the terms,

$$f(x) - [f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2] = O((x-a)^3). \tag{5}$$

The left-hand side is the error you make by replacing the function with the truncated Taylor series. Eq. (5) asserts that this error is *controlled* as $x \to a$: it scales to zero like $(x-a)^3$ or better.[3] In fact, we can similarly get controlled error for any truncation:

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \tfrac{1}{n!}f^{(n)}(a)(x-a)^n + O((x-a)^{n+1}). \tag{6}$$

We call this an *asymptotic* power series: if I truncate the series at some order n, then the error I make scales to zero like $(x-a)^{n+1}$ or faster.[4] Eq. (6) is the statement of "Taylor's theorem": *The Taylor expansion of a smooth function is asymptotic to that function.* Even if you don't study the formal definition of an asymptotic series, make sure you understand the difference from a convergent one:

- To check whether a series is asymptotic, we fix the number of terms n and vary the evaluation point $x \to a$. To check whether a series is convergent, we fix the evaluation point $x \neq a$ and vary the number of terms $n \to \infty$.

Physicists sometimes use the phrase "asymptotic" synonymously with "divergent", because only divergent series force you to think carefully about the asymptotic nature of the approximation. But strictly speaking, all convergent power series are asymptotic as well.

When the function $f(x)$ is known in some convenient form, you can usually figure out how its Taylor series behaves (i.e., which of the three types it is). However, more common in physics is when you are trying to compute some $f(x)$ (e.g. the prediction of your theory) but you don't really know much about it, and all you can get is a series approximation, or perhaps only the first term or two. In this case it's very hard to figure out what type of series it is, or, if you only have a couple terms, even if the function is smooth. Generally, all series should be treated as asymptotic until proven convergent (guilty until proven innoncent!) and even convergent series should be viewed with some suspicion, as they may be missing some important physics (i.e. converging to the wrong answer, even though the asymptotic approximation is good).

0.1 Analytic functions—the Taylor series converges to the function

The smooth functions you are familar with as "elementary functions"—powers, sines and cosines, exponentials, etc.—are all analytic. Analyticity may therefore seem natural, but in a certain sense it is highly unusual. For suppose that you know the value and derivatives of an analytic function at a single point. By definition, the Taylor series will have a non-zero radius of convergence, so you can use it to get the function elsewhere. This may not give it to you *everywhere*, but you can always just pick a new point within initial radius of convergence (where you now know the

[3] This does not guarantee that the approximation is *useful* at any particular value of x, only that it becomes *more useful* as $x \to a$. See below for discussion.

[4] This definition is valid for power series. See Boas 11.10 for the (important) general definition, which allows series involving logs and other non-smooth functions.

function), and repeat the exercise to determine the function in a new region of convergence. It is possible to show that these overlapping regions can always be chosen to include the entire region of analyticity. Thus:

- An analytic function is determined everywhere by its value and all derivatives at a point.

If you've studied the difference between a countable set (like integers) and an uncountable set like real numbers, you will see that this is an extremely constraining property. Analytic functions are evidently described by assigning a number to each non-negative integer (i.e., giving the value and derivatives at a point), whereas normal functions require assigning a number to each real number (the value of the function at each point x). Thus analytic functions are as sparse among functions as the integers are sparse among the reals! For this reason, it should not shock you that many interesting physical phenomena are described by non-analytic functions.

0.2 Smooth, non-analytic type 1—the Taylor series diverges

We said that a function is type 1 non-analytic at a point $x = a$ if its Taylor series there has zero radius of convergence. This happens all the time in physics at special points like the location of an obstacle in a fluid flow or the event horizon of a black hole. Here is a simple example function:

$$f(x) = \int_0^\infty \frac{e^{-t}}{1 + x^2 t} dt. \tag{7}$$

You can see that there might be some funny business at $x = 0$ because the integral ranges to infinite t, where the integrand has qualitatively different end behaviors in the $x = 0$ and $x \neq 0$ cases (e^{-t} versus e^{-t}/t, respectively). There is no way to do this integral in closed form, but it is straightforward to compute the derivatives at $x = 0$. These are

$$f^{(m)}(0) = \begin{cases} 0, & \text{m odd} \\ (-1)^{m/2} \left(\frac{m}{2}\right)! m!, & \text{m even} \end{cases}. \tag{8}$$

The Taylor series is thus

$$f(x) = 1 - (1!)x^2 + (2!)x^4 - (3!)x^6 + (4!)x^8 + \dots \tag{9}$$

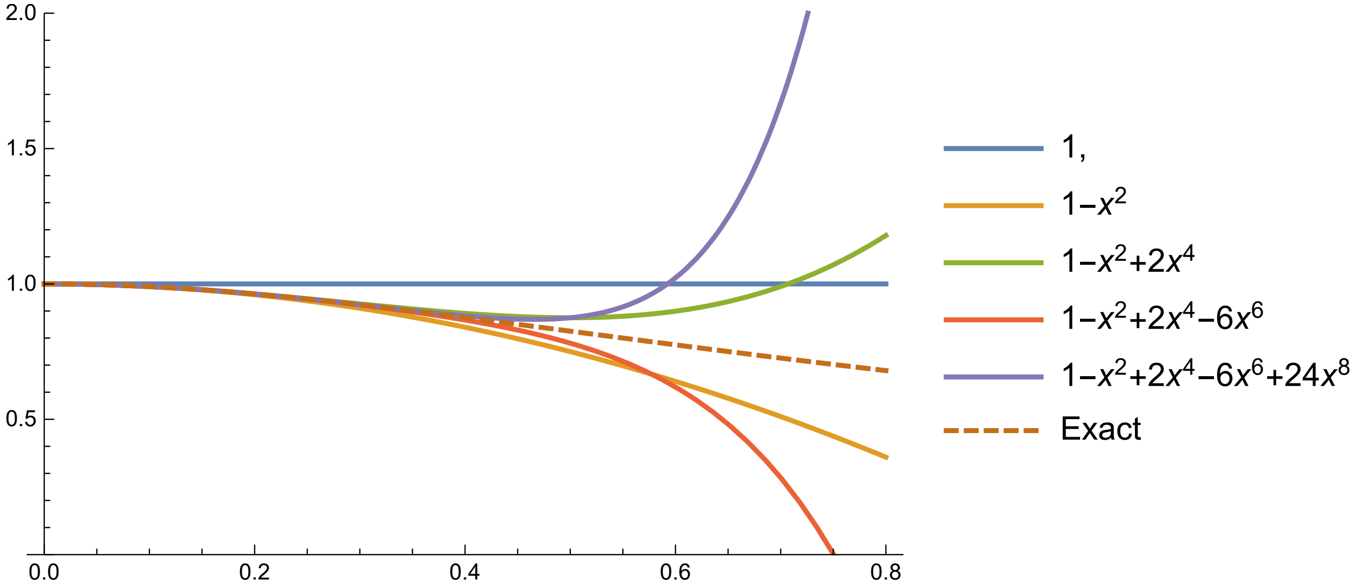
The factorial grows very quickly, and this series diverges for all $x \neq 0$. However, that doesn't mean the series is useless. Since Taylor series are always asymptotic approximations, we may truncate at some order and be confident that the error scales to zero as $x \to 0$ at some given rate. For example, keeping the first five terms, we get

$$f(x) = 1 - (1!)x^2 + (2!)x^4 - (3!)x^6 + (4!)x^8 + O(x^{10}). \tag{10}$$

This statement means that that if I include just the displayed terms, then, as I move x towards zero, the error I make scales to zero like x^{10} or faster. Thus I *expect* that such a truncation will give a decent approximation if I take x to be sufficiently small. But this is just an expectation: there is no guarantee from mathematics that the approximation will be any good at all at any given value of x. Plus, mathematics has no way of knowhing what you mean by good. How small is small, and how good is good? That's up to you, the physicist, to sort out.[5]

[5] You may find this annoying, but that's life. Very often the exact function $f(x)$ is too difficult to compute or even unknown. In this case you have to get clever about validating your approximation—e.g. by working hard to compute $f(x)$ exactly at a few values, or by measuring it experimentally!

To illustrate, let's plot the exact answer together with a few different truncations of the asymptotic series:



For small x, the various truncations all do pretty well, with more terms helping the approximation. But around $x = .5$, adding more terms starts to make things worse: the best approximation around $x \sim (.55, .6)$ is actually the quartic (green) one, with just three terms. In fact, for an asymptotic series, there will always be a "best" number of terms at every x. (If adding more terms always helped you, the series would be convergent!) It may *look* like things are converging at small x, but that's just because we only plotted the first few truncations. Even at $x = .00001$, there is a best number of terms to include, though it will certainly be very many. And probably the first one or two is enough for your purposes, anyway.

0.3 Smooth, non-analytic type 2—the Taylor series converges to something else

The classic example is

$$F(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x}, & x > 0 \end{cases}. \tag{11}$$

The derivatives take the form

$$F^{(n)}(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x}/x^{2n} \times (\text{a polynomial}), & x > 0 \end{cases} \tag{12}$$

Since $e^{-1/x}$ goes to zero faster than any power of x, the formula on the first line limits to zero as $x \to 0$. Thus all derivatives exist everywhere and are continuous: the function is smooth and has a Taylor series. But what is this series? Using Eq. (1) with $a = 0$, we have

$$F(x) \approx 0 + 0 + 0 + 0 + \dots \tag{13}$$

That is, the Taylor series is *zero*. It converges, but it is not equal to the original function $F(x)$ (except at $x = 0$).

Take note that all this is perfectly consistent with the assertion that Taylor series are asymptotic approximcations. For example, suppose we truncate the series after three terms, as in (3),

$$F(x) = 0 + 0 + 0 + O(x^3) = O(x^3). \tag{14}$$

This equation is perfectly true: the difference between the truncated approximation $F \approx 0$ and the exact answer $F(x)$ scales *at least* as fast as x^3 as $x \to 0$, since

$$\lim_{x\to 0} \frac{F(x) - 0}{x^3} = \lim_{x\to 0} \frac{e^{-1/x}}{x^3} = 0. \tag{15}$$

In fact, it is scaling to zero faster than *any* power of x, since we could have truncated at any higher order. Eq. (15) is true when 3 is replaced by any positive integer: $e^{-1/x} = O(x^n)$ for all numbers n.

When I first learned about this function, I thought it was just mathematicians screwing with me with their crazy counter-examples. But in fact, such functions arise in nature. The example I know is the production of electron-positron pairs in a strong electric field ("Schwinger pair production"). The rate is proportional to

$$e^{\frac{-m^2c^3}{\hbar qE}}, \tag{16}$$

where m is the electron mass, q is the electron charge, c is the speed of light, $\hbar$ is Planck's constant, and E is the external field strength.

You may have heard that quantum electrodynamics is extremely well-tested, but this refers only to the formulation of the theory that is *perturbative* in the sense that calculations are done in a power series expansion in q. However, Eq. (16) is invisible to perturbation theory since its Taylor expansion in q is zero. The non-perturbative formulations that predict (16) have not been tested. Verifying (16) experimentally would test an entirely new regime of the theory, and indeed there are big lasers being built right now (2018) to do so.

Sometimes you will run into people who think that the main limitation of series expansions is accuracy: that somehow if you computed enough terms, you would recover all the physics. You can stump these poor misguided souls with Schwinger pair production!